PERIYAR UNIVERSITY

(NAAC 'A++' Grade - State University - NIRF Rank 56 – State Public University Rank - 25)

SALEM - 636 011

CENTRE FOR DISTANCE AND EDUCATION (CDOE)

M.Sc. Mathematics

SEMESTER – I

ELECTIVE – II: MATHEMATICAL PROGRAMMING

(Candidates admitted from 2024 onwards)

Prepared by:

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CONTENT

MATHEMATICAL PROGRAMMING

OBJECTIVE:

The objective of this course is

- the understanding of mathematical structure and properties of the fundamental problem (e.g., linear, non-linear and integer programming, dynamic programming).
- the use of Mathematical Problem algorithms for problem solving but also the design of their variants for special problem cases.
- the formulation and solving of problems arising from the practical, real-life settings.

UNIT I: INTEGER LINEAR PROGRAMMING

Types of integer linear programming problems – concepts of cutting plane – Gomary's all integer cutting plane method – Gomary's mixed integer cutting plane method – Branch and Bound method – Zero-one integer programming.

Dynamic programming: characteristic of dynamic programming problem – developing optimal decision policy – Dynamic programming under certainty – DP approach to solve LPP.

Sections: 1.1 to 1.12

UNIT II: CLASSICAL OPTIMIZATION METHODS

Unconstrained optimization – constrained multi-variable optimization with equality constraints – constrained multi-variable optimization with inequality constraints.

Non-linear programming method: Examples of NLPP – General NLPP – Graphical solution – Quadratic programming – Wolfe's modified simplex methods – Beale's method.

Sections: 2.1 to 2.8

UNIT III: THEORY OF SIMPLEX METHOD

Canonical and standard form of LP-slack and surplus variables – Reduction of any feasible to a basic feasible solution – alternative optimal solution – unbounded solution – optimality conditions – some complications and their resolution – Degeneracy and its resolution.

Sections: 3.1 to 3.8

UNIT IV: REVISED SIMPLEX METHOD

Standard forms for revised simplex Method-Computational procedure for Standard form I - comparison of simplex method and revised simplex Method.

BOUNDED VARIABLES LP PROBLEM: The simplex algorithm.

Sections: 4.1 to 4.6

UNIT V: PARAMETRIC LINEAR PROGRAMMING

Variation in the coefficients c_j, Variations in the Right hand side, b_i.

Goal Programming: Difference between LP and GP approach - Concept of Goal Programming - Goal Programming Model formulation - Graphical Solution Method of Goal Programming –Modified Simplex method of Goal Programming.

Sections: 5.1 to 5.9

BOOKS FOR SUPPLEMENTARY READING AND REFERENCES:

1. J.K. Sharma, *Operations Research*, Theory and Applications, Third Edition (2007) Macmillan India Ltd.
2. Hamdy A. Taha, *Operations Research,* (seventh edition) Prentice-Hall of India Private Limited, NewDelhi,1997.
3. F.S. Hillier & J. Lieberman *Introduction to Operation Research* (7^{th}Edition) Tata-McGraw Hill company, New Delhi, 2001.
4. Beightler. C, D. Phillips, B. Wilde, *Foundations of Optimization* (2^{nd} Edition) Prentice Hall Pvt Ltd., New York, 1979.
5. S.S. Rao - *Optimization Theory and Applications*, Wiley Eastern Ltd., NewDelhi.1990.

UNIT – I

INTEGER LINEAR PROGRAMMING

INTEGER LINEAR PROGRAMMING

Objectives:

After studying this unit, students should be able to learn the limitations of simplex method in deriving integer solution to LPP. Apply cutting plane methods to obtain optimal integer solution value of variables in an LPP. Apply Branch and Bound method to solve integer LPP. Appreciate application of integer LPP problem in several areas of managerial decision-making.

Make distinction between linear programming and dynamic programming approaches for solving a problem. Develop recursive function based on Bellmain's principle of optimality to get an optimal solution of any multi-stage decision problem. To learn various dynamic programming models and their applications in solving a decision-problem. Solve an LPP using the dynamic programming approach.

1.1 Introduction

In linear programming each decision variable as well as slack or surplus variable is allowed to take any real value (fractional also). However, there are some real-life problems in which the fractional value of the decision variable has no significant.

For example, 1.5 men will be working on a project or 1.8 machine will be used in a workshop are meaningless. The integer solution to a problem can be obtained by the rounding off optimum solution of the variable. But this solution may not satisfy all the given constraints. Integer Linear Programming (ILP) deals with linear programme in which some or all of the variables assume integer values.

1.2 Types of Integer Programming Problem (ILPP)

ILPP can be classified in to three categories

Pure Integer Programming Problem

An LPP, in which all decision variables are required to have integer values is called pure (all) integer programming problem.

Mixed Integer Programming Problem

An LPP, in which some, but not all of the decision variables are required to have integer values is called a mixed integer programming problem.

0-1 Integer Programming Problem

An LPP, in which all decision variables must have integer value of 0 or 1 is called a zero – one integer programming problem.

General Model of an ILPP

$$Max\ (or)\ Min Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

Subject to the constraints,

$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \leq (or) = (or) \geq b_i$, $\forall i = 1\ to\ m x_j \geq 0\ and\ integers$, j =1 $to\ n$.

1.3 Enumeration and Cutting Plane Solution Concept

The Cutting- plane method to solve integer LP problem. This method is based on creating a sequence of linear inequalities called cuts. Such a cut reduces a part of the feasible region of the given LP problem, leaving out a feasible region of the integer LP problem. The hyper plane boundary of a cut is called the cutting plane.

Solution Procedure

Relax the solution space of the integer problem by ignoring the integer restrictions. This step converts ILPP into a regular LPP.

Solve the resulting relaxed LPP and identify its optimum point.

Starting from the optimum point add special constraints that will iteratively force the optimum extreme point of the resulting LPP towards the integer restrictions.

Methods to solve ILPP

Gomory's Cutting Plane Method – Branch and Bound Method

1.4 Gomory's All Integer Cutting Plane Method (Or) Fractional Algorithm

Gomory's Cutting Plane method was developed by R.E.Gomory in 1956 to solve ILPP using dual – simplex method. This algorithm has the following properties:

Additional linear constraints never cut-off that portion of the original feasible solution space that contains the feasible integer solution to the original problem.

Each new additional constraint cuts-off the current non–integer optimal solution to the LPP.

Algorithm:

Step-1: Solve the LPP by simplex method ignoring integer requirement of the variables.

Step-2: Test for Optimality:

Examine the optimal solution. If all the given variables have the integer values then the integer optimum solution is obtained and the procedure can be terminated.

If one or more basic variables with integer requirement with fractional value then go to Step (3).

Step-3: Generate Cutting Plane:

Choose row 'r' corresponding to a variable x_r which has largest fractional value f_r.

Let rth constraint equation be,

$$x_r + \sum_{j \neq r} a_{rj} x_j = b_r \text{ ... ... (1)} \qquad \text{(j = 1 to n)}$$

Decompose the coefficients of x_j variables and b_r of equation (1) into integer and non-negative fractional part as follows:

$$x_r + \sum_{j \neq r} \{[a_{rj}] + f_{rj}\} x_j = [b_r] + f_r$$

The corresponding Gomory's cutting Plane constraint is,

$$s_g - \sum_{j \neq r} f_{rj} x_j = -f_r$$

Where S_g is non-negative slack variable, $0 < f_{rj} < 1$ and $0 < f_r < 1$.

Step-4: Obtain the New Solution:

Add the cutting plane constraint at the bottom of the optimum simplex table. Find new optimal solution by using **dual – simplex method**. The process is repeated until **all the basic variable** with integer requirements assume non – negative integer values.

Remark: A basic requirement for the application of this algorithm is that all the coefficients and the R.H.S constants of each constraint must be integer .For example, the constraint,

$x_1 + \frac{1}{2} x_2 \leq \frac{7}{3}$ should be transformed to $6x_1 + 3x_2 \leq 14.$

Where no fractions are present.

Example1.4.1 Find optimum integer solution to the following ILPP

$$MaxZ=2x_1+2x_2$$

$$Subject\ to,\ 5x_1+3x_2\leq 8$$

$$2x_1+4x_2\leq 8$$

$$x_1,x_2\geq 0\ \&\ \text{integers.}$$

Solution: After ignoring the integer restrictions, the standard LPP is,

$$MaxZ=2x_1+2x_2+0x_3+0x_4$$

$$Subject\ to,\ 5x_1+3x_2+x_3=8$$

$$2x_1+4x_2+x_4=8$$

$$x_1,x_2,x_3,x_4\geq 0$$

Applying simplex algorithm,

	C_j	2	2	0	0		
C_B	B.V	x_1	x_2	x_3	x_4	x_B	Ratio
0	x_3	5	3	1	0	8	$\frac{8}{3}=2.6$
0	x_4	2	4	0	1	8	$\frac{8}{4}=2$
$Z_j=C_Bx_j$		0	0	0	0	0	
C_j-Z_j		2	2	0	0	-	
0	x_3	$\frac{7}{2}$	0	1	$-\frac{3}{4}$	2	$\frac{2}{7/2}=\frac{4}{7}$
2	x_2	$\frac{1}{2}$	1	0	$\frac{1}{4}$	2	4
$Z_j=C_Bx_j$		1	2	0	$\frac{1}{2}$	4	
C_j-Z_j		1	0	0	$-\frac{1}{2}$	-	
2	x_1	1	0	$2/7$	$-3/14$	$4/7$	-
2	x_2	0	1	$-1/7$	$5/14$	$12/7$	-
$Z_j=C_Bx_j$		2	2	$2/7$	$2/7$	$32/7$	
C_j-Z_j		0	0	$-2/7$	$-2/7$		

All $C_j - Z_j \le 0$ and $x_B \ge 0$. Hence, solution is optimum and feasible.

Optimum solution after ignoring integer restrictions,

$$x_1 = \frac{4}{7}, x_2 = \frac{12}{7} \text{ and } Z_{max} = \frac{32}{7}$$

Now, $x_1 = \frac{4}{7} = 0 + \frac{4}{7}(f_1)$ and $x_2 = \frac{12}{7} = 1 + \frac{5}{7}(f_2)$

$$\text{Max } \{f_1, f_2\} = \frac{5}{7}$$

The maximum fraction is $\frac{5}{7}$, corresponding to x_2 . Consider x_2 equation for fractional cut.

The x_2 equation is, $x_2 - \frac{1}{7}x_3 + \frac{5}{14}x_4 = \frac{12}{7}$ → $x_2 + \left(-1 + \frac{6}{7}\right)x_3 + \left(0 + \frac{5}{14}\right)x_4 = 1 + \frac{5}{7}$

The corresponding fractional cut equation is, $s_1 - \frac{6}{7}x_3 - \frac{5}{14}x_4 = -\frac{5}{7}$

Including this constraint in the optimum table and applying dual simplex algorithm, we get,

	C_j	2	2	0	0	0	
C_B	B.V	x_1	x_2	x_3	x_4	s_1	x_B
2	x_1	1	0	$2/7$	$-3/14$	0	$4/7$
2	x_2	0	1	$-1/7$	$5/14$	0	$12/7$
0	s_1	0	0	$-6/7$	$-5/14$	1	$-5/7$
$Z_j = C_B x_j$		2	2	$2/7$	$2/7$	0	$32/7$
$C_j - Z_j$		0	0	$-2/7$	$-2/7$	0	-
Ratio		-	-	$\frac{1}{3}$	$\frac{4}{5}$	-	-
2	x_1	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
2	x_2	0	1	0	$\frac{5}{12}$	$-\frac{1}{6}$	$\frac{11}{6}$
0	x_3	0	0	1	$\frac{5}{12}$	$-\frac{7}{6}$	$\frac{5}{6}$
$Z_j = C_B x_j$		2	2	0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{13}{3}$
$C_j - Z_j$		0	0	0	$-\frac{1}{6}$	$-\frac{1}{3}$	

All $C_j - Z_j \le 0$ *and* $x_B \ge 0$. Hence, solution is optimum and feasible.

Optimum solution after ignoring integer restriction,

$$x_1 = \frac{1}{3}, x_2 = \frac{11}{6}, x_3 = \frac{5}{6} \text{ and } Z_{max} = \frac{13}{3}$$

Still the basic variables are non–integer.

Now,

$$x_1 = \frac{1}{3} = 0 + \frac{1}{3}(f_1)$$

$$x_2 = \frac{11}{6} = 1 + \frac{5}{6}(f_2)$$

$$x_3 = \frac{5}{6} = 0 + \frac{5}{6}(f_3)$$

$$\text{Max } \{f_1, f_2, f_3\} = \frac{5}{6}$$

The maximum fraction is $\frac{5}{6}$, corresponding to x_2 and x_3, consider x_2 for fractional cut. The x_2 equation is, $x_2 + \frac{5}{12}x_4 - \frac{1}{6}s_1 = \frac{11}{6}$

$$x_2 + \left(0 + \frac{5}{12}\right)x_4 + \left(-1 + \frac{5}{6}\right)s_1 = 1 + \frac{5}{6}$$

The corresponding fractional cut equation is, $s_2 - \frac{5}{12}x_4 - \frac{5}{6}s_1 = -\frac{5}{6}$. Including this constraint in the optimum table and applying dual simplex algorithm, we get

	C_j	2	2	0	0	0	0	
C_B	B.V	x_1	x_2	x_3	x_4	s_1	s_2	x_B
2	x_1	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{1}{3}$
2	x_2	0	1	0	$\frac{5}{12}$	$\frac{-1}{6}$	0	$\frac{11}{6}$
0	x_3	0	0	1	$\frac{5}{12}$	$-\frac{7}{6}$	0	$\frac{5}{6}$
0	s_2	0	0	0	$-\frac{5}{12}$	$-\frac{5}{6}$	1	$-\frac{5}{6}$
$Z_j = C_B x_j$		2	2	0	$\frac{1}{6}$	$\frac{1}{3}$	0	$\frac{13}{3}$
$C_j - Z_j$		0	0	0	$-\frac{1}{6}$	$-\frac{7}{3}$	0	-

Ratio		-	-	-	$\frac{2}{5}$	$\frac{2}{5}$	-	-
2	x_1	1	0	0	$-\frac{1}{2}$	0	$\frac{2}{5}$	0
2	x_2	0	1	0	$\frac{1}{2}$	0	$-\frac{1}{5}$	2
0	x_3	0	0	1	1	0	$-\frac{7}{5}$	2
0	s_1	0	0	0	$\frac{1}{2}$	1	$-\frac{6}{5}$	1
$Z_j = C_B x_j$		2	2	0	0	0	$\frac{2}{5}$	4
$C_j - Z_j$		0	0	0	0	0	$-\frac{2}{5}$	-

All $C_j - Z_j \le 0$ *and* $x_B \ge 0$. Hence, solution is optimum and feasible. Also the basic variables are integers.

Optimum integer solution is, X_1 , X_2 *and* Max Z= 4

Example1.4.2 Find optimum integer solution to the following ILPP

$MaxZ = x_1 + x_2$

Subject to, $3x_1 + 2x_2 \le 15$; $x_3 \le 2$; $x_1, x_2 \ge 0$ & *integers*.

Solution: The standard LPP is, $\mathrm{MaxZ} = x_1 + x_2 + 0x_3 + 0x_4$

Subject to,

$$3x_1 + 2x_2 + x_3 = 15 \; ;$$

$$x_2 + x_4 = 2 \; ;$$

$$x_1 \, , x_2 \, , x_3 \, , x_4 \ge 0$$

Applying the simplex algorithm, after ignoring the integer restrictions,

	C_j	1	1	0	0		
C_B	B.V	x_1	x_2	x_3	x_4	x_B	Ratio
0	x_3	3	2	1	0	15	$\frac{15}{3} = 5$
0	x_4	0	1	0	1	2	-
$Z_j = C_B x_j$		0	0	0	0	0	
$C_j - Z_j$		1	1	0	0	-	
1	x_1	1	$2/3$	$1/3$	0	5	$\frac{15}{2} = 7.5$
0	x_4	0	1	0	1	2	2
$Z_j = C_B x_j$		1	$2/3$	$1/3$	0	5	
$C_j - Z_j$		0	$1/3$	$-1/3$	0	-	
1	x_1	1	0	$1/3$	$-2/3$	$11/3$	
1	x_2	0	1	0	1	2	
$Z_j = C_B x_j$		1	1	$1/3$	$1/3$	$17/3$	
$C_j - Z_j$		0	0	$-1/3$	$-1/3$	-	

All $C_j - Z_j \leq 0$, the present solution is optimum. Optimum solution after ignoring integer restrictions,

$$x_1 = \frac{11}{3}, x_2 = 2 \text{ and } Z_{max} = \frac{17}{3}$$

Now, $x_1 = \frac{11}{3} = 3 + \frac{2}{3}(f_1)$ and $x_2 = 2$

Consider x_1 equation for fractional cut.

The x_2 equation is, $x_1 + \frac{1}{3}x_3 - \frac{2}{3}x_4 = \frac{11}{3}$

$$x_1 + \left(0 + \frac{1}{3}\right)x_3 + \left(-1 + \frac{1}{3}\right)x_4 = 3 + \frac{2}{3}$$

The corresponding fractional cut equation is $s_1 - \frac{1}{3}x_3 - \frac{1}{3}x_4 = -\frac{2}{3}$

Including the constraint in the optimum table and applying dual simplex algorithm, we get

	C_j	1	1	0	0	0	
C_B	B.V	x_1	x_2	x_3	x_4	s_1	x_B
1	x_1	1	0	$1/3$	$-2/3$	0	$11/3$
1	x_2	0	1	0	1	0	2
0	s_1	0	0	$-1/3$	$-1/3$	1	$-2/3$
$Z_j = C_B x_j$		1	1	$1/3$	$1/3$	0	$17/3$
$C_j - Z_j$		0	0	$-1/3$	$-1/3$	0	
Ratio		0	0	1	1	-	
1	x_1	1	0	0	-1	1	3
1	x_2	0	1	0	1	0	2
0	x_3	0	0	1	1	-3	2
$Z_j = C_B x_j$		1	1	0	0	1	5
$C_j - Z_j$		0	0	0	0	-1	-

All $C_j - Z_j \leq 0$ *and* $x_B \geq 0$.

Hence, solution is optimum and feasible. Also the basic variables are integers.Optimum integer solution is, $x_1 = 3, x_2 = 2$, *and* Max $Z = 5$

Let Us Sum Up

We have learned about Integer linear Programming problem and its types and also to find the solutions of ILPP by using Gomary's all integer cutting plane method.

Check Your Progress

Find the optimum integer solution to the following ILPP

1. Max Z = $2x_1 + 3x_2$

Subject to, $5x_1 + 7x_2 \leq 15$

$4x_1 + 9x_2 \leq 20$;

$x_1 , x_2 \geq 0$ and integers.

2. Max Z = $2x_1 + 20x_2 - 10x_3$

Subject to, $2x_1 + 20x_2 + 4x_3 \leq 15$;

$6x_1 + 20x_2 + 4x_3 = 20$;

$x_1, x_2, x_3 \geq 0$ and integers.

3. Max Z = $7x_1 + 9x_2$

Subject to, $-x_1 + 3x_2 \leq 6$;

$7x_1 + x_2 \leq 35$;

$x_1, x_2 \geq 0$ and integers.

4. Max Z = $3x_1 + 12x_2$

Subject to, $2x_1 + 4x_2 \leq 7$;

$5x_1 + 3x_2 \leq 15$

$x_1 , x_2 \geq 0$ and integers.

1.5 Gomory's Mixed Integer Programming Algorithm

Step-1: Initialization

Solve the LPP by Simplex Method after ignoring integer requirements of the variables.

Step-2: Test for Optimality

Examine the optimal solution. If all the integer restricted basic variable have integer value can terminate the procedure.

If some of the integer restricted basic variables are not integers then go to step-3

Step-3: Generate Cutting Plane

Choose a row 'r' corresponding to a basic variable x_r that has the largest fractional value f_r, among the integer restricted basic variables.

Let the r^{th} constraint equation be,

$$x_r + \sum_{j \in R_+} a_{rj}\, x_j + \sum_{j \in R_-} a_{rj}\, x_j = b_r = [b_r] + f_r$$

where, $R_+ = \{j : a_{rj} \geq 0\} = \{set\ of\ subscripts\ 'j'\ for\ which\ a_{rj} \geq 0\}$

$R_- = \{j : a_{rjj} < 0\} = \{set\ of\ subscripts\ 'j'\ for\ which\ a_{rj} < 0\}$

Corresponding the fractional cut equation is,

$$S_g - \sum_{j \in R_+} a_{rj}\, x_j - \left(\frac{f_r}{f_r - 1}\right) \sum_{j \in R_-} a_{rj}\, x_j = -f_r, \text{ where } 0 < f_r < 1,$$

Step-4: Obtain the new solution

Include the cutting plane generated in step-3 to the bottom of the simplex table. Find a new optimal solution by using dual simplex method. The process is repeated until all integer restricted basic variables are integers.

Example 1.5.1 Solve $\text{Max } Z = -3x_1 + x_2 + 3x_3$

Subject to, $-x_1 + 2x_2 + x_3 \leq 4$; $2x_2 - \frac{3}{2}x_3 \leq 1$; $x_1 - 3x_2 + 2x_3 \leq 3$

$x_1, x_2 \geq 0$ and x_3 is non – negative integer.

Solution: The standard LPP is,

$\text{Max } Z = -3x_1 + x_2 + 3x_3 + 0x_4 + 0x_5 + 0x_6$

$Subject\ to$: $-x_1 + 2x_2 + x_3 + x_4 = 4$;

$$2x_2 - \frac{3}{2}x_3 + x_5 = 1\,;$$

$$x_1 - 3x_2 + 2x_3 + x_6 = 3;$$

$$x_1\,, x_2\,, x_3\,, x_4\,, x_5\,, x_6 \geq 0$$

Ignoring the integer restrictions then the simplex table, we get

	C_j	-3	1	3	0	0	0		
C_B	B.V	x_1	x_2	x_3	x_4	x_5	x_6	x_B	Ratio
0	x_4	-1	2	1	1	0	0	4	4
0	x_5	0	2	$-3/2$	0	1	0	1	-
0	x_6	1	-3	2	0	0	1	3	$3/2$
$Z_j = C_B x_j$		0	0	0	0	0	0	0	
$C_j - Z_j$		-3	1	3	0	0	0		
0	x_4	$-3/2$	$7/2$	0	1	0	$-1/2$	$5/2$	$5/7$
0	x_5	$3/4$	$-1/4$	0	0	1	$3/4$	$13/4$	-
3	x_3	$1/2$	$-3/2$	1	0	0	$1/2$	$3/2$	-
$Z_j = C_B x_j$		$3/2$	$-9/2$	3	0	0	$3/2$	$9/2$	
$C_j - Z_j$		$-9/2$	$11/2$	0	0	0	$-3/2$	-	
1	x_2	$-3/7$	1	0	$2/7$	0	$-1/7$	$5/7$	
0	x_5	$9/14$	0	0	$1/14$	1	$5/7$	$24/7$	
3	x_3	$-1/7$	0	1	$3/7$	0	$2/7$	$18/7$	
$Z_j = C_B x_j$		$-6/7$	1	3	$11/7$	0	$5/7$	$59/7$	
$C_j - Z_j$		$-15/7$	0	0	$-11/7$	0	$-5/7$	-	

All $C_j - Z_j \leq 0$, the Optimum solution is,

$x_1 = 0\,, x_2 = \frac{5}{7}\,, x_3 = \frac{18}{7}\,, x_4 = 0\,, x_5 = \frac{24}{7}\,, \text{Max } Z = \frac{59}{7}$

Here, x_3 is not an integer, hence consider x_3 equation for fractional cut.

x_3 equation is, $x_3 - \frac{1}{7}x_1 + \frac{3}{7}x_4 + \frac{2}{7}x_6 = \frac{18}{7} = 2 + \frac{4}{7}$

Here $R_+ = \{4, 6\}$ and $R_- = \{1\}$

The Gomory's fractional cut equation is,

$$s_1 - \frac{3}{7}x_4 - \frac{2}{7}x_6 - \left(\frac{\frac{4}{7}}{\frac{4}{7}-1}\right)\left(-\frac{1}{7}x_1\right) = -\frac{4}{7}$$

$$s_1 - \frac{3}{7}x_4 - \frac{2}{7}x_6 + \left(\frac{4}{3}\right)\left(-\frac{1}{7}x_1\right) = -\frac{4}{7}$$

$$s_1 - \frac{3}{7}x_4 - \frac{2}{7}x_6 - \frac{4}{21}x_1 = -\frac{4}{7}$$

Put this equation in the optimum table as last row and applying dual simplex algorithm, we get,

	C_j	-3	1	3	0	0	0	0	
C_B	B.V	x_1	x_2	x_3	x_4	x_5	x_6	s_1	x_B
1	x_2	$-3/7$	1	0	$2/7$	0	$-1/7$	0	$5/7$
0	x_5	$9/14$	0	0	$1/14$	1	$5/7$	0	$24/7$
3	x_3	$-1/7$	0	1	$3/7$	0	$2/7$	0	$18/7$
0	s_1	$-4/21$	0	0	$-3/7$	0	$-2/7$	1	$-4/7$
$Z_j = C_B x_j$		$6/7$	1	3	$11/7$	0	$5/7$	0	$59/7$
$C_j - Z_j$		$-15/7$	0	0	$-11/7$	0	$-5/7$	0	-
Ratio		$45/4$	-	-	$11/3$	-	$5/2$	-	
1	x_2	$-1/3$	1	0	$1/2$	0	0	$-1/2$	1
0	x_5	$1/6$	0	0	−1	1	0	$5/2$	2
3	x_3	$-1/3$	0	1	0	0	0	1	2
0	x_6	$2/3$	0	0	$3/2$	0	1	$-7/2$	2
$Z_j = C_B x_j$		$-4/3$	1	3	$1/2$	0	0	$5/2$	7
$C_j - Z_j$		$-5/3$	0	0	$-1/2$	0	0	$-5/2$	

All $C_j - Z_j \leq 0$ and $x_B \geq 0$.

Hence, solution is optimum and feasible. Also, the basic variable x_3 is an integer. The Optimum solution is, $x_1 = 0, x_2 = 1, x_3 = 2, \text{Max } Z = 7$.

Example 1.5.2: Solve: Max Z = $4x_1 + 6x_2 + 2x_3$

Subject to, $4x_1 - 4x_2 \leq 5$;

$-x_1 + 6x_2 \leq 5$;

$-x_1 + x_2 + x_3 \leq 5$;

$x_2 \geq 0$ & x_1, x_3 are non-negative integers.

Solution:

The standard LPP is, Max Z = $4x_1 + 6x_2 + 2x_3 + 0x_4 + 0x_5 + 0x_6$

Subject to, $4x_1 - 4x_2 + 0x_3 + x_4 = 5$

$x_1 + 6x_3 + x_5 = 5$

$-x_1 + x_2 + x_3 + x_6 = 5$

$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

Ignoring the integer restrictions and applying simplex method, we get

	C_j	4	6	2	0	0	0		
C_B	B.V	x_1	x_2	x_3	x_4	x_5	x_6	x_B	Ratio
0	x_4	4	-4	0	1	0	0	5	-
0	x_5	-1	6	0	0	1	0	5	5/6
0	x_6	-1	1	1	0	0	1	5	$5/1$
$Z_j = C_B x_j$		0	0	0	0	0	0	0	
$C_j - Z_j$		4	6	2	0	0	0		
0	x_4	$10/3$	0	0	1	2/3	0	$25/3$	$5/2$
6	x_2	$-1/6$	1	0	0	1/6	0	$5/6$	-
0	x_6	$-5/6$	0	1	0	-1/6	1	$25/6$	-
$Z_j = C_B x_j$		-1	6	0	0	1	0	5	
$C_j - Z_j$		5	0	2	0	-1	0	-	
4	x_1	1	0	0	$3/10$	1/5	0	$5/2$	
6	x_2	0	1	0	$1/20$	1/5	0	$5/4$	
0	x_6	0	0	1	$1/4$	0	1	$25/4$	

$Z_j = C_B x_j$		4	6	0	3/2	2	0	$35/2$	
$C_j - Z_j$		0	0	2	$-3/2$	-2	0	-	
4	x_1	1	0	0	$3/10$	1/5	0	$5/2$	
6	x_2	0	1	0	$1/20$	1/5	0	$5/4$	
2	x_3	0	0	1	$1/4$	0	1	$25/4$	
$Z_j = C_B x_j$		4	6	2	2	2	2	30	
$C_j - Z_j$		0	0	0	-2	-2	-2	-	

All $C_j - Z_j \leq 0$, the Optimum solution, after ignoring integer restriction is,

$x_1 = \frac{5}{2}, x_2 = \frac{5}{4}, x_3 = \frac{25}{4}, x_4 = 0, x_5 = 0, x_6 = 0$, Max Z = 30

Here, x_1 & x_3 are restricted to take integer values, here both are not integers.

$$x_1 = \frac{5}{2} = 2 + \frac{1}{2}(f_1)$$

$$x_3 = \frac{25}{4} = 6 + \frac{1}{4}(f_2)$$

Max $\{f_1, f_2\} = \left\{\frac{1}{2}, \frac{1}{4}\right\} = \frac{1}{2}$ corresponding to x_1

Hence consider x_1 equation for fractional cut.

x_1 equation is, $x_1 + \frac{3}{10}x_4 + \frac{1}{5}x_5 = \frac{5}{2} = 2 + \frac{1}{2}$

Here $R_+ = \{4, 5\}, R_- = \{\ \}$

The Gomory's fractional cut equation is, $s_1 - \frac{3}{10}x_4 - \frac{1}{5}x_5 = -\frac{1}{2}$

Include this equation in the optimum table as last row and applying dual simplex algorithm, we get

	C_j	4	6	2	0	0	0	0	
C_B	B.V	x_1	x_2	x_3	x_4	x_5	x_6	s_1	x_B
4	x_1	1	0	0	$3/10$	1/5	0	0	$5/7$
6	x_2	0	1	0	$1/20$	1/5	0	0	$24/7$
2	x_3	0	0	1	$1/4$	0	1	0	$18/7$
0	s_1	0	0	0	$-3/10$	-1/5	0	1	$-1/2$
$Z_j = C_B x_j$		4	6	2	2	2	2	0	30
$C_j - Z_j$		0	0	0	-2	-2	-2	0	-
Ratio		-	-	-	$-20/3$	-10	-	-	
4	x_1	1	0	0	0	0	0	1	2
6	x_2	0	1	0	0	1/6	0	1/6	7/6
2	x_3	0	0	1	0	-1/6	1	5/6	35/6
0	x_4	0	0	0	1	2/3	0	$-10/3$	5/3
$Z_j = C_B x_j$		4	6	2	0	2/3	2	$20/3$	80/3
$C_j - Z_j$		0	0	0	0	-2/3	-2	$-20/3$	

All $C_j - Z_j \leq 0$ and $x_B \geq 0$. Hence, solution is optimum and feasible.

$x_1 = 2, x_2 = \frac{7}{6}, x_3 = \frac{35}{6}, x_4 = \frac{5}{3}, x_5 = 0, x_6 = 0$, Max $Z = \frac{80}{3}$

Here, x_1 & x_3 are restricted to take integer values, but x_3 is not integer.

Hence consider x_3 equation for fractional cut.

x_3 equation is, $x_3 - \frac{1}{6}x_5 + x_6 + \frac{5}{6}s_1 = \frac{35}{6} = 5 + \frac{5}{6}$

Here $R_+ = \{6, 1\}, R_- = \{5\}$

The Gomory's fractional cut equation is, $s_2 - \left\{x_6 + \frac{5}{6}s_1\right\} - \left(\frac{\frac{5}{6}}{\frac{5}{6}-1}\right)\left(-\frac{1}{6}\right)x_5 = -\frac{5}{6}$

Include this equation in the optimum table as last row and applying dual simplex algorithm, we get,

	C_j	4	6	2	0	0	0	0	0	
C_B	B.V	x_1	x_2	x_3	x_4	x_5	x_6	s_1	s_2	x_B
4	x_1	1	0	0	0	0	0	1	0	2
6	x_2	0	1	0	0	1/6	0	1/6	0	7/6
2	x_3	0	0	1	0	-1/6	1	5/6	0	35/6
0	x_4	0	0	0	1	2/3	0	$-10/3$	0	5/3
0	s_2	0	0	0	0	-5/6	-1	-5/6	1	-5/6
$Z_j = C_B x_j$		4	6	2	0	2/3	2	$20/3$	0	80/3
$C_j - Z_j$		0	0	0	0	-2/3	-2	$-20/3$	0	-
Ratio		-	-	-	-	-4/5	2	-8	-	
4	x_1	1	0	0	0	0	0	1	0	2
6	x_2	0	1	0	0	0	0	0	1/5	1
2	x_3	0	0	1	0	0	1	1	-1/5	6
0	x_4	0	0	0	1	0	0	-4	4/5	1
0	x_5	0	0	0	0	1	-6/5	1	-6/5	1
$Z_j = C_B x_j$		4	6	2	0	0	2	6	4/5	26
$C_j - Z_j$		0	0	0	0	0	-2	-6	-4/5	

The Optimum solution is, $x_1 = 2$, $x_2 = 1$, $x_3 = 6$ and Max Z = 26.

Example 1.5.3 Find optimum solution to the following ILPP

Max Z = $7x_1 + 9x_2$

Subject to, $-x_1 + 3x_2 \leq 6$;

$7x_1 + x_2 \leq 35$;

$x_2 \geq 0$ & x_1 is an integer.

Solution:

The standard LPP is, Max Z = $7x_1 + 9x_2 + 0x_3 + 0x_4$

Subject to, $-x_1 + 3x_2 + x_3 \leq 6$;

$7x_1 + x_2 + x_4 \leq 35$;

$x_1, x_2, x_3, x_4 \geq 0$

Applying the simplex algorithm, after ignoring the integer restrictions,

	C_j	7	9	0	0		
C_B	B.V	x_1	x_2	x_3	x_4	x_B	Ratio
0	x_3	-1	**3**	1	0	6	$\frac{6}{3}$
0	x_4	7	1	0	1	35	35
$Z_j = C_B x_j$		0	0	0	0	0	
$C_j - Z_j$		7	9	0	0		
9	x_2	$-1/3$	1	$1/3$	0	2	--
0	x_4	$22/3$	0	$-1/3$	1	33	$\frac{9}{2}$
$Z_j = C_B x_j$		-3	9	3	0	18	
$C_j - Z_j$		10	0	-3	0		
9	x_2	0	1	$7/22$	$1/22$	$7/2$	
7	x_1	1	0	$-1/22$	$3/22$	$9/2$	
$Z_j = C_B x_j$		7	9	$28/11$	$15/11$	63	
$C_j - Z_j$		0	0	$-28/11$	$-15/11$		

All $C_j - Z_j \le 0$, the present solution is optimum. Optimum solution after ignoring integer restrictions,

$$x_1 = \frac{9}{2}, \quad x_2 = \frac{7}{2} \ and \ Z_{max} = 63$$

Now, x_1 is restricted to take integer value, bet it is not integer.

Consider x_1 equation for fractional cut.

The x_1 equation is, $x_1 - \frac{1}{22}x_3 + \frac{3}{22}x_4 = \frac{9}{2} = 4 + \frac{1}{2}$

The corresponding fractional cut equation is,

$$s_1 - \frac{3}{22}x_4 - \left(\frac{\frac{1}{2}}{\frac{1}{2} - 1}\right)\left(-\frac{1}{22}\right)x_3 = -\frac{1}{2}$$

$$s_1 - \frac{3}{22}x_4 - \frac{1}{22}x_3 = -\frac{1}{2}$$

Including this constraint in the optimum table and applying dual simplex algorithm, we get

	C_j	7	9	0	0	0	
C_B	B.V	x_1	x_2	x_3	x_4	s_1	x_B
9	x_2	0	1	$7/22$	$1/22$	0	$7/2$
7	x_1	1	0	$-1/22$	$3/22$	0	$9/2$
0	s_1	0	0	$-1/22$	$-3/22$	1	$-1/2$
$Z_j = C_B x_j$		7	9	$28/11$	$15/11$	0	63
$C_j - Z_j$		0	0	$-28/11$	$-15/11$	0	
Ratio		-	-	56	10	-	
9	x_2	0	1	$10/33$	0	$1/3$	$10/3$
7	x_1	1	0	$-1/11$	0	1	4

0	x_4	0	0	$1/3$	1	$-22/3$	$11/3$
$Z_j = C_B x_j$		7	9	$29/11$	0	10	58
$C_j - Z_j$		0	0	$-29/11$	0	-10	-

All $C_j - Z_j \leq 0$ and $x_B \geq 0$.

Hence, solution is optimum and feasible. Also the basic variables are integers.

Optimum integer solution is, $x_1 = 4, x_2 = \frac{10}{3}$ and Max Z = 58.

Example 1.5.4 Solve $\text{Min Z} = 2x_1 + x_2$

Subject to, $3x_1 + x_2 \geq 3$; $4x_1 + 3x_2 \geq 6$; $x_1 + 2x_2 \leq 3$

$x_1, x_2 \geq 0$ & x_1 is an integer.

Solution: This problem can easily be solved by dual simplex method.

LPP in right type after ignoring integer restriction is:

Max w = $-2x_1 - x_2$

Subject to, $-3x_1 - x_2 \leq -3$; $-4x_1 - 3x_2 \leq -6$; $x_1 + 2x_2 \leq 3$; $x_1, x_2 \geq 0$

	C_j	-2	-1	0	0	0	
C_B	B.V	x_1	x_2	x_3	x_4	x_5	x_B
0	x_3	-3	-1	1	0	0	-3
0	x_4	-4	-3	0	1	0	-6
0	x_5	1	2	0	0	1	3
$Z_j = C_B x_j$		0	0	0	0	0	0
$C_j - Z_j$		-2	-1	0	0	0	-
Ratio		$1/2$	$1/3$	-	-	-	-
0	x_3	$-5/3$	0	1	$-1/3$	0	-1
-1	x_2	$4/3$	1	0	$-1/3$	0	2
0	x_5	$-5/3$	0	0	$2/3$	1	-1
$Z_j = C_B x_j$		$-4/3$	-1	0	$1/3$	0	-2

$C_j - Z_j$		$-2/3$	0	0	$-1/3$	0	-
Ratio		$2/5$	-	-	-	-	-
0	x_3	0	0	1	-1	-1	0
-1	x_2	0	1	0	$1/5$	$4/5$	$6/5$
-2	x_1	1	0	0	$-2/5$	$-3/5$	$3/5$
$Z_j = C_B x_j$		-2	-1	0	$3/5$	$2/5$	$-12/5$
$C_j - Z_j$		0	0	0	$-3/5$	$-2/5$	-

All $C_j - Z_j \leq 0$ and $x_B \geq 0$.

Hence, solution is optimum and feasible.

$x_1 = \frac{3}{5}, x_2 = \frac{6}{5}, x_5 = 0, x_4 = 0, x_5 = 0$ and $Max\ W = -\frac{12}{5}$.

Here, x_1 is restricted to take integer value, but x_1 is not an integer.

Hence consider x_1 equation for fractional cut.

x_1 equation is, $x_1 - \frac{2}{5}x_4 - \frac{3}{5}x_5 = \frac{3}{5} = 0 + \frac{3}{5}$

Here $R_+ = \{\ \}$, $R_- = \{4, 5\}$ and $f_1 = \frac{3}{5}$

The Gomory's fractional cut equation is,

$$s_1 - \left\{\left(\frac{\frac{3}{5}}{\frac{3}{5} - 1}\right)\left[-\frac{2}{5}x_4 - \frac{3}{5}x_5\right]\right\} = -\frac{3}{5}$$

$$s_1 - \left\{\left(-\frac{3}{2}\right)\left[-\frac{2}{5}x_4 - \frac{3}{5}x_5\right]\right\} = -\frac{3}{5}$$

$$s_1 - \frac{3}{5}x_4 - \frac{9}{10}x_5 = -\frac{3}{5}$$

Include this equation in the optimum table as last row and applying dual simplex algorithm, we get,

	C_j	-2	-1	0	0	0	0	
C_B	B.V	x_1	x_2	x_3	x_4	x_5	s_1	x_B
0	x_3	0	0	1	-1	-1	0	0
-1	x_2	0	1	0	$1/5$	$4/5$	0	$6/5$
-2	x_1	1	0	0	$-2/5$	$-3/5$	0	$3/5$
0	s_1	0	0	0	$-3/5$	$-9/10$	1	$-3/5$
$Z_j = C_B x_j$		-2	-1	0	$3/5$	$2/5$	0	$-12/5$
$C_j - Z_j$		0	0	0	$-3/5$	$-2/5$	0	-
Ratio		-	-	-	1	$4/9$	-	-
0	x_3	0	0	1	$-1/3$	0	$-10/9$	$2/3$
-1	x_2	0	1	0	$-1/3$	0	$8/9$	$2/3$
-2	x_1	1	0	0	0	0	$-2/3$	1
0	x_5	0	0	0	$2/3$	1	$-10/9$	$2/3$
$Z_j = C_B x_j$		-2	-1	0	$1/3$	0	$4/9$	$-8/3$
$C_j - Z_j$		0	0	0	$-1/3$	0	$-4/9$	-

All $C_j - Z_j \leq 0$ and $x_B \geq 0$. Hence, solution is optimum and feasible.

$x_1 = 1$, $x_2 = \frac{2}{3}$, Max W = $-\frac{8}{3}$ and Min Z = $\frac{8}{3}$.

Let us Sum Up

We have learned about mixed Integer linear Programming problem and also to find the solutions of mixed Integer linear Programming problem by using Gomary's mixed integer cutting plane method.

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Check Your Progress

5. Solve: $Max\ Z = \boldsymbol{x_1} + \boldsymbol{x_2}$

Subject to, $2x_1 + 5x_2 \leq 16$; $6x_1 + 5x_2 \leq 30$; $x_1, x_2 \geq 0$, non – negative integer.

6. Solve: $Min\ Z = 5x_1 + 4x_2$

Subject to, $4x_1 + 2x_2 \geq 6$; $2x_1 + 3x_2 \geq 8$; $x_1, x_2 \geq 0$, non – negative integer

7. Solve: Max Z = $3x_1 + x_2 + 3x_3$

Subject to, $-x_2 + 2x_2 + x_3 \leq 4$; $4x_2 - 3x_3 \leq -2$; $x_1 - 3x_2 + 2x_3 \leq 3$

$x_1, x_2, x_3 \geq 0$ and x_1, x_3 are non – negative integers.

8. Solve: Max Z = $2x_1 + x_2$

Subject to, $10x_1 + 10x_2 \leq 9$; $10x_1 + 5x_2 \geq 1$; $x_1, x_2 \geq 0$, non – negative integer.

1.6 BRANCH AND BOUND METHOD

The Branch and Bound method developed first by A H Land and A G Doig is used to solve all-integer, mixed-integer and zero-one linear programming problems. The concept behind this method is to divide the feasible solution space of an LP problem into smaller parts called sub problems and then evaluate corner (extreme) points of each sub problem for an optimal solution.

The branch and bound method starts by imposing bounds on the value of objective function that help to determine the sub problem to be eliminated from consideration when the optimal solution has been found. If the solution to a sub problem does not yield an optimal integer solution, a new sub problem is selected for branching. At a point where no more sub problem can be created, an optimal solution is arrived at.

The branch and bound method for the profit-maximization integer LP problem can be summarized in the following steps:

The Procedure

Step 1: Initialization

Consider the following all integer programming problem

$$\text{Maximize } Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

subject to the constraints

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + c_{2n}x_n &= b_2 \\ \vdots \qquad\qquad & \quad \vdots \qquad\qquad \text{(LP – A)} \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

and $x_j \geq 0$ and non-negative integers.

Obtain the optimal solution of the given LP problem ignoring integer restriction on the variables.

(i) If the solution to this LP problem (say LP-A) is infeasible or unbounded, the solution to the given all-integer programming problem is also infeasible or unbounded, as the case may be.

(ii) If the solution satisfies the integer restrictions, the optimal integer solution has been obtained. If one or more basic variables do not satisfy integer requirement, then go to Step 2. Let the optimal value of objective function of LP-A be Z_1. This value provides an initial upper bound on objective function value and is denoted by Z_U.

(iii) Find a feasible solution by rounding off each variable value. The value of objective function so obtained is used as a lower bound and is denoted by Z_L.

Step 2: Branching step

(i) Let x_k be one basic variable which does not have an integer value and also has the largest fractional value.

(ii) Branch (or partition) the LP-A into two new LP sub-problems (also called *nodes*) based on integer values of x_k that are immediately above and below its non-integer value. That is, it is partitioned by adding two mutually exclusive constraints:

$$x_k \leq [\, x_k \,] \quad \text{and} \quad x_k \geq [\, x_k \,] + 1$$

to the original LP problem. Here $[x_k]$ is the integer portion of the current non-integer value of the variable x_k.This is obviously is done to exclude the non-integer value of the variable x_k. The two new LP sub-problems are as follows:

LP sub-problem B

Max $Z = \sum_{j=1}^{n} c_j x_j$

Subject to $\sum_{j=1}^{n} a_{ij} x_j = b_i$

$x_k \leq [x_k]$

and $\quad x_j \geq 0$

LP sub-problem C

Max $Z = \sum_{j=1}^{n} c_j x_j$

Subject to $\sum_{j=1}^{n} a_{ij} x_j = b_i$

$x_k \leq [x_k] + 1$

and $\quad x_j \geq 0$

Step 3: Bound step Obtain the optimal solution of sub problems B and C. Let the optimal value of the objective function of LP-B be Z_2 and that of LP-C be Z_3. The best integer solution value becomes the lower bound on the integer LP problem objective function value (Initially this is the rounded off value). Let the lower bound be denoted by Z_L.

Step 4: Fathoming step Examine the solution of both LP-B and LP-C

(i) If a sub problem yields an infeasible solution, then terminate the branch.

(ii) If a sub problem yields a feasible solution but not an integer solution, then return to Step 2.

(iii) If a sub problem yields a feasible integer solution, examine the value of the objective function. If this value is equal to the upper bound, an optimal solution has been reached. But if it is not equal to the upper bound but exceeds the lower bound, this value is considered as new upper bound and return to Step 2.

Finally, if it is less than the lower bound, terminate this branch.

Step 5: Termination The procedure of branching and bounding continues until no further sub-problem remains to be examined. At this stage, the integer solution corresponding to the current lower bound is the optimal all-integer programming problem solution.

Remark The above algorithm can be represented by an enumeration tree. Each node in the tree represents a sub problem to be evaluated. Each branch of the tree creates a new constraint that is added to the original problem.

Example 1.6.1

Solve the following all integer programming problem using the branch and bound method.

Maximize $Z = 2x_1 + 3x_2$

subject to the constraints

$$\text{(i)}\ \ 6x_1 + 5x_2 \leq 25, \qquad \text{(ii)}\ x_1 + 3x_2 \leq 10$$

and $x_1,\ x_2 \geq 0$ and integers.

Solution:

Relaxing the integer conditions, the optimal non-integer solution to the given integer LP problem obtained by graphical method as shown in Fig.1.61 is: $x_1 = 1.92$, $x_2 = 2.69$ and max $Z_1 = 11.91$. The value of Z_1 represents *initial upper bound* as: $Z_L = 11.91$. Since value of variable x_2 is non-integer, therefore selecting it to decompose (branching) the given problem into two sub-problems by adding two new constraints $x_2 \leq 2$ and $x_2 \geq 3$ to the constraints of original LP problem as follows:

LP sub-problem B

Max $Z = 2x_1 + 3x_2$

Subject to (i) $6x_1 + 5x_2 \leq 25,$

(ii) $x_1 + 3x_2 \leq 10,$

(iii) $x_2 \leq 2,$

and $x_1, x_2 \geq 0$ integer.

LP sub-problem c

Max $Z = 2x_1 + 3x_2$

Subject to (i) $6x_1 + 5x_2 \leq 25,$

(ii) $x_1 + 3x_2 \leq 10,$

(iii) $x_2 \geq 3,$

and $x_1, x_2 \geq 0$ integer.

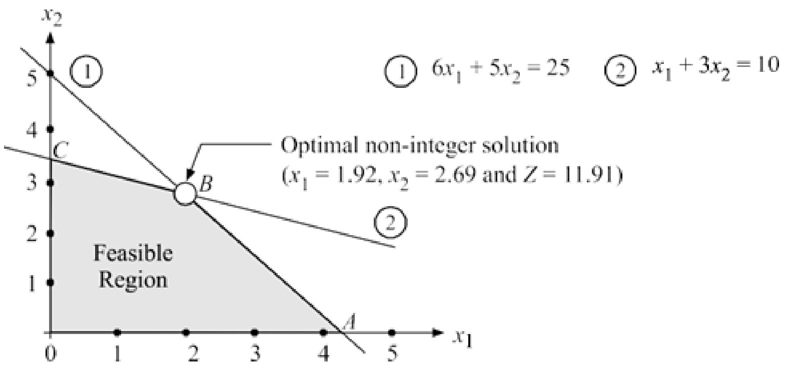


Fig. 1.6.1

Sub-problem B and C are solved graphically as shown in Fig. 1.6.2.

The feasible solutions are:

Sub-problem B : $x_1 = 2.5$, $x_2 = 2$ and Max $Z_2 = 11$

Sub-problem C : $x_1 = 1$, $x_2 = 3$ and Max $Z_3 = 11$

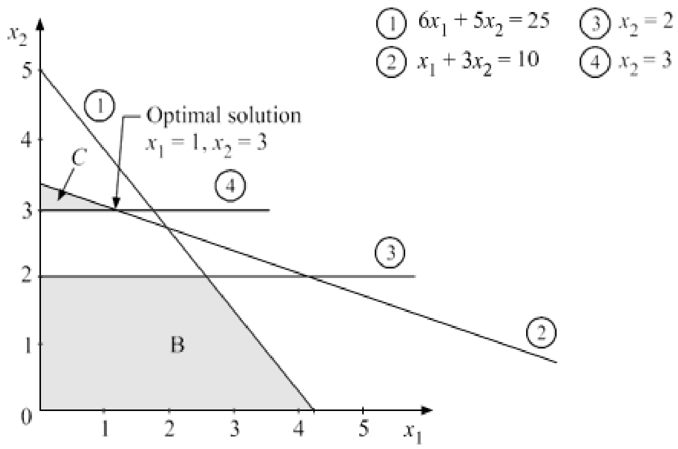


Fig. 1.6.2

The value of decision variables at one of the extreme point of feasible region (solution space) of LP sub-problem C, are: $x_1 = 1$ and $x_2 = 3$. Since these are integer values, so there is no need to further decompose (branching) this sub-problem. The value of objective function, Max Z_L = 11 becomes lower bound on the maximum value of objective function, Z for future solutions.

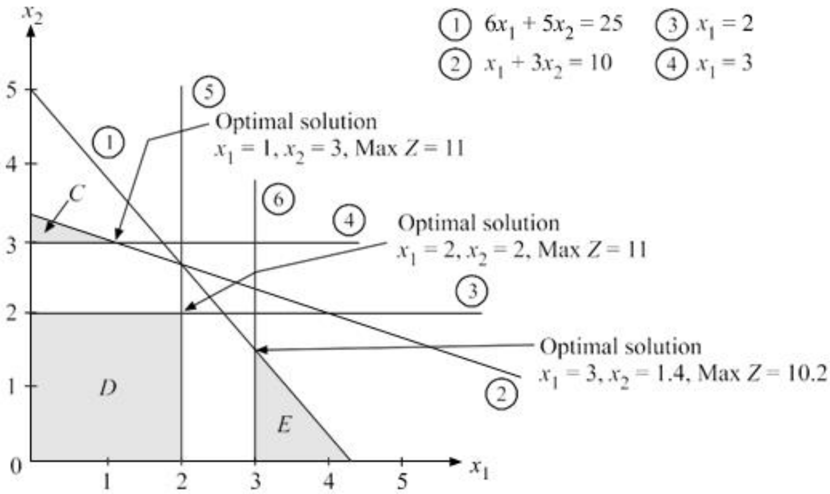


Fig. 1.6.3

LP sub-problem *B* is further subdivided into two LP sub-problems *D* and *E* (shown in fig 1.6.3) by taking variable x_1 = 2.5. Adding two new constraints $x_1 \leq 2$ and $x_1 \geq 3$ to sub-problem *B*. Also Max Z = 11 is also not inferior to the Z_L = 11.

LP sub-problem D	LP sub-problem E
Max $Z = 2x_1 + 3x_2$	Max $Z = 2x_1 + 3x_2$
Subject to (i) $6x_1 + 5x_2 \leq 25,$	Subject to (i) $6x_1 + 5x_2 \leq 25,$
(ii) $x_1 + 3x_2 \leq 10,$	(ii) $x_1 + 3x_2 \leq 10,$
(iii) $x_2 \leq 2,$	(iii) $x_2 \leq 2,$
(iv) $x_1 \leq 2$	(iv) $x_1 \geq 3,$
and $x_1, x_2 \geq 0$ integer.	and $x_1, x_2 \geq 0$ integer.

Sub-problems *D* and *E* are solved graphically as shown in Fig. 1.6.3.

The feasible solutions are:

Sub-problem *D* : x_1 = 2, x_2 = 2 and max Z_4 = 10.

Sub-problem E : x_1 = 3, x_2 = 1.4 and max Z_5 = 10.2

The solution of LP sub-problem *D* is satisfying integer value requirement of variables but is inferior to the solution of LP sub-problem *E* in terms of value of objective function, Z_5 = 10.2. Hence the value of lower bound Z_L = 11 remains unchanged and sub-problem D is not considered for further decomposition.

Since the solution of sub-problem E is non-integer, it can be further decomposed into two sub-problemsby considering variable, x_2. But the value of objective function (Z_5 = 10.2) is inferior to the lower boundand hence this does not give a solution better than the one already obtained. The sub-problem E is alsonot considered for further branching. Hence, the best available solution corresponding to sub-problem $C$is the integer optimal solution: x_1 = 1, x_2 = 3 and Max Z = 11 of the given integer LP problem. The entire branch and bound procedure for the given Integer LP problem in Fig. 1.6.4

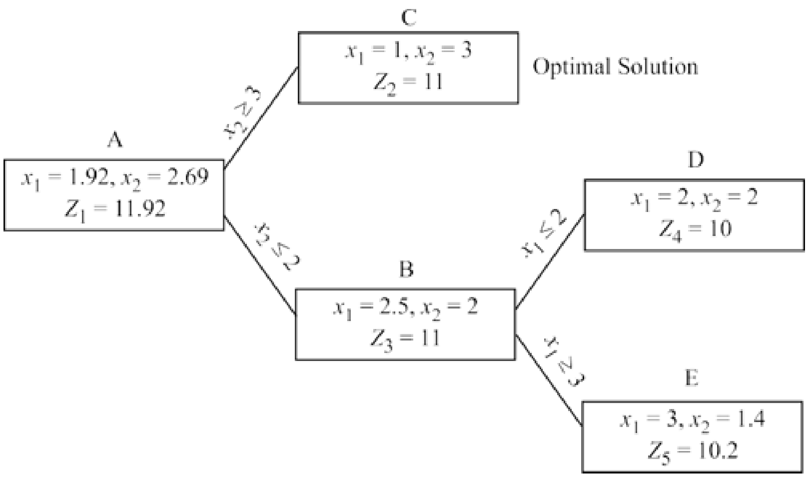


Fig. 1.6.4.

Example 1.6.2 Solve the following all-integer programming problem using the branch and bound method. Maximize $Z = 3x_1 + 5x_2$

subject to the constraints

(i) $2x_1 + 4x_2 \leq 25$, (ii) $x_1 \leq 8$, (iii) $2x_2 \leq 10$ and $x_1, x_2 \geq 0$ and integers.

Solution:

Relaxing the integer requirements, the optimal non-integer solution of the given Integer LP problem obtained by the graphical method, as shown in Fig. 1.6.5, is: x_1 = 8, x_2 = 2.25 and Z_1 = 35.25. The value of Z_1 represents the *initial upper bound*, Z_U = 35.25 on the value of the objective function. This means that the value of the objective function in the subsequent steps should not exceed 35.25.

The *lower bound* Z_L = 34 is obtained by the rounded off solution values to x_1 = 8 and x_2 = 2.

The variable x_2 (= 2.25) is the non-integer solution value, therefore, it is selected for dividing the given LP-A problem into two sub problems LP-B and LP-C by adding two new constraints: $x_2 \leq 2$ and $x_3 \geq 3$, to the constraints of given LP problem as follows:

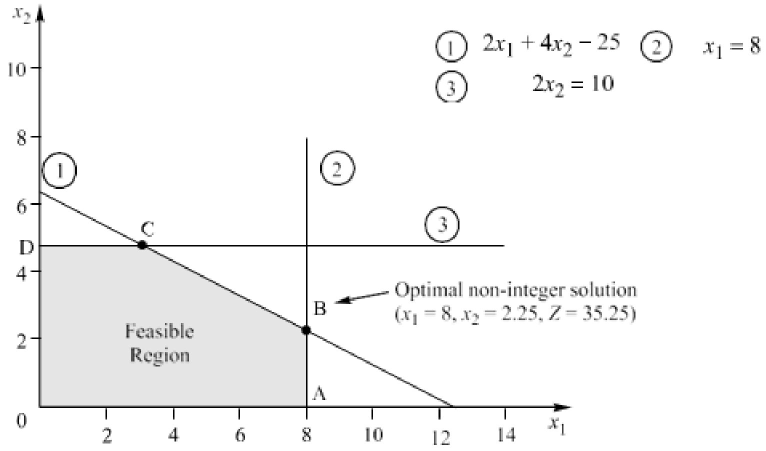


Fig. 1.6.5

LP sub-problem B

Max $Z = 3x_1 + 5x_2$

Subject to (i) $2x_1 + 4x_2 \leq 25$,

(ii) $x_1 \leq 8$,

(iii) $2x_2 \leq 10$ (redundant),

(iv) $x_2 \leq 2$

and $x_1, x_2 \geq 0$ integer.

LP sub-problem C

Max $Z = 3x_1 + 5x_2$

Subject to (i) $2x_1 + 4x_2 \leq 25$,

(ii) $x_1 \leq 8$,

(iii) $2x_2 \leq 10$,

(iv) $x_2 \geq 3$

and $x_1, x_2 \geq 0$ integer.

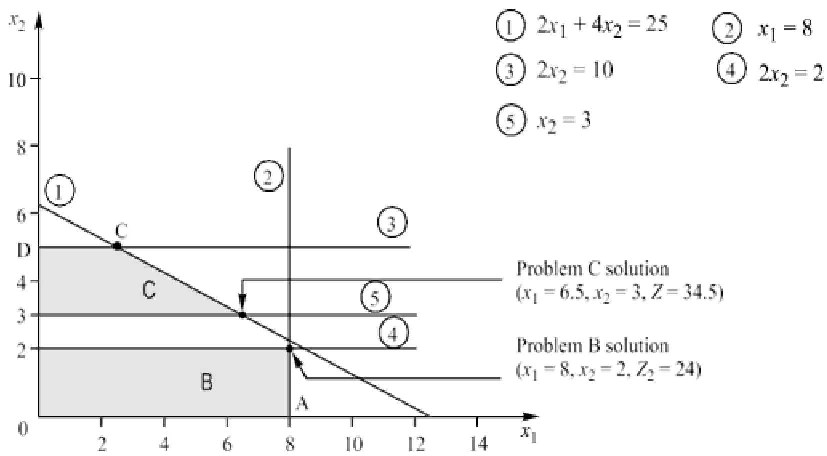


Fig. 1.6.6

Sub problems B and C are solved graphically as shown in Fig. 1.6.6.

The feasible solutions are:

Sub problem B : x_1 = 8, x_2 = 2, and Max Z_2 = 34

Sub problem C : x_1 = 6.5, x_2 = 3, and Max Z_3 = 34.5

Since solution of the sub problem B is satisfying the integer value requirement of variables but value of objective function $Z_2 < Z_3$, therefore this problem is not considered for further branching. However,if $Z_3 \le Z_2$, then no further branching would have been possible for sub problem C.

The sub problem C is now branched into two new sub problems: D and E, by taking variable, x_1 = 6.5. Adding two new constraints $x_1 \le 6$ and $x_1 \le 7$ to sub problem *C*. The two sub problems *D* and *E* are stated as follows:

LP sub-problem D	LP sub-problem E
Max $Z = 3x_1 + 5x_2$	Max $Z = 3x_1 + 5x_2$
Subject to (i) $2x_1 + 4x_2 \le 25,$	Subject to (i) $2x_1 + 4x_2 \le 25,$
(ii) $x_1 \le 8,$ (redundant)	(ii) $x_1 \le 8,$
(iii) $2x_2 \le 10,$	(iii) $2x_2 \le 10,$
(iv) $x_2 \ge 3,$	(iv) $x_2 \le 3$
(v) $x_1 \le 6$	(v) $x_1 \ge 7$
and $x_1, x_2 \ge 0$ integer.	and $x_1, x_2 \ge 0$ integer.

Sub problems D and E are solved graphically as shown in Fig. 1.6.7.

The feasible solutions are:

Sub problem D: x_1 = 6, x_2 = 3.25 and Max Z_4 = 34.25.

Sub problem E: No feasible solution exists because constraints $x_1 \ge 7$ and $x_2 \ge 3$

Do not satisfy the first constraint. So this branch is terminated.

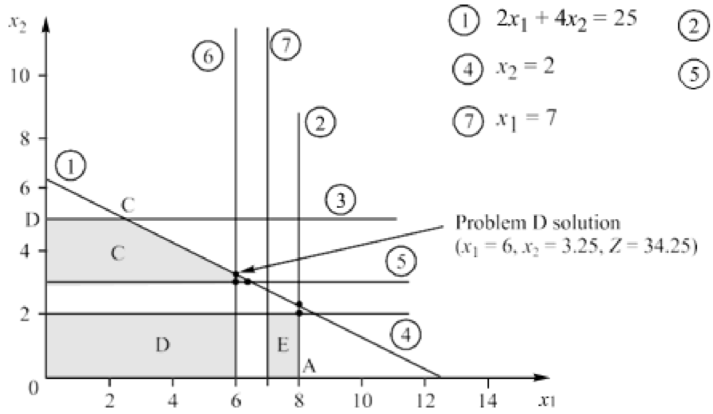


Fig. 1.6.7

The non-integer solution obtained at sub-problem D yields an upper bound of 34.25 instead of 34.50 and also greater than Z_2 (an upper bound for sub-problem B).

Once again we create sub-problems F and G from sub-problem D with two new constraints $x_2 \leq 3$ and $x_2 \geq 4$, as shown in Fig. 1.6.3.

LP sub-problem F

Max $Z = 3x_1 + 5x_2$

Subject to (i) $2x_1 + 4x_2 \leq 25$,

(ii) $x_1 \leq 8$,

(iii) $2x_2 \leq 10$ (redundant)

(iv) $x_2 \geq 3$,

(v) $x_1 \leq 6$

(vi) $x_2 \leq 3$

and $x_1, x_2 \geq 0$ integer.

LP sub-problem G

Max $Z = 3x_1 + 5x_2$

Subject to (i) $2x_1 + 4x_2 \leq 25$,

(ii) $x_1 \leq 8$,

(iii) $2x_2 \leq 10$,

(iv) $x_2 \geq 3$ (redundant)

(v) $x_1 \leq 6$

(vi) $x_2 \geq 4$

and $x_1, x_2 \geq 0$ integer.

The graphical solution to sub problems F and G as shown in Fig. 1.6.8 is as follows:

Sub problem F: $x_1 = 6$, $x_2 = 3$ and Max $Z_5 = 33$.

Sub problem G: $x_1 = 4.25$, $x_2 = 4$ and Max $Z_6 = 33.5$

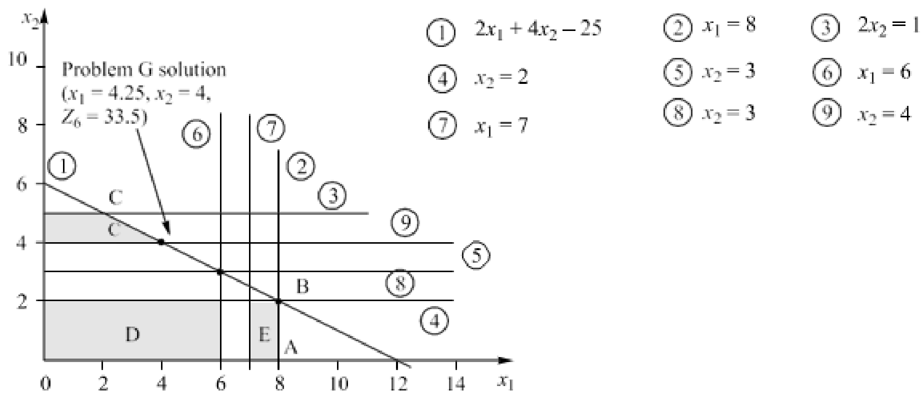


Fig.1.6.8

The solution at node G is non-integer, no additional branching is required from this node because $Z_6 < Z_4$. The branch and bound algorithm thus terminated and the optimal integer solution is: x_1 = 8, x_2 = 2 and Z = 34 yielded at node B.

The branch and bound procedure for the given Integer LP problem is shown in Fig. 1.6.9

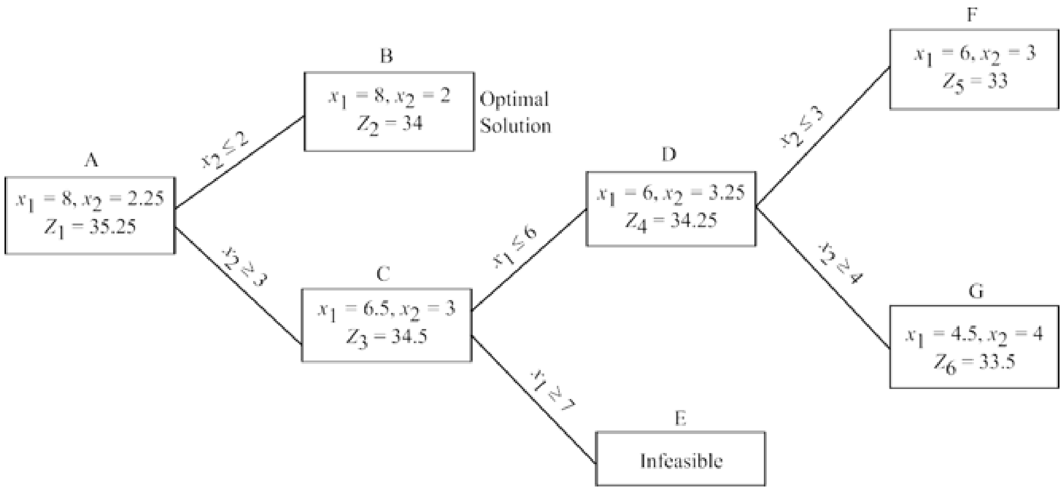


Fig. 1.6.9

Example 1.6.3. Solve the following all-integer programming problem using the branch and bound method. Minimize $Z = 3x_1 + 2.5x_2$

subject to the constraints

(i) $x_1 + 2x_2 \geq 20$ (ii) $3x_1 + 2x_2 \geq 50$

and $x_1, x_2 \geq 0$ and integers.

Solution: Relaxing the integer requirements, the optimal non-integer solution of the given integer LP problem, obtained by the graphical method, is: x_1 = 15, x_2 = 2.5 and

Z_1 = 51.25. This value of Z_1 represents the initial lower bound, Z_L = 51.25 on the value of the objective function, i.e. the value of the objective function in the subsequent steps cannot be less than 51.25.

The variable x_2 (= 2.5) is the only non-integer solution value and is therefore is selected for dividing the given problem into two sub problems: B and C. In order to eliminate the fractional part of x_2 = 2.5, two new constraints $x_2 \leq 2$and $x_2 \geq 3$are created by adding in the given set of constraints as shown below:

LP sub-problem B

Max $Z = 3x_1 + 2.5x_2$

Subject to (i) $x_1 + 2x_2 \geq 20$,

(ii) $3x_1 + 2x_2 \geq 50$

(iii) $x_2 \leq 2$,

and $x_1, x_2 \geq 0$ integer.

LP sub-problem c

Max $Z = 3x_1 + 2.5x_2$

Subject to (i) $x_1 + 2x_2 \geq 20$,

(ii) $3x_1 + 2x_2 \geq 50$,

(iii) $x_2 \geq 3$,

and $x_1, x_2 \geq 0$ integer

Sub problems B and C are solved graphically. The feasible solutions are:

Sub problem B: x_1 = 16, x_2 = 2 and Min Z_2 = 53.

Sub problem C: x_1 = 14.66, x_2 = 3 and Min Z_3 = 51.5.

Since the solution of sub problem B is all-integer, therefore no further decomposition (branching) of this sub problem is required. The value of Z_2 = 53 becomes the new lower bound. A non-integer solution of sub- problem C and also $Z_3 < Z_2$ indicates that further decomposition of this problem need to be done in order to search for a desired integer solution. However, if $Z_3 \geq Z_2$, then no further branching was needed from sub-problem C. The second lower bound takes on the value Z_L = 51.5 instead of Z_L = 51.25 at node A.

Dividing sub problem C into two new sub problems: D and E by adding constraints $x_1 \leq 14$ and $x_1 \geq 15$, as follows:

LP sub-problem D

Max $Z = 3x_1 + 2.5x_2$

Subject to (i) $x_1 + 2x_2 \geq 20$,

(ii) $3x_1 + 2x_2 \geq 50$

(iii) $x_2 \geq 3$,

(iv) $x_1 \leq 14$

and $x_1, x_2 \geq 0$ integer.

LP sub-problem C

Max $Z = 3x_1 + 2.5x_2$

Subject to (i) $x_1 + 2x_2 \geq 20$,

(ii) $3x_1 + 2x_2 \geq 50$,

(iii) $x_2 \geq 3$,

(iv) $x_1 \geq 15$

and $x_1, x_2 \geq 0$ integer

Sub problems D and E are solved graphically. The feasible solutions are:

Sub problem D: $x_1 = 14$, $x_2 = 4$ and Min $Z_4 = 52$.

Sub problem E: $x_1 = 15$, $x_2 = 3$ and Min $Z_5 = 52.5$.

The feasible solutions of both sub problems *D* and *E* are all-integer and therefore branch and bound procedure is terminated. The feasible solution of sub problem *D* is considered as optimal basic feasible solution because this solution is all-integer and the value of the objective function is the lowest amongst al such values.

The branch and bound procedure for the given problem is shown in Fig.1.6.10.

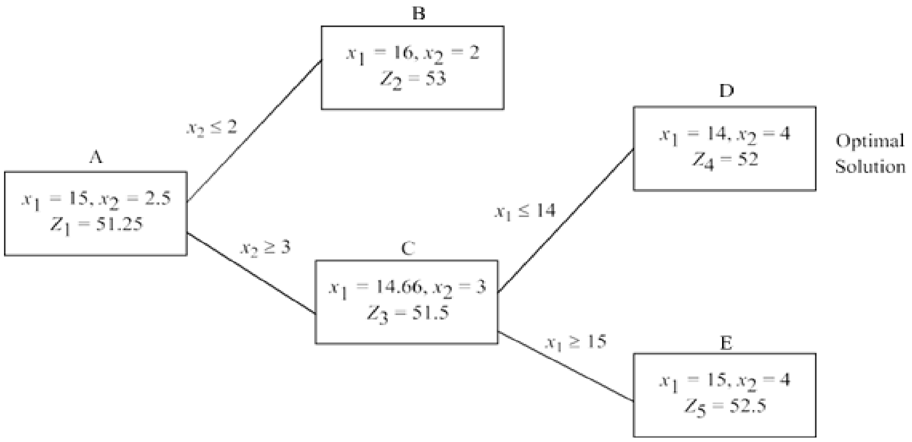


Fig. 1.6.10

1.7 APPLICATIONS OF ZERO-ONE INTEGER PROGRAMMING

A large number of real-world problems such as capital budgeting problem, fixed cost problem, sequencing problem, scheduling problem, location problem, travelling salesman problem, etc., require all or some of the decision variables to assume the value of either zero or one. A few such problems are discussed below.

The zero-one integer programming problem is stated as:

Minimize $Z = \sum_{j=0}^{n} c_j x_j$

Subject to the constraints

$$\sum_{j=1}^{n} a_{ij} x_j \ge b_i; \quad i = 1, 2, \ldots\ldots, m$$

and $x_j = 0$ or 1.

1.7.1 Capital Budgeting Problem

Such problems cover the problem of allocating limited funds to various investment portfolios in order to maximize the net gain.

Example 1.7.1. A corporation is considering four possible investment opportunities. The following table presents information about the investment (in Rs thousand) profits:

Project	*Present Value of Expected Return*	*Capital Required Year-wise by Projects*		
		Year 1	*Year 2*	*Year 3*
1	6,500	700	550	400
2	7,000	850	550	350
3	2,250	300	150	100
4	2,500	350	200	–
Capital available for investment		1,200	700	400

In addition, projects 1 and 2 are mutually exclusive and project 4 is contingent on the prior acceptance of project 3. Formulate an integer programming model to determine which projects should be accepted and which should be rejected in order to maximize the present value from the accepted projects.

Model formulation: Let us define decision variables as:

$$x_j = \begin{cases} 1 & if\ project\ j\ is\ accepted \\ 0 & if\ project\ j\ is\ rejected \end{cases}$$

Integer LP model

Maximize (Total present value) $Z = 6{,}500x_1 + 7{,}000x_2 + 2{,}250x_3 + 2{,}500x_4$

subject to the constraints

(i) Expenditure in years 1, 2 and 3

(i) $700x_1 + 850x_2 + 300x_3 + 350x_4 \leq 1{,}200$

(ii) $550x_1 + 550x_2 + 150x_3 + 200x_4 \leq 700$

(iii) $400x_1 + 350x_2 + 100x_3 \leq 400$

(iv) $x_1 + x_2 \geq 1$,

(v) $x_4 - x_3 \leq 1$

and $x_j = 0$ or 1.

1.7.2 Fixed Cost (or Charge) Problem

In certain projects, while performing a particular activity or set of activities, the fixed costs (fixed charge or setup costs) are incurred. In such cases, the objective is to minimize the total cost (sum of fixed and variable costs) associated with an activity:

The general fixed cost problem can be stated as:

$$\text{Minimize } Z = \sum_{j=0}^{n}\left(c_j x_j + F_j y_j\right)$$

Subject to the constraints

$$\sum_{j=1}^{n} a_{ij}x_j \geq b_i; \quad i = 1, 2, \ldots\ldots, m$$

$$x_j \leq M y_j \text{ or } x_j - M y_j \leq 0; j = 1, 2, \ldots\ldots, n$$

and $$x_j \geq 0 \text{ for all } j\text{ , } y_j = 0 \text{ or } 1 \text{ for all } j$$

Where M = a large number so that $x_j \leq M$

x_j = level of activity j

F_j = fixed cost associated with activity $x_j > 0$

c_j = variable cost associated with activity $x_j > 0$

Example 1.7.2. Consider the following production data:

Product	Profit per Unit (Rs)	Direct Labour Requirement (hours)
1	8	15
2	10	14
3	7	17

Fixed Cost (Rs)	Direct Labour Requirement
10,000	up to 20,000 hours
20,000	20,000–40,000 hours
30,000	40,000–70,000 hours

Formulate an integer programming problem to determine the production schedule so as to maximize the total net profit.

Model formulation: Let us define decision variables as:

x_1, x_2 and x_3 = number of units of products 1, 2 and 3, respectively to be produced

y_j = fixed cost (in Rs); j = 1, 2, 3.

Integer LP model

$$\text{Maximize } z = 8x_1 + 10x_2 + 7x_3 - 10{,}000y_1 - 20{,}000y_2 - 30{,}000y_3$$

subject to the constraints

(i) $15\,x_1 + 14x_2 + 17x_3 \leq 20{,}000\,y_1 + 40{,}000\,y_2 + 70{,}000y_3$

(ii) $y_1 + y_2 + y_3 = 1$

and $x_j \geq 0$ for all j , $y_j = 0$ or 1, for j = 1, 2, 3.

1.7.3 Plant Location Problem

Suppose there are *m* possible sites (locations) at which the plants could be located. Each of these plants produces a single commodity for *n* customers (markets or demand points), each with a minimum demand for b_j units (j = 1, 2, . . ., *n*). Suppose at i^{th} location, the fixed setup cost (expenses associated with constructing and operating a plant) of a plant is f_i (i = 1, 2, . . . *m*). The production capacity of each plant is limited to a_i units. The unit transportation cost from plant *i* to customer *j* is c_{ij}. The problem now is to decide the location of plants in such a way that the sum of the fixed setup costs and transportation cost is lowest (minimum).

Let x_{ij} be the amount shipped from plant *i* to customer *j*, and y_i be the new variable associated with each of the plant locations, such that

$$y_i = \begin{cases} 1\ , if\ plant\ is\ called\ at\ the\ i^{th}\ location \\ 0\ \ , otherwise \end{cases}$$

The value of f_i is assumed to be fixed and independent of the amount of x_{ij} shipped so long as $x_{ij} > 0$, i.e. for $x_{ij} = 0$, the value $f_i = 0$. Thus, the objective is to minimize the total cost (variable + fixed) of settings up and operating the network of transportation routes.

The general mathematical 0–1 integer programming model of plant location problem

can be stated as follows:

$$\text{Minimize } Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} + \sum_{i=1}^{m} f_i y_i$$

subject to the constraints

$$\sum_{i=1}^{m} x_{ij} = b_j \quad ; \quad j = 1, 2, \ldots, n \tag{16}$$

$$\sum_{j=1}^{n} x_{ij} \leq y_i u_i \; ; \quad i = 1, 2, \ldots, m \tag{17}$$

$$\sum_{j=1}^{n} x_{ij} \leq a_i \quad ; \quad i = 1, 2, \ldots, m \tag{18}$$

and $\quad x_{ij} \geq 0$, for all i and j.

where u_i = capacity of plant i.

Constraints (16) indicate that each customer's demand is met. If all the shipping costs are positive, then it never pays to send more than the needed amount. In such a case replace the inequality sign by an equality sign. Inequality (17) indicates that there is no need to ship from a plant which is not operating. The capacity u_i of plant i represent the maximum amount of commodity that may be shipped from it. Inequality (18) controls the production capacity of a plant, i to exceed beyond,

1.8 DYNAMIC PROGRAMMING (MULTISTAGE PROGRAMMING)

Dynamic Programming is a mathematical procedure designed primarily to improve the computational efficiency of select mathematical programming problem by decomposing them in to smaller and hence computationally simple sub-problem. Dynamic Programming typically solves the problem in stages, with each stage involving exactly one optimizing variable. The solution of the Dynamic Programming Problem (DPP) is achieved sequentially starting from one stage to the next till the final stage is reached. The computations at the different stages are linked through recursive computations in a manner that yields a feasible optimal solution to the entire problem. This technique was developed by Richard E. Bellman in 1950.

Dynamic Programming differs from LPP in two ways:

i. In DP, there is no algorithm as in LP that can be used to solve all problems. GP is a technique that allows to break up the given problem in to a sequence of easier and smaller sub-problems which are then solved in stages.

ii. LP gives one time period solution whereas GP considers decision making over time and solves each sub-problem optimally.

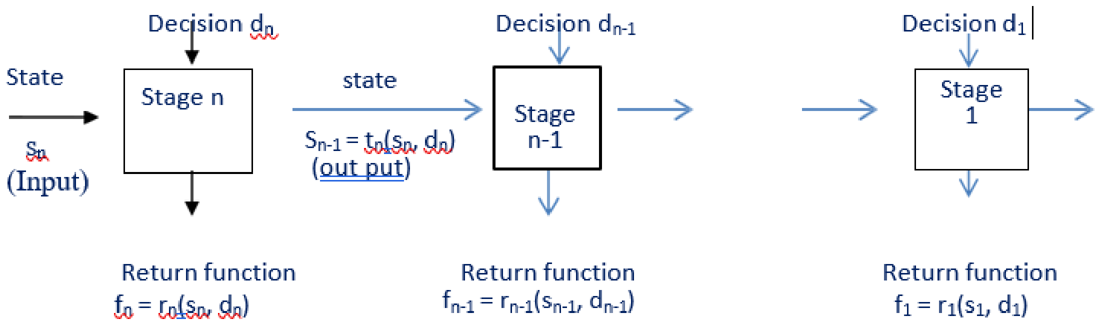
1.9 Dynamic Programming terminology

Stage: The DPP can be decomposed or divided in to a sequence of smaller sub-problems called stages. At each stage there are a number of decision alternatives and a decision is made by selecting the most suitable alternative. Stages very often represent different time periods in the planning period of the problem, places, people or other entities. For example, in replacement problem, each year is a stage; in the salesman allocation problem, each territory represents a stage; in an LPP each variable is a stage.

State: Each stage in a DPP has a certain number of states associated with it. These states represent various conditions of the decision process at a stage. The variables which specify the condition of the decision process or describe the states of the system at a particular stage are called state variables. These variables provide information for analyzing the possible effects that the current decision could have upon future courses of action. At any stage of the decision-making process there could be a finite or infinite number of states. For example, a specific city is referred to as a state variable in any stage of the shortest path problem.

Return function: At each stage, a decision is made which can affect the state of the system at the next stage and help in arriving at the optimal solution at the current stage. Every decision that is made has its own merit in terms of worth of benefit associated with it and can be described in an algebraic equation form. Generally, this equation is called a return function. This return function in general depends on the state variable as well as the decision made at a particular stage. An optimal policy or decision at a stage yields optimal return for a given value of the state variable.

For a multistage decision process, functional relationship among state, stage and decision may be described as shown below:



Where n = stage number.

s_n = State input to stage n from stage n+1. Its value is the states of the system resulting from the previous (n+1) stage decision.

d_n= decision variable at stage n (independent of previous stage). It represents the range of alternatives that can be selected from when making a decision at stage n.

$f_n = r_n (s_n, d_n)$ = return (objective) function for stage n.

Transition function: Suppose that there are n stages at which a decision is to made.These n stages are all interconnected by the relation called transition function. It is defined by

$$s_{n-1} = t_n(s_n, d_n) = s_n * d_n$$

i.e., Output at stage n = (Input to stage n) * (Decision at stage n)

Where * represent any mathematical operation namely addition, subtraction, division or multiplication. t_n represents a state transformation function and its form depends on the particular problem to be solved.

1.10 Developing Optimal Decision policy

Policy: A particular sequence of alternatives adopted by the Decision Maker (DM) in a multistage decision problem is called a policy. The optimal policy is the sequence of alternatives that achieves the decision maker's objective.

Bellman's Principle of optimality: The solution of a DPP is based on Bellman's principle of optimality, which states "The optimal policy must be one such that regardless of how a particular state is reached, all later decisions (choices) proceeding from that state must be optimal".

Solution procedure: The solution procedure is based on (i) Backward induction process or (ii) Forward induction process. In the first process, the problem is solved by solving the problem in the last stage and working backwards towards the first stage, making optimal decisions at each stage of the problem. In certain cases, forward induction process is used to solve a problem by first solving the initial stage of the problem and working towards the last stage, making an optimal decision at each stage of the problem.

The one stage return function is given by $f_1 = r_1(s_1, d_1)$ and the optimal value of f_1 under the state variable s_1 can be obtained by selecting a suitable decision variable d_1.

i.e., $f_1^*(s_1) = \underset{d_1}{opt}\{r_1(s_1, d_1)\}$

The range of d_1 is determined by s_1, but s_1 is determined by what has happened in stage 2. Then in stage 2, the return function will take the form

$$f_2^*(s_2) = \underset{d_2}{opt}\{r_2(s_2, d_2) * f_1^*(s_1)\}; \; s_1 = t_2(s_2, d_2)$$

By continuing the above logic recursively for a general n stage problem, we have

$$f_n^*(s_n) = \underset{d_n}{opt}\{r_n(s_n, d_n) * f_{n-1}^*(s_{n-1})\}; \; s_{n-1} = t_n(s_n, d_n)$$

The General Algorithm:

Step-1: Identify the problem decision variables and specify objective function to be optimized under certain limitations, if any.

Step-2: Divide the given problem in to a number of smaller sub-problems (or stages). Identify the state variable at each stage and write down the transformation function as a function of the state variable and decision variable at the next stage.

Step-3: Write down a general recursive relationship for computing the optimal policy. Decide whether to follow the forward or the backward method to solve the problem.

Step-4: Construct appropriate tables to show the required values of the return function at each stage.

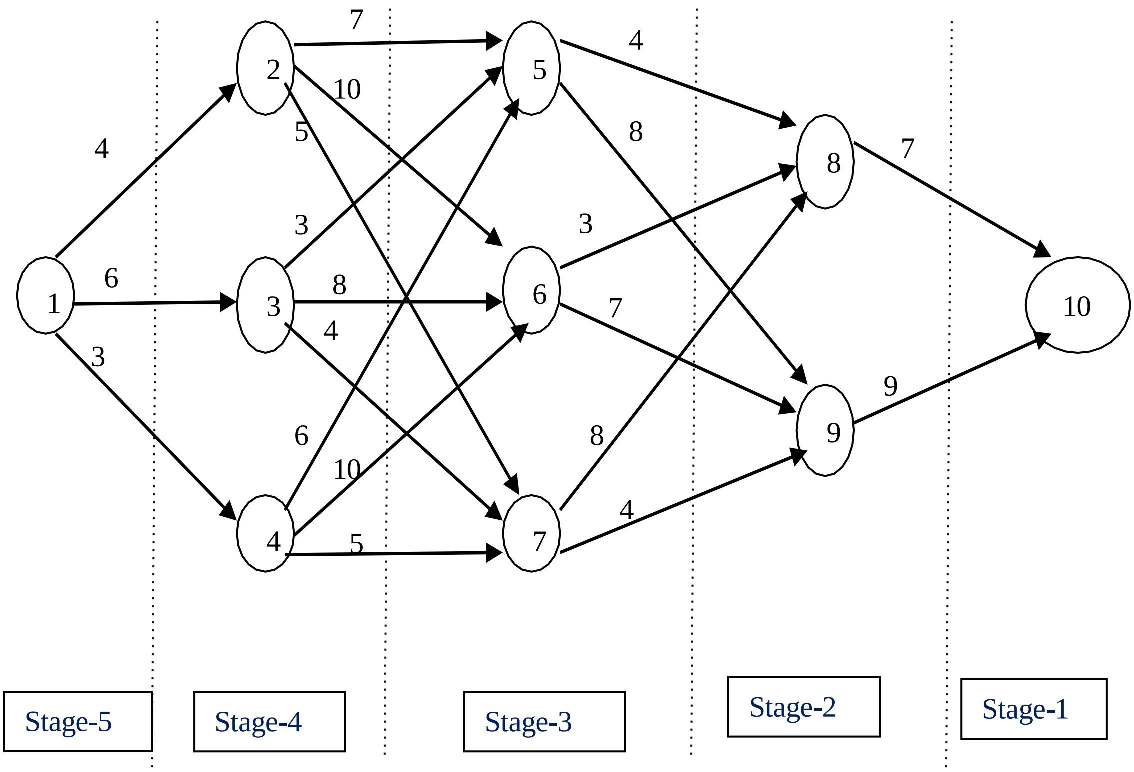
Step-5: Determine the overall optimal policy or decisions and its value at each stage. There may be more than one such optimal policy.

1.11 Dynamic Programming under Certainty

Model – 1: Shortest Route Problem

Example 1.11.1

A Salesman located in a city A decided to travel to city B. He knew the distances of alternative routes from city A to city B. He then constructs a highway network map as shown below. The city of origin A is city -1. The destination city B is city 10. Other cities through which the salesman will have to pass through are numbered 2 to 9. The arrow representing routes between cities and distances in kms are indicated on each route. The salesman's problem is to find the shortest route that covers all the selected cities from A to B.



Solution:

d_n= Decision variable that define the immediate destinations when there are n stages to go (n = 1,2,3,4)

s_n = State variable describe a specific city at any stage.

, = Distance associated with the state variable, s_n and decision variable, d_n, for the current nth stage.

f_n (s_n, d_n) = Minimum total distance for the last n stages, given that salesman is in state s_n and selects d_n as immediate destination.

f_n* (s_n) = Optimal value of the path (minimum distance) when the salesman is in state s_n with n more stages to go for reaching the final stage(destination).

The forward recursion relationship for this problem is:

$$f_1^*(s_1) = \min_{d_1}\{f_1(s_1, d_1)\} = \min_{d_n}\{D_{s_1,d_1}\}$$

$$f_n^*(s_n) = \min_{d_n}\{f_n(s_n, d_n)\} = \min_{d_n}\{D_{s_n,d_n} + f_{n-1}^*(d_n)\};\ \ n = 1, 2, 3, 4$$

Where $f_{n-1}^*(d_n)$ is the optimal distance for the previous stages.

Stage -1

Decision d_1 / State s_1	$f_1(s_1, d_1) = D_{s_1, d_1}$	Minimum distance f_1^* (s_1)	Optimal decision d_1
	10		
8	7	7	10
9	9	9	10

Stage -2

Decision d_2 / *State S_2*	$f_2(s_2, d_2) = D_{s_2,d_2} + f_1^*(d_2)$		Minimum distance f_2^* (s_2)	Optimal decision d_2
	8	9		
5	4+7=11	8+9=17	11	8
6	3+7=10	7+9=16	10	8
7	8+7=15	4+9=13	13	9

Continuing the process for stage -3 and 4, we get

Stage -3

Decision d_3 / State S_3	$f_3(s_3, d_3) = D_{s_3,d_3} + f_2^*(d_3)$			Minimum distance $f_3^*(s_3)$	Optimal decision d_3
	5	6	7		
2	7+11=18	10+10=20	5+13=18	18	5 or 7
3	3+11=14	8+10=18	4+13=17	14	5
4	6+11=17	10+10=20	5+13=18	17	5

Stage -4

Decision d_4 / State S_4	$f_4(s_4, d_4) = D_{s_4,d_4} + f_3^*(d_4)$			Minimum distance $f_4^*(s_4)$	Optimal decision d_4
	2	3	4		
1	4+18=22	6+14=20	3+17=20	20	3 or 4

There are two optimal paths: 1 – 3 – 5 – 8 – 10 and 1 – 4 – 5 – 8 – 10.
The value of the path is 20 K.M.

Model – 2: Single additive constraint, multiplicative separable return.

Example 1.11.2 (Optimal Sub-division problem)

Divide quantity b in to n parts so as to maximize their product. Or Let (b) be the maximum value. Then show that

$f_1(b) = b$ and $f_n(b) = \underset{0 < z < b}{Max} \{z f_{n-1}(b - z)\}$. Hence find $f_n(b)$ and the division that maximizes it.

Solution:

Let x_j be the j[th] part of the quantity b (j = 1, 2, …, n). Then the problem becomes

$$Max\ (b) = x_1 . x_2 . \dots x_n$$

Subject to $x_1 + x_2 + \dots + x_n = b$

$x_j > 0$, j = 1, 2, … , n

Here each part x_j, (j = 1, 2, … , n) of b may be regarded as a stage.

The recursive equation can be obtained as follows:

For n = 1: $f_1(b) = x_1 \ or \ b \ldots \ldots \ldots$ (1) (True)

For n = 2 (Stage-2):

The quantity b is divided into two parts, say $x_1 = z \ and \ x_2 = b - z$,

$$\text{then } f_2(b) = \max_{0<z<b} \{z(b-z)\} = \max_{0<z<b} \{zf_1(b-z)\} \ldots \ldots \ldots (2), using\ (1).$$

For n = 3 (Stage-3): The maximum product of b divided into three parts,

$$\text{i.e., } f_3(b) = \max_{0<z<b} \{zf_2(b-z)\} \ldots \ldots \ldots (3)$$

Similarly, for Stage-n: The recursive equation for n is,

$$f_n(b) = \max_{0<z<b} \{zf_{n-1}(b-z)\} \ldots \ldots \ldots (4)$$

Solution of the recursive equation:

$$\text{For } n = 2: \ f_2(b) = \max_{0<z<b} \{z(b-z)\} \ldots \ldots \ldots (2)$$

Let $(z) = z\,(b - z)$

$F'(z) = 0 \ \Rightarrow z.(-1) + (b - z) = 0 \Rightarrow -2z + b = 0 \ \Rightarrow z = \frac{b}{2}$

$F''(z) = -1 + (-1) = -2 < 0$

Therefore, the function $z\,(b - z)$ attains its maximum value for z = $\frac{b}{2}$, satisfying the condition 0 < z < b. Hence (2) ⟹ $f_2(b) = \frac{b}{2}\left(b - \frac{b}{2}\right) = \left(\frac{b}{2}\right)^2$.

Hence optimal policy is $\left(\frac{b}{2}, \frac{b}{2}\right)$ & $f_2(b) = \left(\frac{b}{2}\right)^2$. …… (5)

For $n = 3$: $f_3(b) = \underset{0<z<b}{Max}\{z.f_2(b-z)\}$ ……… (3)

$= \underset{0<z<b}{Max}\{z.(\frac{b-z}{2})^2\}$ using (5)

Let $F(z) = z.(\frac{b-z}{2})^2$

$F'(z) = 0 \Rightarrow \frac{1}{4}[z.2(b-z)(-1) + (b-z)^2] = 0 \Rightarrow (b-z)(-2z+b-z) = 0$

$As\ b \neq z, b - 3z = 0 \Rightarrow z = \frac{b}{3}.$

The maximum value of z. $(\frac{b-z}{2})^2$ is attained for z = $\frac{b}{3}$, satisfying the condition 0 < z < b.

$$\text{Hence (3)} \Rightarrow f_3(b) = \frac{b}{3}.\frac{1}{4}\left(b-\frac{b}{3}\right)^2 = \frac{b}{3}.\frac{1}{4}\left(\frac{2b}{3}\right)^2 = \left(\frac{b}{3}\right)^3$$

Hence optimal policy is ($\frac{b}{3}$, $\frac{b}{3}$, $\frac{b}{3}$) & $f_3(b) = \left(\frac{b}{3}\right)^3$.

In general, for n stage problem the optimal policy is ($\frac{b}{n}$, $\frac{b}{n}$,…. $\frac{b}{n}$) & $f_n(b) = \left(\frac{b}{n}\right)^n$, n = 1,2,3..

Example 1.11.3 Determine the value of u_1, u_2 *and* u_3 so as to

$Maximize\ z = u_1.u_2.u_3$ subject to $u_1 + u_2 + u_3 = 10;\ u_1, u_2, u_3 \geq 0$

Solution:

Let u_j be the j[th] part of the quantity b (10)(j = 1, 2, 3). Then the problem becomes:

$Max\ (b) = u_1.u_2.\ u_3$

Subject to $u_1 + u_2 + u_3 = 10 = b$

$u_j \geq 0$, j = 1, 2, 3

Here each part u_j, (j = 1, 2, 3) of b may be regarded as a stage.

The recursive equation can be obtained as follows:

For n = 1: $f_1(b) = u_1 \text{ or } b \ldots \ldots \ldots$ (1)

For n = 2 (Stage-2):

The quantity b is divided into two parts, say $u_1 = z \text{ and } u_2 = b - z$,

$$\text{then } f_2(b) = \underset{0 < z < b}{Max} \{z(b - z)\} = \underset{0 < z < b}{Max} \{z f_1(b - z)\} \ldots \ldots \ldots (2), \text{using } (1).$$

For n = 3 (Stage-3): The maximum product of b divided into three parts,

$$\text{i.e., } f_3(b) = \underset{0 < z < b}{Max} \{z f_2(b - z)\} \ldots \ldots \ldots (3)$$

Solution of the recursive equation:

$$\text{For } n = 2: \; f_2(b) = \underset{0 < z < b}{Max} \{z(b - z)\} \ldots \ldots \ldots (2)$$

Let $(z) = (b - z)$

$F'(z) = 0 \Rightarrow z.(-1) + (b - z) = 0 \Rightarrow -2z + b = 0 \Rightarrow z = \frac{b}{2}$

$F''(z) = -1 + (-1) = -2 < 0$

Therefore, the function z (b - z) attains its maximum value for $z = \frac{b}{2} = \frac{10}{2} = 5$, satisfying the condition $0 < 5 < 10$. Hence (2) $\Longrightarrow f_2(b) = \frac{b}{2}\left(b - \frac{b}{2}\right) = \left(\frac{b}{2}\right)^2 = 25$.

Hence optimal policy is $\left(\frac{b}{2}, \frac{b}{2}\right)$ & $f_2(b) = \left(\frac{b}{2}\right)^2$.

For n = 3: $f_3(b) = \underset{0<z<b}{Max}\{z.f_2(b-z)\}$ ……… (3)

$= \underset{0<z<b}{Max}\{z.(\frac{b-z}{2})^2\}$ using (5)

Let $F(z) = z.(\frac{b-z}{2})^2$

$F'(z) = 0 \Rightarrow \frac{1}{4}[z.2(b-z)(-1) + (b-z)^2] = 0 \Rightarrow (b-z)(-2z+b-z) = 0$

$As\ b \neq z, b - 3z = 0 \Rightarrow z = \frac{b}{3}$

The maximum value of z. $(\frac{b-z}{2})^2$ is attained for z = $\frac{b}{3}$, satisfying the condition 0 < z < b.

$$\text{Hence (3)} \Rightarrow f_3(b) = \frac{b}{3}.\frac{1}{4}\left(b-\frac{b}{3}\right)^2 = \frac{b}{3}.\frac{1}{4}\left(\frac{2b}{3}\right)^2 = \left(\frac{b}{3}\right)^3$$

Hence optimal policy is $(\frac{b}{3}, \frac{b}{3}, \frac{b}{3})$ & $f_3(b) = (\frac{b}{3})^3$.

Therefore, $u_1 = u_2 = u_3 = \frac{10}{3}$ and hence $Max\{u_1, u_2, u_3\} = (\frac{10}{3})^3$.

Another method:

Let us define state variable x_j, (j = 1, 2, 3) such that

$x_3 = u_1 + u_2 + u_3 = 10\ at\ stage - 3$

$x_2 = x_3 - u_3 = u_1 + u_2\ at\ stage - 2$

$x_1 = x_2 - u_2 = u_1\ at\ stage - 1$

The recursive equation is

$f_3(x_3) = \underset{u_3}{Max}\{u_3.f_2(x_2)\}$ …(1)

$f_2(x_2) = \underset{u_2}{Max}\{u_2.f_1(x_1)\}$ …(2)

$f_1(x_1) = u_1 = x_2 - u_2$ …. (3)

Using (3) in (2), we get

$f_2(x_2) = \underset{u_2}{Max}\{u_2.(_2 - u_2)\}$ …(4)

Let f (u_2) = $u_2(_2 - u_2)$

$f'(u_2) = 0 \Rightarrow u_2(-1) + (x_2 - u_2) = 0 \Rightarrow -2u_2 + x_2 = 0 \Rightarrow u_2 = \frac{x_2}{2}$

(4) ⇒ $f_2(x_2) = \frac{x_2}{2}(_2 - \frac{x_2}{2})^2$ … (5)

Using (5) in (1), $f_3(x_3) = \underset{u_3}{Max}\{ u_3.(\frac{x_2}{2})^2 \} = \underset{u_3}{Max}\{ u_3.(\frac{x_3-u_3}{2})^2 \}$ …(6)

Let g (u_3) = $\frac{1}{4} u_3 (x_3 - u_3)^2$

$$g'(u_3) = 0 \Rightarrow u_3.2(x_3 - u_3)(-1) + (x_3 - u_3)^2 = 0 \Rightarrow (x_3 - u_3)(-2u_3 + x_3 - u_3) = 0$$

$$\Rightarrow -3u_3 + x_3 = 0 \Rightarrow u_3 = \frac{x_3}{3} = \frac{10}{3}$$

$$u_2 = \frac{x_2}{2} \Rightarrow \frac{(x_3-u_3)}{2} = \frac{10-\frac{10}{3}}{2} = \frac{10}{3}.$$

$$u_1 = x_2 - u_2 = \frac{20}{3} - \frac{10}{3} = \frac{10}{3}$$

Therefore, , $u_1 = u_2 = u_3 = \frac{10}{3}$ and hence $Max\{u_1, u_2, u_3\} = (\frac{10}{3})^3$.

1.12 SOLVING LPP USING DYNAMIC PROGRAMMING

An LPP in n decision variable and m constraints can be converted in to an **n-stage** dynamic programming problem with **m-state** parameters.

Consider the LPP $\quad Maximize\ z = \sum_{j=1}^{n} c_j x_j$

Subject to the constraints $\sum_{j=1}^{n} a_{ij} x_j \leq b_j, \quad i = 1\ to\ m$

$x_j \geq 0, j = 1\ to\ n$

Let $(b_1, b_2, \dots \dots .., b_m)$be the state vector and $f_1\ (b_1, b_2, \dots \dots .., b_m)$ be the optimal value of the objective function for stages $j,\ j+1, \dots \dots, n$ given the stage$(b_1, b_2, \dots \dots .., b_m)$. Using backward recursive equation, we shall optimize the last stage first and then last but one etc.

$$f_n = (b_1, b_2, \dots \dots .., b_m) = \max_{0 \leq a_{in} x_n \leq b_i} \{c_n x_n\}, \quad i = 1\ to\ m$$

$$f_j = (b_1, b_2, \dots \dots .., b_m) = \max_{\substack{0 \leq a_{ij} x_j \leq b_j \\ i=1\ to\ m, j=1\ to\ n-1}} \{c_j x_j + f_{j+1}(b_1 - a_{1j}\, x_j, b_2 - a_{2j} x_j, \dots \dots .., b_m - a_{mj} x_j)\}.$$

Similarly, we can use forward recursive equation,

$$f_1 = (b_1, b_2, \dots \dots .., b_m) = \max_{0 \leq a_{in} x_1 \leq b_i} \{c_n x_n\}, \quad i = 1\ to\ m$$

$f_j = (b_1, b_2, \dots \dots .., b_m) =$

$$\max_{\substack{0\le a_{ij}x_j\le b_j \\ i=1\ to\ m, j=2\ to\ n}} \{c_j x_j + f_{j-1}(b_1 - a_{1j}\,x_j, b_2 - a_{2j}x_j, \dots \dots .., b_m - a_{mj}x_j)\}$$

Example 1.12.1: Solve the following LPP using Dynamic Programming

$Maximize\ z = 3x_1 + 2x_2$

Subject to $2x_1 + x_2 \le 40;\ x_1 + x_2 \le 24;\ 2x_1 + 3x_2 \le 60; x_1, x_2 \ge 0.$

Solution:

$Number\ of\ variables\ = 2 \Rightarrow\ Number\ of\ stage\ = 2$

$Number\ of\ constraints\ = 3 \Rightarrow\ Number\ of\ state\ variable\ = 3$

Let $(b_1, b_2, b_3) = (40, 24, 60)$ be the state vector and $f_j(b_1, b_2, b_3)$ be the optimal value of the objective function. Now we are use to solve Backward recursive formula.

Stage $j = 2$;

$$f_2(b_1, b_2, b_3) = \text{Max}_{\substack{0\ \le x_2 \le\ b_1 \\ 0\ \le x_2\ \le b_2 \\ 0\ \le 3x_2\ \le b_3}} \{2x_2\}$$

$$= 2 \text{Max}_{\substack{0\ \le x_2\ \le b_1 \\ 0\ \le x_2\ \le b_2 \\ 0\ \le x_2 \le \frac{b_3}{3}}} \{x_2\}$$

$$= 2\ Min\left\{b_1, b_2, \frac{b_3}{3}\right\} \quad \dots (1)$$

Stage $j = 1$;

$$f_1(b_1, b_2, b_3) = \text{Max}_{\substack{0\ \le 2x_2 \le\ b_1 \\ 0\ \le x_1 \le\ b_2 \\ 0\ \le 2x_1 \le\ b_3}} \{3x_2 + f_2(b_1 - a_{11}x_1, b_2 - a_{21}x_1, b_3 - a_{31}x_1)\}$$

$$= \text{Max}_{\substack{0\le x_1 \le \frac{b_1}{2} \\ 0\le x_1 \le b_2 \\ 0\le x_1 \le \frac{b_3}{2}}} \{3x_1 + f_2(b_1 - 2x_1, b_2 - x_1, b_3 - 2x_1)\}$$

$$= \text{Max}_{\substack{0\le x_1 \le \frac{b_1}{2} \\ 0\le x_1 \le b_2 \\ 0\le x_1 \le \frac{b_3}{2}}} \{3x_1 + 2\ Min(b_1 - 2x_1, b_2 - x_1, \frac{b_3 - 2x_1}{3})\} \quad \dots (2)$$

$$f_1(40, 24, 60) = \text{Max}_{\substack{0\le x_1 \le 20 \\ 0\le x_1 \le 24 \\ 0\le x_1 \le 30}} \{3x_1 + 2\ Min\left\{40 - 2x_1, 24 - x_1, \frac{60 - 2x_1}{3}\right\}\} \quad \dots (3)$$

Now $\text{Min}_{0\ \le x_1 \le\ 20}\left\{40 - 2x_1, 24 - x_1, \frac{60-2x_1}{3}\right\}$ =? (Intervals $0 \le x_1 \le 24\ and\ 0 \le x_1 \le 30$ are ignored)

x_1	$x_2 = 40 - 2x_1$	$x_2 = 24 - x_1$	$x_2 = \dfrac{60 - 2x_1}{3}$
0	40	24	20*
10	20	14	40/3=13.3*
12	16	12*	12*
15	10	9*	10
16	8*	8*	28/3=9.3
20	0*	4	20/3=6.6

Hence $\text{Min}_{0 \le x_1 \le 20}\left\{40 - 2x_1, 24 - x_1, \frac{60-2x_1}{3}\right\} = \begin{cases} \frac{60-2x_1}{3}, 0 \le x_1 \le 12 \\ 24 - x_1, 12 \le x_1 \le 16 \\ 40 - 2x_1, 16 \le x_1 \le 20 \end{cases}$ … (4)

$$f_1(40,24,60) = Max\begin{Bmatrix} 3x_1 + 2\left(\frac{60-2x_1}{3}\right), if\ \ 0 \le x_1 \le 12 \\ 3x_1 + 2(24 - x_1), if\ \ 12 \le x_1 \le 16 \\ 3x_1 + 2(40 - 2x_1), if\ 16 \le x_1 \le 20 \end{Bmatrix} \text{ using (4) and (3)}$$

$$f_1(40,24,60) = Max\begin{Bmatrix} 40 + \left(\frac{5}{3}\right)x_1, if\ \ 0 \le x_1 \le 12 \\ 48 + x_1, if\ \ 12 \le x_1 \le 16 \\ 80 - x_1, if\ 16 \le x_1 \le 20 \end{Bmatrix}$$

$$f_1(40,24,60) = Max\begin{Bmatrix} 60, & at\ \ x_1 = 12 \\ 64, & at\ \ x_1 = 16 \\ 64, & at\ \ x_1 = 16 \end{Bmatrix}$$

$f_1^*(40,24,60) = 64, \quad at\ \ x_1^* = 16$

$x_2^* = \text{Min}_{0 \le x_1 \le 20}\left\{40 - 2x_1^*, 24 - x_1^*, \frac{60-2x_1^*}{3}\right\} = Min\ \{8, 8, 14\} = 8.$

Optimal solution is $x_1^* = 16,\ \ x_2^* = 8,\ \ z_{max} = 64.$

Remark: We can also solve the same problem using forward recursive equation.

Stage $j = 1$;

$$f_1(b_1, b_2, b_3) = \text{Max}_{\substack{0 \le 2x_1 \le b_1 \\ 0 \le x_1 \le b_2 \\ 0 \le 2x_1 \le b_3}}\{3x_1\}$$

$$= 3 \underset{\substack{0 \le x_1 \le \frac{b_1}{2} \\ 0 \le x_1 \le b_2 \\ 0 \le x_1 \le \frac{b_3}{3}}}{\text{Max}} \{x_1\}$$

$$= 3\, Min \left\{\frac{b_1}{2}, b_2, \frac{b_3}{3}\right\} \quad \ldots \ (1)$$

Stage $j = 2$;

$$f_2(b_1, b_2, b_3) = \underset{\substack{0 \le x_2 \le b_1 \\ 0 \le x_1 \le b_2 \\ 0 \le 3x_1 \le b_3}}{\text{Max}} \{2x_2 + f_1(b_1 - a_{12}x_2, b_2 - a_{22}x_2, b_3 - a_{32}x_3)\}$$

$$= \text{Max}_{\substack{0 \le x_2 \le b_1 \\ 0 \le x_2 \le b_2 \\ 0 \le x_2 \le \frac{b_3}{2}}} \{2x_1 + f_1(b_1 - x_2, b_2 - x_2, b_3 - 3x_2)\}$$

$$= \text{Max}_{\substack{0 \le x_2 \le b_1 \\ 0 \le x_2 \le b_2 \\ 0 \le x_2 \le \frac{b_3}{2}}} \{2x_2 + 2\, Min(\frac{b_1 - x_2}{2}, b_2 - x_2, \frac{b_3 - 3x_2}{2})\} \quad \ldots \ (2)$$

$$f_2(40, 24, 60) = \text{Max}_{\substack{0 \le x_1 \le 40 \\ 0 \le x_1 \le 24 \\ 0 \le x_1 \le 20}} \{2x_2 + 3\, Min \left\{\frac{40 - x_2}{2}, 24 - x_2, \frac{60 - 3x_2}{2}\right\}\} \quad \ldots \ (3)$$

Now $\text{Min}_{0 \le x_2 \le 20} \left\{\frac{40 - x_2}{2}, 24 - x_2, \frac{60 - 3x_2}{2}\right\}$ =? (Intervals $0 \le x_2 \le 24$ *is* ignored)

x_2	$x_1 = \frac{40 - x_2}{2}$	$x_1 = 24 - x_2$	$x_1 = \frac{60 - 3x_2}{2}$
0	20*	24	30
6	17	18	21
8	16*	16*	18
10	15	14*	15
12	14	12*	12*
14	13	10	9*
20	10	4	0*

Hence $\text{Min}_{0 \le x_2 \le 20} \left\{\frac{40 - x_2}{2}, 24 - x_2, \frac{60 - 3x_2}{2}\right\} = \left\{\begin{matrix} \frac{40 - x_2}{2}, 0 \le x_2 \le 8 \\ 24 - x_2, 8 \le x_2 \le 12 \\ \frac{60 - 3x_2}{2}, 12 \le x_2 \le 20 \end{matrix}\right\} \ldots \ (4)$

$$f_2(40,24,60) = Max \begin{Bmatrix} 2x_1 + 3\left(\frac{40-x_2}{2}\right), if \;\; 0 \le x_2 \le 8 \\ 2x_1 + 3(24 - x_2), if \;\; 8 \le x_2 \le 12 \\ 2x_1 + 3\left(\frac{60-3x_2}{2}\right), if \;\; 12 \le x_2 \le 20 \end{Bmatrix}$$ using (4) and (3)

$$f_2(40,24,60) = Max \begin{Bmatrix} 60 + \left(\frac{1}{2}\right)x_2, if \;\; 0 \le x_2 \le 8 \\ 72 - x_2, if \;\; 8 \le x_2 \le 12 \\ 90 - \left(\frac{5}{2}\right)x_2, if \;\; 12 \le x_2 \le 20 \end{Bmatrix}$$

$$f_2(40,24,60) = Max \begin{Bmatrix} 64, & at \;\; x_2 = 8 \\ 64, & at \;\; x_2 = 8 \\ 60, & at \;\; x_2 = 12 \end{Bmatrix}$$

$$f_1^*(40,24,60) = \; 64, \quad at \;\; x_2^* = 8$$

$$x_1^* = \text{Min}_{0 \le x_2 \le 20} \left\{\frac{40-x_2^*}{2}, 24 - x_2^*, \frac{60-3x_2^*}{3}\right\} = Min\,\{16,16,18\} = 16$$

Optimal solution is $x_1^* = 16, x_2^* = 8, z_{max} = 64$.

Example 1.12.2 Solve the following LPP using Dynamic Programming

$Maximize\ z = 3x_1 + 5x_2$

Subject to $x_1 \le 4; x_2 \le 6;\ 3x_1 + 2x_2 \le 18; x_1, x_2 \ge 0$

Solution:

$Number\ of\ variables\ = 2 \Rightarrow\ Number\ of\ stage\ = 2$

$Number\ of\ constraints\ = 3 \Rightarrow\ Number\ of\ state\ variable\ = 3$

Let $(b_1, b_2, b_3) = (4, 6, 18)$ be the state vector and $f_j(b_1, b_2, b_3)$ be the optimal value of the objective function. Now we are use to solve Backward recursive formula.

Stage $j = 2$;

$$f_2(b_1,b_2,b_3) = \max_{\substack{0 \le x_2 \le b_2 \\ 0 \le 2x_2 \le b_3}} \{5x_2\}$$

$$= 5 \max_{\substack{0 \le x_2 \le b_2 \\ 0 \le 2x_2 \le \frac{b_3}{2}}} \{x_2\}$$

$$= 5\ Min\left\{b_2, \frac{b_3}{3}\right\} \ldots (1)$$

Stage $j = 1$;

$$f_1(b_1,b_2,b_3) = \max_{\substack{0 \le x_1 \le b_1 \\ 0 \le 3x_1 \le b_3}} \{3x_1 + f_2(b_1 - x_1, b_2, b_3 - 3x_1)\}$$

$$= \max_{\substack{0 \le x_1 \le b_2 \\ 0 \le 2x_1 \le \frac{b_3}{2}}} \{3x_1 + 5\ Min(b_2, \frac{b_3 - 3x_1}{2})\}$$

$$f_1(4,6,18) = \max_{\substack{0\le x_1\le 4 \\ 0\le x_1\le \frac{18}{3}}} \{3x_1 + 5\, Min \left\{6,\ \frac{18-3x_1}{2}\right\}\} \quad \dots (2)$$

$\text{Min}_{0\le x_1\le 4}\left\{6, \frac{18-3x_1}{2}\right\}$ =? , (Intervals $0 \le x_1 \le 18/3$ is ignored)

x_1	$x_2 = 6$	$x_2 = \dfrac{18-3x_1}{2}$
0	6*	9
1	6*	15/2=7.5
2	6*	6*
3	6	4.5*
4	6	3*

$$\text{Min}_{0\le x_1\le 4}\left\{6,\ \frac{18-3x_1}{2}\right\} = \begin{Bmatrix} 6, if\ 0 \le x_1 \le 2 \\ \frac{18-3x_1}{2}, if\ 2 \le x_1 \le 4 \end{Bmatrix} \dots (3)$$

$$f_1(4,6,18) = Max \begin{Bmatrix} 3x_1 + 5(6)\ if\ 0 \le x_1 \le 2 \\ 3x_1 + 5(\frac{18-3x_1}{2}), if\ 2 \le x_1 \le 4 \end{Bmatrix} \text{ using (3) and (2)}$$

$$f_1(4,6,18) = Max \begin{Bmatrix} 3x_1 + 30\ if\ 0 \le x_1 \le 2 \\ 45 - \frac{9}{2}x_1, if\ 2 \le x_1 \le 4 \end{Bmatrix}$$

$$f_1(4,6,18) = Max \begin{Bmatrix} 36, & at\ x_1 = 2 \\ 36, & at\ x_1 = 2 \end{Bmatrix} = 36$$

$$f_1^*(4,6,18) = 36, \quad at\ x_1^* = 2$$

$$x_2^* = \text{Min}_{0\le x_1\le 4}\left\{6, \frac{18-3x_1^*}{2}\right\} = Min\,\{6,6\} = 6$$

Optimal solution is $x_1^* = 2, x_2^* = 6, z_{max} = 36.$

Let Us Sum Up

We have learned about branch and bound method which is used to solve all integer, mixed integer and zero-one linear programming problems. Also learned about characteristics of Dynamic programming problems and how to solve it.

Check Your Progress

9. Solve the following LPP using Dynamic Programming

$Maximize\ z = 4x_1 + 14x_2$

Subject to $2x_1 + 7x_2 \leq 21;\ 7x_1 + 2x_2 \leq 21; x_1, x_2 \geq 0$

10. Solve the following LPP using Dynamic Programming.

$Maximize\ z = 3x_1 + 2x_2 + 5x_3$

Subject to

$$x_1 + 2x_2 + x_3 \leq 430;$$
$$3x_1 + 2x_3 \leq 460;$$
$$x_1 + 4x_2 \leq 420;$$
$$x_1, x_2, x_3 \geq 0$$

Unit Summary

In this unit, an extension of linear programming, referred to as integer linear programming, was introduced where few or all variables must be an integer. If all variables of a problem are integers, then such problems are referred to as all-integer linear programming problems. If some, but not necessarily all, variables are integers, then such problems are referred to as mixed integer linear programming problems. Most integer programming applications involve 0-1 variables.

The number of applications of integer linear programming continues to grow rapidly due to the availability of integer linear programming software packages. The study of integer linear programming is helpful when fractional values for the variables are not permitted and rounding off their values may not provide an optimal integer solution; Integer LP programming facilitates developing mathematical models with variables assume either value 0 or 1. Capital budgeting, fixed cost, plant location, etc., are few examples where 0-1 integer programming techniques are extensively used to find an optimal solution..

Dynamic programming is an approach in which the problem is broken down into a number of smaller sub problems called stages. These sub problems are then solved sequentially until the original problem is finally solved. A particular sequence of alternatives (courses of action) adopted by the decision-maker in a multistage decision problem is called a policy.

The optimal policy, therefore, is the sequence of alternatives that achieves the decision-maker's objective. The solution of a dynamic programming problem is based upon

Bellman's principle of optimality (recursive optimization technique), which states: The optimal policy must be one such that, regardless of how a particular state is reached, all later decisions proceeding from that state must be optimal. Based on this principle of optimality, the best policy is derived by solving one stage at a time, and then sequentially adding a series of one-stage-problems are solved until the overall optimum of the initial problem is obtained.

The solution procedure is based on a backward induction process and forward induction process. In the first process, the problem is solved by solving the problem in the last stage and working backwards towards the first stage, making optimal decisions at each stage of the problem.

Glossary

- LPP- Linear programming problem
- ILPP- Integer Linear programming problem
- MIPP- Mixed integer programming problem
- DPP- Dynamic programming problem

Self – Assessment Questions

True or False

1. When a new constraint is added to a non-integer optimal simplex table, the new table represents an infeasible solution because of the negative value in the xB column of the new constraint.

2. The branch and bound terminates where the upper and lower bounds are identical and that value is the solution to the problem.

3. One disadvantage of the cutting plane integer programming method is that each new cut includes an artificial variable.

4. While using branch and bound method, decision regarding which subproblem needs decomposition is heuristic rule.

5. Integer programming always require more iterations of the simplex method than corresponding linear programming

ACTIVITIES

True or False

1. In integer programming, any non-integer variable can be picked up to enter the solution.
2. The branch and bound method is a modified form of enumeration method because in a maximization LP problem, all solutions that will result in return greater than the current upper bound are not considered.
3. Along a branch and bound minimization tree, the lower bound do not increase objective function value.
4. Alternate optimal solutions do not occur in integer programming.
5. While adding additional constraint to an integer linear programming a feasible integer solution is not eliminated.

Suggested Readings

1. J. K. Sharma, *Operations Research*, Theory and Applications, Third Edition (2007)Macmillan India Ltd
2. Hamdy A. Taha, *Operations Research,* (seventh edition) Prentice - Hall of IndiaPrivate Limited, New Delhi, 1997.
3. F.S. Hillier & J.Lieberman *Introduction to Operation Research* (7th Edition) Tata-McGraw Hill ompany, New Delhi, 2001.
4. Beightler. C, D.Phillips, B. Wilde ,*Foundations of Optimization* (2nd Edition) PrenticeHall Pvt Ltd., New York, 1979

UNIT – II

CLASSICAL OPTIMIZATION METHODS

CLASSICAL OPTIMIZATION METHODS

Objectives:

After studying this unit, students should able to use differential calculus-based methods to obtain an optimal solution of problems that involve continuous and differential functions. Derive necessary and sufficient conditions for obtaining an optimal solution for unconstrained and constrained, single and multivariable, optimization problems, with equality and inequality constraints.

Make distinction between local, global and inflection extreme points. Derive and use Kuhn-Tucker conditions necessary for an optimal value of an objective function subject to inequality constraints. Use graphical method to solve a non-linear programming model. Appreciate the use of some of the non-linear programming techniques such as quadratic programming, separable programming, geometric programming, stochastic programming, etc., for solving non-linear programming problems.

2.1 Introduction

The classical optimization methods are used to obtain an optimal solution of certain types of problems that involve continuous and differentiable functions. These methods are analytical in nature and make use of differential calculus to find points of maxima and minima for both unconstrained and constrained continuous objective functions. In this chapter, we shall discuss the necessary and sufficient conditions for obtaining an optimal solution of

(i) Unconstrained single and multiple variable optimization problems and

(ii) Constrained multivariable optimization problems with equality and inequality constraints.

2.2 UNCONSTRAINED OPTIMIZATION

2.2.1 Optimizing Single – Variable Functions

Figure 2.1 depicts the graph of a continuous function $y = f(x)$ of single independent variable x in the domain (a,b). The domain is the range of values of x. The domain limits (or end points) are generally called stationary (or critical) points. There are two categories of stationary points:

(i) Inflection points

(ii) Extreme points

Extreme points are further classified as (a) local (or relative) extreme

(b) global (or absolute) extreme

Local extreme points represent the maximum or minimum values of the function in the given range of values of the variable. In figure 2.1, points a, x_1, x_2, x_3, x_4, x_5 and b are all extrema of f(x). The classical approach to the theory of maxima and minima does not provide a direct method of obtaining global (or absolute) maximum (or minimum) value of a function. It only provides the method for determining the local (or relative) maximum and minimum values.

Mathematically, a function y = f(x) is said to achoeve its maximum value at a point, $x = x_0$ if $f(x_0 + h) – f(x_0) < 0$ or $f(x_0 + h) < f(x_0)$

where h is a sufficiently small number in the neighbourhood of the point $x = x_0$. In other words, the point x_0 is a local minimum if the value of f(x) at every point in the neighbourhood of x_0 does not exceed $f(x_0)$.

Similarly, a function f(x) is said to achieve its minimum value at a point $x = x_0$ if

$f(x_0 + h) – f(x_0) > 0$ or $f(x_0 + h) > f(x_0)$.

When a function has several local maximum and minimum values, the global minimum (in case of cost minimization) or global maximum (in case of profit maximization) is obtained bu comparing the values of the function at various extreme points (including the limits of the domian). The global minimum value of a function is the minimum value among all local minimum values of the function in the domain. Similarly, the global minimum value of a function is the maximum value among all local maximum values of the function in the domain. In figure 2.1, the point E i.e., $f(x_4)$ represents the global maximum, whereas the point F i.e., $f(x_5)$ represents the global minimum.

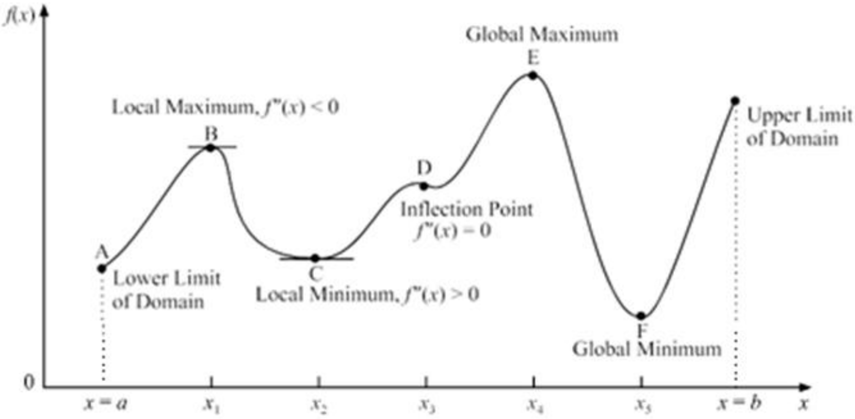


Figure 2.1: Local and Global Optimum

The global maximum (or minimum) of a function over the larger interval can also occur at an end point of the interval rather than at any local (relative) maximum or minimum point. It is possible for a local maximum value of a function to be less than a local minimum value of the function.

2.2.2 Conditions for Local Minimum and Maximum Value

Theorem 2.1: (Necessary condition): A necessary condition for a point x_0 to be the local extrema (local maximum and minimum) of a function y = f(x) defined in the interval $a \leq x \leq b$ is that the first derivative of f(x) exists as a finite number at $x = x_0$ and $f'(x_0) = 0$.

Proof: Let y = f(x) be a given function that can be expanded in the neighbourhood of $x = x_0$ by Taylor's theorem. Let at $x = x_0$ the value of f(x) be $f(x_0)$.

Consider two values of x, namely +h and –h in the neighbourhood and either side of $x = x_0$ (h being very small). If maximum is at $x = x_0$, then from definition, $f(x_0) > f(x_0 + h)$ and $f(x_0) > f(x_0 - h)$. i.e., $f(x_0 + h) - f(x_0)$ and $f(x_0 - h) - f(x_0)$ are both negative for maximum at $x = x_0$. Further, if minimum is at $x = x_0$, then $f(x_0) < f(x_0 + h)$ and $f(x_0) < f(x_0 - h)$. i.e., $f(x_0 + h) - f(x_0)$ and $f(x_0 - h) - f(x_0)$ are both positive for minimum at $x = x_0$. By using Taylor's theorem, we have:

$$f(x_0 + h) = f(x_0) + h\,f'(x_0) + \frac{h^2}{2!}\,f''(x_0) + \cdots + \frac{h^n}{n!}\,f^n(x_0) + R_n(x_0 + \theta h)\,; \quad 0 < \theta < 1$$ or

$$f(x_0 + h) - f(x_0) = h\,f'(x_0) + \frac{h^2}{2!}\,f''(x_0) + \cdots + \frac{h^n}{n!}\,f^n(x_0) + R_n(x_0 + \theta h)$$ ➔ (1) where

$R_n(x_0 + \theta h) = \frac{h^{n+1}}{(n+1)!}\,f^{n+1}(x_0 + \theta h)$ and is called the remainder.

The expressions $f'(x_0)$ and $f''(x_0)$ represent the first and second derivative of $f(x)$

at x = x_0. Similarly, $f(x_0-h)=f(x_0)-h\,f'(x_0)+\frac{h^2}{2!}f''(x_0)-\cdots$

$$f(x_0-h)-f(x_0)=-h\,f'(x_0)+\frac{h^2}{2!}f''(x_0)-\cdots \quad \rightarrow (2)$$

If h is very small, then neglecting the terms of higher order, we get,

$f(x_0+h)-f(x_0)=h\,f'(x_0)$ ➔ (3) and

$f(x_0-h)-f(x_0)=-h\,f'(x_0)$ ➔ (4)

For x = x_0 to be a local maximum or minimum value, the sign of $f(x_0+h)-f(x_0)$ and $f(x_0-h)-f(x_0)$ must be the same for all x = $x_0 \pm$ h. Thus from Equations (3) and (4) if $f(x_0+h)-f(x_0)$ and $f(x_0-h)-f(x_0)$ have the same sign, then $f'(x_0)$ should be zero; otherwise they will have different signs. Hence the necessary condition for ant function f(x) to have local optimum value at any extreme poiny x = x_0 is that its first derivative $f'(x_0)$ = 0.

Remark: The distinction between a local minimum and local maximum can also be seen by examining the direction of change of first derivative. $f'(x_0)$ at x = x_0.

(i) If the sign of $f'(x_0)$ changes from positive to negative as x increases in the neighbourhood of x = x_0, then the value of f(x) will be a local maximum.

(ii) If the sign of $f'(x_0)$ changes from negative to positive as x increases in the neighbourhood of x = x_0, then the value of f(x) will be a local minimum.

Theorem 2.2 (Sufficient condition) If an extreme point x = x_0 of f(x), the first (n – 1) derivatives of it become zero, i.e., $f'(x_0)=f''(x_0)=\cdots=f^{n-1}(x_0)=0$ and $f^{(n)}(x_0)\neq 0$, then:

(i) Local maximum of f(x) occurs at x = x_0, if $f^{(n)}(x_0)$ < 0, for n even,

(ii) Local minimum of f(x) occurs at x = x_0, if $f^{(n)}(x_0)$ > 0, for n even,

(iii) Point of inflection occurs at x = x_0 if $f^{(n)}(x_0)\neq 0$, for n odd.

Proof: From theorem 2.1 at an extreme point x = x_0, $f'(x_0)$ = 0. Then from equations (1) and (2), we have

$f(x_0+h)-f(x_0)=\frac{h^2}{2!}\,f''(x_0)$ ➔ (5) and

$f(x_0-h)-f(x_0)=\frac{h^2}{2!}\,f''(x_0)$ ➔ (6)

neglecting powers of h higher than second. Here, the following three possible cases may arise:

Case 1: If $f''(x_0) > 0$, then both $f(x_0 + h) - f(x_0)$ and $f(x_0 - h) - f(x_0)$ are positive and hence local minimum value of f(x) exists at x = x_0.

Case 2: If $f''(x_0) < 0$, then both $f(x_0 + h) - f(x_0)$ and $f(x_0 - h) - f(x_0)$ are negative and hence local maximum value of f(x) exists at x = x_0.

Case 3: If $f''(x_0) = 0$, then no information is obtained about the maximum or minimum value of f(x). i.e., in this case, the function f(x) may have a local maximum, a local minimum or a point of inflection. Hence, if If $f''(x_0) = 0$, then we examine successively higher order derivatives of f(x) at x = x_0 until we find a derivative such that If $f^{(n)}(x_0) \neq 0$, n ≥ 2.

If If $f^{(n)}(x_0) < 0$, for n even, then f(x) has local maximum value at x = x_0. If If $f^{(n)}(x_0) > 0$, for n even, then f(x) has local minimum value at x = x_0. If n is odd, then x = x_0 is the point of inflection (or saddle point).

The necessary and sufficient conditions for the existence of local maximum, local minimum and point of inflection are summarized in Table 2.1. The entire preceding discussion is summarized in fig 2.2.

Necessary condition	**Sufficient Condition**	**Nature of Function**	**Conclusion**
$f'(x_0) = 0$	$f'(x_0) = f''(x_0) = \cdots = f^{n-1}(x_0) = 0$ and $f^{(n)}(x_0) < 0$, n even	Concave	Local maximum at x = x_0
$f'(x_0) = 0$	$f'(x_0) = f''(x_0) = \cdots = f^{n-1}(x_0) = 0$ and $f^{(n)}(x_0) > 0$, n even	Convex	Local minimum at x = x_0
$f'(x_0) = 0$	$f'(x_0) = f''(x_0) = \cdots = f^{n-1}(x_0) = 0$ and $f^{(n)}(x_0) \neq 0$, n odd	-	Point of inflection at x = x_0

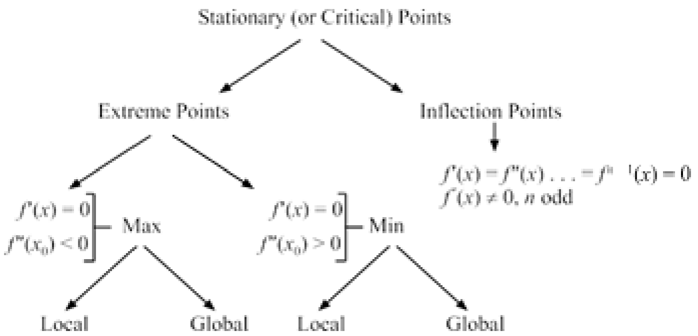


Figure 2.2: Determination of Critical Point

It becomes easy to find the maximum or minimum values when the function is either convex or concave. If a function is convex, the first derivative set equal to zero must give at least one local minimum. The value of the function at the end points of the domain may still be the global minimum. Similarly, if a function is concave, the first derivative set equal to zero must give at least one local maximum. It is due to this reason that functions most commonly found in business are assumed to be either concave or convex.

Summary of the procedure: The procedure to determine the global minimum or maximum is summarized in the following steps:

1) Compute first derivative, $\frac{dy}{dx}$ and equate it with zero.
2) Solve the equation $\frac{dy}{dx} = 0$ for $x = x_0$.
3) Substitute the value $x = x_0$, $x = a$ and $x = b$ in the original equation and determine $f(x_0)$, $f(a)$ and $f(b)$.
4) Compare these values to determine global minimum and maximum respectively.

Remarks:

1) A local minimum of a convex function on a convex set is also a global minimum of that function.
2) A local maximum of a concave function on a convex set is also a global maximum of that function.
3) A local minimum of a strictly convex function on a convex set is also a unique global minimum of that function.
4) A local maximum of a strictly concave function on a convex set is also a unique global maximum of that function.

Example 2.1 A trader receives *x* units of an item at the beginning of each month. The cost of carrying x units per month is given by:

$$C(x) = \frac{c_1 x^2}{2n} + \frac{c_2(20n - x)^2}{2n}$$

where c_1 = cost per day of carrying a unit of item in stock (= Rs 10)

c_2 = cost per day of shortage of a unit of item (= Rs 150)

n = number of units of item to be supplied per day (= 30)

Determine the order quantity x that would minimize the cost of inventory.

Solution:

The necessary condition for a function to have either minimum or maximum value at a point is that its first derivative should be zero. Thus,

$$\frac{dC(x)}{dx} = \frac{c_1 x}{n} - \frac{c_2(20n - x)}{n} = 0$$

Therefore, $x = \frac{20nc_2}{c_1+c_2} = \frac{(20)(30)(150)}{10+150} = 562.5$

The nature of the extreme point given by x is determined by considering the second derivative.

$$\frac{d^2C(x)}{dx^2} = \frac{c_1}{n} + \frac{c_2}{n} > 0$$

Since the value of the second derivative is positive, therefore, x = 562.5 is a local minimum point. By substituting the value of x in the objcetive function C(x), we get

C(x = 562.5) = Rs. 56,249.37 ; C(x = 0) = Rs. 9,00,000

$\lim_{x\to\infty} C(x) = \infty$

It follows that, a global minimum value for C(x) occurs at x = 562.5.

Example 2.2 A firm has a total revenue function, R = 20x – 2x^2, and a total cost function, c = x^2 – 4x + 20, where x represents the quantity. Find the revenue maximizing output level and the corresponding vaue of profit, price and total revenue.

Solution: The necessary condition for a revenue function R to have maximum value at a point is that: $\frac{dR}{dx} = 0$ and $\frac{d^2R}{dx^2} < 0$.

Since R = 20x – 2x^2, therefore $\frac{dR}{dx}$ = 0 gives 20 – 4x = 0 (or) x = 5. Also $\frac{d^2R}{dx^2} = -4$ (< 0). Since the value of second derivative is negative, the revenue will be maximum at an output level, x = 5.

The profit function is given by: π = R – C = (20x – 2x^2) – (x^2 – 4x + 20) = 24x – 3x^2 – 20

Thus, the total profit at x = 5 will be: P = 24(5) – 3(5^2) – 20 = 25

The price of a product is given by P = $\frac{\pi}{x}$ = 20 – 2x = 10, at x = 5. The maximum revenue at x = 5, is R = 20(5) – 2(5^2) = 50.

Example 2.3 The total profit y, in rupees, of a drug comoany from the manufacturing and sale of x drug bottles is given by, y = $-\frac{x^2}{400}$ + 2x – 80.

(a) How many drug bottles must the company sell in order to achieve the maximum profit

(b) What is the profit per druf bottle when this maximum is achieved ?

Solution: Given: y = $-\frac{x^2}{400}$ + 2x – 80. Therefore, $\frac{dy}{dx} = -\frac{2x}{400} + 2 = -\frac{x}{200} + 2$
The first order condition for maximum value of y is $\frac{dy}{dx} = 0$, i.e., $-\frac{x}{200} + 2 = 0$ (or) x = 400.
Since $\frac{d^2y}{dx^2} = -\frac{1}{200}$ (< 0), therefore the company must sell x = 400 drug bottles in order to achieve the maximum profit, which is equal to $y = -\frac{400^2}{400} + 2(400) - 80$ = Rs. 320.

Example 2.4 The efficiency E of a small manufacturing concern depends on the workers W and is given by 10E = $-\frac{W^3}{40} + 30W - 392$. Find the strength of the workers that would give the maximum efficiency.

Solution: Given: 10E = $-\frac{W^3}{40} + 30W - 392$ (or) E = $-\frac{W^3}{400} + 3W - 39.2$. Therefore,

$$\frac{dE}{dW} = -\frac{3W^2}{400} + 3$$

The first order condition for maximum value of E is $\frac{dE}{dW} = 0$, i.e., $-\frac{3W^2}{400} + 3 = 0$ (or) W = $\pm$ 20. (neglecting W = –20 because workers cannot be negative in number). Also $\frac{d^2E}{dW^2} = -\frac{6W}{400}$ (< 0) at W = 20 (a second order condition for maxima), therefore the efficiency of the workers shall be maximum when they are W = 20 in number.

Example 2.5 The cost of fuel for running a train is proportional to the cube of the speed generated in kilo meter per hour. When the speed is 12 km/h, the cost of fuel is Rs. 64 /h. If the other charges are fixed, namely Rs. 2000 /h, find the most economical speed of the train for running a distance of 100 km.

Solution: Let x km/h be the speed of the train. Then the cost of fuel = kx^3, where k is constant of proportionality.

Given that it costs Rs. 64 per hour at 12km/hr. Therefore, 64 = $k(12^3)$.

⇨ k = $\frac{64}{12^3} = 0.037$. Hence, cost of fuel = Rs. $0.037(x^3)$ per hour. The fuel for running a distance of 100 km is: $0.037(x^3)\left(\frac{100}{x}\right) = 3.7x^2$.

Also, the fixed cost = $2000\left(\frac{100}{x}\right)$.

If C is the cost of running 100 km, then, c = $3.7x^2 + 2000\left(\frac{100}{x}\right)$

$\frac{dC}{dx} = 7.4x - 2000\left(\frac{100}{x^2}\right)$ and $\frac{d^2C}{dx^2} = 7.4 + 2000\left(\frac{200}{x^3}\right)$

Solving the first equation above, we get x = 30. For this value of x, $\frac{d^2C}{dx^2}$ > 0, i.e., C is minimum. Thus, the most economic speed of the train should be 30 km/hr.

Example 2.6 The production function of a commodity is given by: Q = 40F + $3F^2$ - $\left(\frac{F^3}{3}\right)$, where Q is the total output and F is the units of inputs.

(a) Find the number of units of input required to give the maximum output.

(b) Find the maximum value of marginal product.

(c) Verify that when the average product is maximum, it is equal to marginal product.

Solution: (a) We have, Q = 40F + $3F^2$ - $\left(\frac{F^3}{3}\right)$; F ≥ 0.

For maximum or minimum input level, $\frac{dQ}{dF} = 40 + 6F - F^2$

Now, $\frac{dQ}{dF} = 0$ ➔ $40 + 6F - F^2 = 0$ ➔ F = 4 (or) –10.

$\frac{d^2Q}{dF^2} = 6 - 2F$

For F = 4, $\frac{d^2Q}{dF^2} = 6 - 2F$ = –2 (< 0) and for F = –10, $\frac{d^2Q}{dF^2} = 6 - 2F$ = 14 (> 0).

Thus, the output Q is maximum when F = 4 units of inout are used.

(b) The marginal product is given by, MP = $\frac{dQ}{dF}$ = 40 + 6F – F^2

For maximum or minimum value of marginal product, $\frac{d}{dF}(MP)$ = 6 – 2F = 0 ➔ F = 3.

Also $\frac{d^2}{dF^2}(MP) = -2$ (< 0), i.e., MP is maximum at F = 3.

(c) The average product is given by: AP = $\frac{Q}{F} = \frac{1}{F}\left(40F + 3F^2 - \frac{F^3}{3}\right) = 40 + 3F - \frac{F^2}{3}$

For maximum or minimum value of average product,

$\frac{d}{dF}(AP) = 3 - \frac{2}{3}F = 0$. i.e., F = $\frac{9}{2}$.

Also $\frac{d^2}{dF^2}(AP) = -\frac{2}{3}$ (< 0), i.e., AP is maximum at F = $\frac{9}{2}$.

Maximum value of AP at F = $\frac{9}{2}$ is$40 + 3\left(\frac{9}{2}\right) - \frac{1}{3}\left(\frac{9}{2}\right)^2 = \frac{187}{4}$ and maximum value of MP at F = $\frac{9}{2}$ is also $\frac{187}{4}$.

This shows that when AP is maximum, it is equal to MP.

2.2.3 Optimizing Multivariable Functions

To optimize a multivariable function, we use the concept ofpartial derivatives. This is because partial derivatives measure the change in the dependent variable due to unit change in one of the independent variables, while keeping constant the effect of all other independent variables. The necessary and sufficient conditions for local optimum (maximumor minimum) of constrained multivariable functions may be described as follows:

Taylor's series expansion of a multivariable function:

Let f(x) be a real valued continuous and differentiable function of x in E^n. Ket (x + h) be a point in the neighbourhood of x such that: $h = (h_1, h_2, \ldots, h_n)^T$ and $x + h = (x_1 + h_1, x_2 + h_2, \ldots x_n + h_n)^T$ where $x = (x_1, x_2, \ldots, x_n)^T$. Then f(x) can be expressed as a power series involving the differentials of f(x) itself to Taylor's series.

$f(x + h) = f(x_1 + h_1, x_2 + h_2, \ldots x_n + h_n)$

$$= f(x) + \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i}\right) h_i + \frac{1}{2!} \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\frac{\partial^2 f}{\partial x_i \, \partial x_j}\right) h_i h_j \quad \rightarrow (7)$$

Now we define the gradient vector of f(x) , denoted by ∇ f(x), as folloes. The n^{th} gradient vector whoes components are the partial derivatives of f(x) and Hessian matrix H(x) of order n evaluted at $x + \theta h$ $(0 < \theta < 1)$ are as follows:

$$\nabla f(x) = \left[\frac{\partial f(x)}{\partial x_1}, \frac{\partial f(x)}{\partial x_2}, \ldots, \frac{\partial f(x)}{\partial x_n}\right]^T \text{ and } H(x) = \begin{bmatrix} \frac{\partial^2 f(x)}{\partial x_1^2} & \frac{\partial^2 f(x)}{\partial x_1 \, \partial x_2} & \cdots & \frac{\partial^2 f(x)}{\partial x_1 \, \partial x_n} \\ \frac{\partial^2 f(x)}{\partial x_2 \, \partial x_1} & \frac{\partial^2 f(x)}{\partial x_2^2} & \cdots & \frac{\partial^2 f(x)}{\partial x_2 \, \partial x_n} \\ \frac{\partial^2 f(x)}{\partial x_n \, \partial x_1} & \frac{\partial^2 f(x)}{\partial x_n \, \partial x_2} & \cdots & \frac{\partial^2 f(x)}{\partial x_n^2} \end{bmatrix}$$

Using the above definitions, we can write:

$$f(x + h) - f(x) = \nabla f(x)\, h + \frac{1}{2!} h^T H(x)\, h \; ; \; x + \theta h \; (0 < \theta < 1) \quad \rightarrow (8)$$

Theorem 2.3: The necessary condition for the continuous function f(x) to have an extreme point at $x = x_0$ is that gradient $\nabla f(x_0) = 0$. i.e., $\frac{\partial f(x_0)}{\partial x_1} = \frac{\partial f(x_0)}{\partial x_2} = \cdots = \frac{\partial f(x_0)}{\partial x_n} = 0.$

Proof: Putting $x = x_0$ in (8), we get: : $f(x_0 + h) - f(x_0) = \nabla f(x_0)\, h + \frac{1}{2!} h^T H(x_0)\, h$; $x = x_0 + \theta h$ $(0 < \theta < 1)$.

As we know that the term $h^T H(x_0) h$ contains terms of order h^2 and hence the term $h^T H(x_0) h$ tends to zero as $h \rightarrow 0$. Thus, the sign of $f(x_0 + h) - f(x_0)$ depends upon ths sign of

∇ f(x) h. Again, the sign of ∇ f(x_0) h depends upon the sign of h. Hence, f(x_0 + h) – f(x_0) will be positive or negative according to whether h is positive or negative, respectively. This contradicts our assumption that x_0 is an extreme point. It follows that for x_0 to be an extreme point it is necessary that ∇ f(x_0) = 0. In other words, the partial derivatives of f(x) with respect to x_i (i = 1, 2, …, n) must be zero at the exterme point x_0.

Theorem 2.4: A sufficient condition for a stationary point x_0 to be an extreme point is that the Hessian matrix H(x), evaluted at x_0 is:

(a) positive definite when x_0 is a minimum point and

(b) negative definite when x_0 is a maximum point.

Proof: Putting x = x_0 in (8), we get: : $f(x_0 + h) – f(x_0) = \nabla f(x_0) h + \frac{1}{2!} h^T H(x_0) h$ ➔ (9)

$x = x_0 + \theta h$ $(0 < \theta < 1)$.

Since x_0 is a stationary point, therefore from Theorem 2.2 we have ∇ f(x_0). Thus (9) becomes $f(x_0 + h) – f(x_0) = \frac{1}{2} h^T H(x_0) h$ at $x = x_0 + \theta h$ $(0 < \theta < 1)$.

Now the sign of f(x_0 + h) – f(x_0) depends upon the sign of the quadratic expression $\frac{1}{2} h^T H(x_0) h$ whereas the sign of $\frac{1}{2} h^T H(x_0) h$ varies with the choice of h. Let the extreme point x_0 be a local minimum. Then by definition, f(x_0 + h) – f(x_0) will be positive. Hence for x_0 to be a local minimum, the expression

$\frac{1}{2} h^T H(x_0) h = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right) h_i h_j$; $x = x_0 + \theta h$ is positive.

Since the second partial derivative is continuous, i.e., $\frac{\partial^2 f(x)}{\partial x_i \partial x_j} = \frac{\partial^2 f(x)}{\partial x_j \partial x_i}$ for all i, j = 1, …, n in the neighbourhood of the point x_0, it will have the same sign for all sufficiently small h in the neighbourhood of $x_0 + \theta h$. The quadratic expression $h^T H(x_0) h$ is positive only if the Hessian matrix H(x_0) is positive definite at x = x_0. It follows that a sufficient condition for a stationary point x_0 to be a local minimum is that the Hessian matrix evaluated at the same point to be positive definite.

Similarly, it can also be proved that H(x_0) is negative definite for a maximization case.

Remarks:

(1) The different types of test can also be used to identify local maxima or minima by examining the minors of the matrix H(x).

(a) H(x) is positive definite if all its leading principal minors pf order 1 x 1 are positive. In this case the extreme point is a local minimum. A principal minor of H(x) is the determinant of a square submatrix whose elements lie on the diagonal of H(x), whereas leading principal minor is one whose (1,1) element is the (1,1) element of H(x).

(b) H(x) is negative definite, if the signs of all even leading principal mirros is positive.

(c) If signs of determinants do not meet conditions (i) and (ii), then the extreme point may either a maximum or a minimum or neither. In this case the matrix H(x) is termed as semi – definite or indefinite.

(2) Summary of results:

Necessary Condition	Sufficient Condition	Conclusion
$\nabla f(x_0) = 0$	$H(x_0)$ is positive definite	Local minimum at $x = x_0$
$\nabla f(x_0) = 0$	$H(x_0)$ is negative definite	Local maximum at $x = x_0$
$\nabla f(x_0) = 0$	$H(x_0)$ is indefinite	Point of inflection at $x = x_0$

Example 2.7 Find the second order Taylor's series approximation of the function:
$f(x_1, x_2) = x_1^2 x_2 + 5x_1 e^{x_2}$ about the point $x_0 = [1, 0]^T$

Solution: The second order Taylor's series approximation of the function $f(x_1, x_2) = x_1^2 x_2 + 5x_1 e^{x_2}$ is:

$$f(x_1, x_2) = f\begin{bmatrix}1\\0\end{bmatrix} + \nabla f\begin{bmatrix}1\\0\end{bmatrix} h + \frac{1}{2!} h^T H(x) h, \text{ where } x = x_0 + \theta h, \text{ and}$$

$$h = \begin{bmatrix}h_1\\h_2\end{bmatrix} = \begin{bmatrix}x_1\\x_2\end{bmatrix} - \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}x_1 - 1\\x_2\end{bmatrix}$$

$$x_0 + \theta h = \begin{bmatrix}1\\0\end{bmatrix} + \theta \begin{bmatrix}x_1 - 1\\x_2\end{bmatrix} = \begin{bmatrix}1 + \theta x_1 - \theta\\\theta x_2\end{bmatrix}$$

$$\nabla f(x_0) = \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}\right] = (2x_1 x_2 + 5e^{x_2}, x_1^2 + 5x_1 e^{x_2})$$

For $x_0 = [1,0]^T$, the value of $\nabla f(x_0) = [5,6]$

$$H(x) = \begin{bmatrix}\frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2}\\\frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2}\end{bmatrix} = \begin{bmatrix}2x_2 & 2x_1 + 5e^{x_2}\\2x_1 + 5e^{x_2} & 5x_1 e^{x_2}\end{bmatrix}$$

Substituting the values in $f(x_1, x_2)$, we get:

$$f(x_1, x_2) = 5 + [5,6]\begin{bmatrix}x_1 - 1\\x_2\end{bmatrix} + \frac{1}{2!}\begin{bmatrix}x_1 - 1\\x_2\end{bmatrix}^T \begin{bmatrix}2x_2 & 2x_1 + 5e^{x_2}\\2x_1 + 5e^{x_2} & 5x_1 e^{x_2}\end{bmatrix}\begin{bmatrix}x_1 - 1\\x_2\end{bmatrix}$$

Example 2.8 Consider the function $f(x) = x_1 + 2x_2 + x_1x_2 - x_1^2 - x_2^2$. Determine the maximum or minimum point (if any) of the function.

Solution: The necessary condition for local optimum (maximum or minimum) value is that gradient $\nabla f(x) = \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}\right] = 0$

i.e., $\frac{\partial f}{\partial x_1} = 1 + x_2 - 2x_1 = 0$ and $\frac{\partial f}{\partial x_2} = 2 + x_1 - 2x_2 = 0$. The solution of these simultaneous equations is: $x_0 = \left(\frac{4}{3}, \frac{5}{3}\right)$.

Ths sufficient condition can be verified by considering the Hessian matric as follows:

$$H(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$$

$$\text{Det } A_1 = \left|\frac{\partial^2 f}{\partial x_1^2}\right| = -2 \text{ and Det } A_2 = \begin{vmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{vmatrix} = 3$$

Since the signs of the principal minor determinants of H(x) are alternating, matrix H(x) is negative definits and the point $x_0 = \left(\frac{4}{3}, \frac{5}{3}\right)$ is the local maximum of the function f(x).

Let Us Sum Up

We have learned about single- optimizing variable function, conditions for local extrema of a function defined in an interval related concepts, examples. And also optimizing multi variable functions by using the concept of partial derivatives.

Check Your Progress

11. Examine the following function for extreme points:

    (a) $f(x_1, x_2) = 3x_1^2 + x_2^2 - 10$

    (b) $f(x_1, x_2) = 100(x_1 - x_2^2)^2 + (1 - x_1)^2$

    (c) $f(x_1, x_2) = x_1^3 + x_2^3 + 2x_1^2 + 4x_2^2$

12. Show that a cube curve whose equation is of the form: $y = ax^3 + bx^2 + cx + d$, where $a, b, c, d \neq 0$, has one and only one point of inflection.

13. The demand function for a particular commodity is, $\rho = 15e^{-x/3}$, where ρ is the price per unit and x is the number of units demanded. Determine the price and the quantity for which the revenue (R) is maximum.

14. If the total revenue (R) and total cost (C) function of a firm are given by $R = 30x - x^2$ and $C = 20 + 4x$, where x is the output, find the equilibrium level output of the firm. What is the maximum profit ?

15. There are 60 newly built apartments. All these would be occupied at rent of Rs. 4,500 per month. But one apartment is likely to remain vacant for every Rs. 150 increase in rent. An occupied apartment requires Rs. 6 month for maintenance. Find the relationship between profit and the number of unoccupied apartments. What is the number of vacant apartments for which the profit is maximum?

2.3 CONSTRAINED MULTIVARIABLE OPTIMIZATION WITH EQUALITY CONSTRAINTS

In this section, we shall discuss the problem of optimizing a continuous and differentiable function subject to equality constraints.

Optimize (max or min) $Z = f(x_1, x_2, \ldots, x_n)$ subject to the constraints

$g_i(x) = 0$; i = 1, 2, …, m → (10)

where $x = (x_1, x_2, \ldots, x_n)$ → (11) and $g_i(x) = h_i(x) - b_i$; b_i is a constant.

Here, it is assumed that m < n to get the solution.

There are various methods for solving the above defined problem. But in this section, we shall discuss only two methods:

(i) Direct Substitution Method (ii) Lagrange Multipliers Method

2.3.1 Direct Substitution Method

Since the constraint set $g_i(x)$ is also continuous and differentiable, any variable in the constraint set can be expressed in terms of the remaining variables. Then it is substituted into the objective function. The new objective function, so obtained, is not subjected to any constraints and hence its optimum value can be obtained by the unconstrained optimization method, discussed in the previous section.

Sometimes this method is not convenient, particularly when there are more than two variables in the objective function and are subject to constraints.

Example 2.9 Find the optimum solution of the following constrained multivariable problem. Minimize $Z = x_1^2 + (x_2 + 1)^2 + (x_3 - 1)^2$ subject to the constraint $x_1 + 5x_2 - 3x_3 = 6$ and $x_1, x_2, x_3 \geq 0$

Solution: Since the given problem has three variables and one equality constraint, any one of the variables can be removed from Z with the help of the equality constraint. Let us choose variable x_3 to be eliminated from Z. Then, from the equality constraint, we have: $x_3 = \frac{x_1+5x_2-6}{3}$

Substituting the value of x_3 in the objective function, we get:

Z (or) f(x) = $x_1^2 + (x_2 + 1)^2 + \frac{1}{9}(x_1 + 5x_2 - 9)^2$

The necessary condition for minimum of Z is that the gradient

$$\nabla f(x) = \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}\right] = 0$$

$\frac{\partial Z}{\partial x_1} = 2x_1 + \frac{2}{9}(x_1 + 5x_2 - 9) = 0$ → (12)

$\frac{\partial Z}{\partial x_2} = 2(x_1 + 1) + \frac{10}{9}(x_1 + 5x_2 - 9) = 0$ → (13)

On solving these equations, we get $x_1 = \frac{2}{5}$ and x_1 = 1.

To find whether the solution, so obtained, is minimum or not, we must apply the sufficiency condition by forming a Hessian matrix. The Hessian matrix for the given objective function is H(x1 , x2) = $\begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} \frac{20}{9} & \frac{10}{9} \\ \frac{10}{9} & \frac{20}{9} \end{bmatrix}$

Since the matric is symmetric and principal diagonal elements are positive, H(x_1 , x_2) is positive definite and the objective function is convex. Hence, the optimum solution to the given problem is, $x_1 = \frac{2}{5}$, x_1 = 1, $x_1 = -\frac{1}{5}$ and Min Z = $\frac{28}{5}$

2.3.2 Lagrange Multipliers Method

In this method an additional variable in each of the given constraints is added. Thus, if the problem has n variables and m equality constraints, then m is additional variables are to be added so that the problem would have n + m variables. Before discussing the general method, let us illistrate its salient features through the following simple problem that involves onlt three variables:

Necessary condition for a problem with n = 3 and m = 1

Consider the NLP problem:

Optimize (max or min) Z = f(x_1, x_2, x_3) → (14)

subject to the constraint g(x_1, x_2, x_3) = 0 → (15)

Let an optimum value of Z occur at a point (x_1, x_2, x_3) = (a, b, c) at which at least one of the partial derivatives $\frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}, \frac{\partial g}{\partial x_3}$ does not vanish. Thus, we may proceed as follows:

(i) Choose one variable say x_3 in constraint (15) and express it in terms of the remaining two variabes such that x_3 = h (x_1 , x_2)

(ii) Substitute the value of x_3 into the objective function (14), we get:

$Z = f\{(x_1, x_2), h(x_1, x_2)\}$

From unconstrained optimization methods, we know that the necessary condition for local optimum is that all first derivatives with respect to x_1 and x_2 must be zero; i.e.,

$$\frac{\partial Z}{\partial x_j} = 0 \; ; j = 1, 2 \qquad \rightarrow (16)$$

Applying the chain rule for differentiation on (16) we get:

$$\frac{\partial Z}{\partial x_j} = \frac{\partial f}{\partial x_j} + \frac{\partial f}{\partial x_3} \cdot \frac{\partial h}{\partial x_j} \; ; j = 1, 2$$

But from (15), we have: $\frac{\partial g}{\partial x_j} + \frac{\partial g}{\partial x_3} \cdot \frac{\partial h}{\partial x_j} = 0 \; ; j = 1, 2$

$$\frac{\partial h}{\partial x_j} = -\frac{\frac{\partial g}{\partial x_j}}{\frac{\partial g}{\partial x_3}} \; ; \frac{\partial g}{\partial x_3} \neq 0 \; ; j = 1, 2$$

at the point $(x_1, x_2, x_3) = (a, b, c)$

since optimum occurs at the point (a, b, c), we have:

$$\frac{\partial Z}{\partial x_j} = \frac{\partial f}{\partial x_j} - \left[\frac{\partial f}{\partial x_3} \cdot \left\{\frac{\frac{\partial g}{\partial x_j}}{\frac{\partial g}{\partial x_3}}\right\}\right] = 0 \text{ at } (x_1, x_2, x_3) = (a, b, c) \qquad \rightarrow (17)$$

As $\frac{\partial g}{\partial x_3} \neq 0$, we define a quantity λ, called Lagrange multiplier as given below. The value of λ represents theamount of change in the objective function due to the per unit change in the constraint li,it. i.e.,

$$\frac{\partial f}{\partial x_3} - \lambda \frac{\partial g}{\partial x_3} = 0 \; at \; (x_1, x_2, x_3) = (a, b, c)$$

$$(or) \qquad \lambda = \frac{\frac{\partial f}{\partial x_3}}{\frac{\partial g}{\partial x_3}}$$

Equation (17) can now be written as:

$$\frac{\partial Z}{\partial x_j} = \frac{\partial f}{\partial x_j} - \lambda \frac{\partial g}{\partial x_j} = 0 \; ; j = 1, 2 \qquad \rightarrow (18)$$

at $(x_1, x_2, x_3) = (a, b, c)$ and the constraint equation $g(x_1, x_2, x_3) = 0$ → (19)

is also satisfied at the extreme (or critical) points, $x_1 = a$, $x_2 = b$ and $x_3 = c$. The conditions (18) and (19) are called necessary conditions for a local optimum, provided not all $\frac{\partial g}{\partial x_j}$, j = 1, 2 become zero at the extreme points.

The necessary conditions given by (18) and (19) can be obtained very easily by forming a function L, called the Lagrange function, as:

$$L(x_j, \lambda) = f(x_j) - \lambda \, g(x_j), \; j = 1, 2, 3 \qquad \rightarrow (20)$$

We must, then, partially differentiate $L(x_j, \lambda)$ with respect to x_j (j = 1, 2, 3) and λ and equate them with zero. The following equations provide the necessary conditions for local optimum:

$$\frac{\partial L}{\partial x_j} = \frac{\partial f}{\partial x_j} - \lambda \frac{\partial g}{\partial x_j} = 0 \text{ ; j = 1, 2, 3} \qquad \rightarrow (21)$$

$$\frac{\partial L}{\partial \lambda} = g(x_j) = 0 \text{ ; j = 1, 2, 3}$$

These equations can be solved for the unkown x_j (j = 1, 2, 3) and λ.

Remark: The necessary conditions, so obtained, become sufficient conditions for a maximum (or minimum) if f(x) is concave (or convex), with equality constraints.

Example 2.10 Obtain the necessary condition for the optimum solution of the following problem: Minimize $f(x_1, x_2) = 3e^{2x_1+1} + 2e^{x_2+5}$ subject to the constraint $g(x_1, x_2) = x_1 + x_2 - 7 = 0$ and $x_1, x_2 \geq 0$

Solution: Forming the Lagrangian function, we obtain

$$L(x_1, x_2, \lambda) = f(x_1, x_2) - \lambda\, g(x_1, x_2) = 3e^{2x_1+1} + 2e^{x_2+5} - \lambda\,(x_1 + x_2 - 7)$$

The necessary conditions for the minimum of $f(x_1, x_2)$ are given by:

$$\frac{\partial L}{\partial x_1} = 6e^{2x_1+1} - \lambda = 0 \;(or)\; \lambda = 6e^{2x_1+1}$$

$$\frac{\partial L}{\partial x_2} = 2e^{x_2+5} - \lambda = 0 \;(or)\; \lambda = 2e^{x_2+5}$$

$$\frac{\partial L}{\partial \lambda} = -(x_1 + x_2 - 7) = 0$$

On solving these three equations in three unknowns, we obtain:

$x_1 = \left(\frac{1}{3}\right)(11 - \log 3)$ and $x_1 = 7 - \left(\frac{1}{3}\right)(11 - \log 3)$.

Necessary conditions for a general problem:

Consider the non – linear programming problem:

Optimize *Z* = *f* (**x**) subject to the constraint $h_i(x) = b_i$ or $g_i(x) = h_i(x) - b_i = 0$ where i = 1, 2, …, m and m ≤ n ; $x \in E^n$.

The necessary conditions (21) for a function to have a local optimum at the given points can be extended to the case of a general problem with *n* variables and m equality constraints.

Multiply each constraint with an unknown variable λ_i (i = 1, 2, . . ., m) and subtract each from the objective function, $f(x)$ to be optimized. The new objective function now becomes:

$$L(x,\lambda) = f(x) - \sum_{i=1}^{m} \lambda_i g_i(x) \; ; x = (x_1, x_2, \dots, x_n)^T$$

where $m < n$. The function L(x, λ) is called the *Lagrange function.*

The necessary conditions for an unconstrained optimum of $L(x, \lambda)$, i.e. the first derivatives, with respect to x and λ of $L(x, \lambda)$ must be zero, are also necessary conditions for the given constrained optimum of $f(x)$, provided that the matrix of partial derivatives $\frac{\partial g_i}{\partial x_j}$ has rank m at the point of optimum.

The necessary conditions for an optimum (max or min) of $L(x, \lambda)$ or $f(x)$ are the $m + n$ equations to be solved for $m + n$ unknown ($x_1, x_2, \dots, x_n ; \lambda_1, \lambda_2, \dots, \lambda_m$)

$$\frac{\partial L}{\partial x_j} = \frac{\partial f}{\partial x_j} - \sum_{i=1}^{m} \lambda_i \frac{\partial g_i}{\partial x_j} = 0 \; ; j = 1, 2, \dots, n$$

$$\frac{\partial L}{\partial x_j} = -g_i \; ; \qquad\qquad j = 1, 2, \dots, m$$

These ($m + n$) necessary conditions also become sufficient conditions for a maximum (or minimum) of the objective function $f(x)$, in case it is concave (or convex) and the constraints are equalities, respectively.

Sufficient conditions for a general problem: Let the Lagrangian function for a general NLP problem, involving n variables and m ($< n$) constraints, be

$$L(x,\lambda) = f(x) - \sum_{i=1}^{m} \lambda_i g_i(x)$$

Further, the necessary conditions

$$\frac{\partial L}{\partial x_j} = 0 \; and \; \frac{\partial L}{\partial \lambda_i} = 0 \; ; for \; all \; i \; and \; j$$

for an extreme point to be local optimum of $f(x)$ is also true for optimum of L(x, λ).

Let there exist points x and l that satisfy the equations

$\nabla L(x,\lambda) = \nabla f(x) - \sum_{i=1}^{m} \lambda_i g_i(x) = 0$ and $g_i(x) = 0$; i = 1, 2, …, m

Then the sufficient condition for an extreme point x to be a local minimum (or local maximum) of f(x) subject to the constraints $g_i(x) = 0$ (i = 1, 2, ..., m) is that the determinant of the matrix (also called Bordered Hessian *matrix*)

$$D = \begin{bmatrix} Q & H \\ H^T & 0 \end{bmatrix}_{(m+n)\times(m+n)}$$

is positive (or negative), where

$$Q = \left[\frac{\partial^2 L(x\,,\lambda)}{\partial x_i \partial x_j}\right]_{n\text{Ÿ}n} ; H = \left[\frac{\partial g_i(x)}{\partial x_j}\right]_{m\times n}$$

Conditions for maxima and minima: The sufficient condition for the maxima and minima is determined by the signs of the last (n – m) principal minors of matrix *D*. i.e.,

(1) If starting with principal minor of order (*m* + 1), the extreme point gives the maximum value of the objective function when signs of last (*n* – *m*) principal minors alternate, starting with $(-1)^{m+n}$ sign.

(2) If starting with principal minor of order (2*m* + 1), the extreme point gives the maximum value of the objective function when all signs of last (*n* – *m*) principal minors are the same and are of $(-1)^m$ type.

Example 2.11 Solve the following problem by using tge method of Lagrangian multipliers. Minimize Z = $x_1^2 + x_2^2 + x_3^2$ subject to the constraints (i) $x_1 + x_2 + 3x_3 = 2$ (ii) $5x_1 + 2x_2 + x_3 = 5$ and $x_1, x_2 \geq 0$.

Solution: The Lagrangian function is

$$L(x\,,\lambda) = x_1^2 + x_2^2 + x_3^2 - \lambda_1(x_1 + x_2 + 3x_3 - 2) - \lambda_2(5x_1 + 2x_2 + x_3 - 5)$$

The necessary conditions for the minimum of Z give us the following:

$$\frac{\partial L}{\partial x_1} = 2x_1 - \lambda_1 - 5\lambda_2 = 0\,; \frac{\partial L}{\partial x_2} = 2x_2 - \lambda_1 - 2\lambda_2 = 0\,; \frac{\partial L}{\partial x_3} = 2x_3 - 3\lambda_1 - \lambda_2 = 0$$

$$\frac{\partial L}{\partial \lambda_1} = -(x_1 + x_1 + 3x_3 - 2) = 0\,; \frac{\partial L}{\partial \lambda_2} = -(5x_1 + 2x_1 + x_3 - 5) = 0$$

The solution of these simultaneous equations gives:

$$x = (x_1\,, x_2\,, x_3) = \left(\frac{37}{46}\,, \frac{16}{46}\,, \frac{13}{46}\right)\,;\ \lambda = (\lambda_1\,, \lambda_2) = \left(\frac{2}{23}\,, \frac{7}{23}\right)\ and\ Z = \frac{193}{250}$$

To see that this solution corresponds to the minimum of Z, apply the sufficient condition with the help of a matrix:

$$D = \begin{bmatrix} 2 & 0 & 0 & 1 & 5 \\ 0 & 2 & 0 & 2 & 2 \\ 0 & 0 & 2 & 3 & 1 \\ 1 & 1 & 3 & 0 & 0 \\ 5 & 2 & 1 & 0 & 0 \end{bmatrix}$$

Since m = 2, n = 3, so n – m = 1 and 2m + 1 = 5, only one minor of D of order 5 needs to be evaluted and it must have a positive sign; $(–1)^m = (–1)^2 = 1$. Since |D| = 460 > 0, the extreme point, x = (x_1, x_2, x_3) corresponds to the minimum of Z.

Necessary and sufficient conditions when concavity (convexity) of objective function is not known, with single equality constraint:

Let us consider the non – linear programming problem that involves n decision variables and a single constraint: Optimize Z = g(x) subject to the constraint g(x) = h(x) – b = 0 ; x = $(x_1 , x_2, \ldots, x_n)^T \geq 0$

Multiply each constraint by Lagrange multiplier λ and sibtract it from the objective function. The new unconstrained objective function (Lagrange function) becomes: L(x, λ) = f(x) – λ g(x)

The necessary conditions for an extreme point to be an optimum (max or min) point are:

$$\frac{\partial L}{\partial x_j} = \frac{\partial f}{\partial x_j} - \lambda \frac{\partial g}{\partial x_j} = 0\,;j = 1, 2, \ldots, n$$

$$\frac{\partial L}{\partial \lambda} = -g(x) = 0$$

From the first condition we obtain the value of λ as: $\lambda = \dfrac{\frac{\partial f}{\partial x_j}}{\frac{\partial g}{\partial x_j}}\,; j = 1, 2, \ldots, n$

The sufficient conditions for determining whether theoptimal solution, so obtained, is either maximum or minimum, need computation of the value of (n – 1) principal minors, of the determinant, for each extreme point, as follows:

$$\Delta_{n+1} = \begin{vmatrix} 0 & \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} & \cdots & \frac{\partial g}{\partial x_n} \\ \frac{\partial g}{\partial x_1} & \frac{\partial^2 f}{\partial x_1^2} - \lambda \frac{\partial^2 g}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} - \lambda \frac{\partial^2 g}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} - \lambda \frac{\partial^2 g}{\partial x_1 \partial x_n} \\ \frac{\partial g}{\partial x_2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} - \lambda \frac{\partial^2 g}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} - \lambda \frac{\partial^2 g}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} - \lambda \frac{\partial^2 g}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{\partial g}{\partial x_n} & \frac{\partial^2 f}{\partial x_n \partial x_1} - \lambda \frac{\partial^2 g}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} - \lambda \frac{\partial^2 g}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} - \lambda \frac{\partial^2 g}{\partial x_n^2} \end{vmatrix}$$

If the signs of minors $\Delta_3, \Delta_4, \Delta_5$ are alternatively positive and negative, then the extreme point is a local maximum. But if signs of all minors $\Delta_3, \Delta_4, \Delta_5$ are negative, then the extreme point is a local minimum.

Example 2.12 Use the method of Lagrangian multipliers to solve the following NLP problem. Does thesolution maximize or minimize the objective function ?

Optimize Z = $2x_1^2 + x_2^2 + 3x_3^2 + 10x_1 + 8x_2 + 6x_3 - 100$ subject to the constraint $g(x) = x_1 + x_2 + x_3 = 20$ and $x_1, x_2, x_3 \geq 0$.

Solution: Lagrangian function can be formulated as:

$$L(x, \lambda) = 2x_1^2 + x_2^2 + 3x_3^2 + 10x_1 + 8x_2 + 6x_3 - 100 - \lambda(x_1 + x_2 + x_3 - 20) = 0$$

The necessary conditions for maximum or minimum are:

$$\frac{\partial L}{\partial x_1} = 4x_1 + 10 - \lambda = 0\,; \frac{\partial L}{\partial x_2} = 2x_2 + 8 - \lambda = 0\,; \frac{\partial L}{\partial x_1} = 4x_1 + 10 - \lambda = 0\,;$$

$$\frac{\partial L}{\partial \lambda} = -(x_1 + x_2 + x_3 - 20) = 0$$

Putting the value of x_1, x_2, x_3 in the last equation and solving for λ, we get λ = 30. Substituting the value of λ in the other three equations, we get an extreme point: $(x_1, x_2, x_3) = (5, 11, 14)$.

To prove the sufficient condition of whether the extreme point solution gives maximum or minimum value of the objective function we evalute (n – 1) principal minors as follows:

$$\Delta_3 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 4 & 0 \\ 1 & 0 & 2 \end{vmatrix} = -6$$

$$\Delta_4 = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 4 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 6 \end{vmatrix} = 48$$

Since the sign of Δ_3 and Δ_4 are alternative, therefore extreme point: (x_1, x_2, x_3) = (5, 11, 14) is a local maximum. At this point the value of objective function is Z = 281.

Interpretation of the Lagrange Multiplier

The value of Lagrange multiplier, which was introduced as an additional variable, can be used to provide valuable information about the sensitivity of an optimal value of the objective function to changes in resource levels (right – hand – side values of the constraints).

Let Us Sum Up

We have studied about constrained multi-variable optimization with equality constraints from the interpretation of the Lagrange multiplier.

Check Your Progress

Obtain the solution of the following problems by using the method of Lagrangian multipliers:

16. Min $Z = -2x_1^2 + 5x_1x_2 - 4x_1^2 + 18x_1$

    Subject to $x_1 + x_2 = 7$ and $x_1, x_2 \geq 0$.

17. Min $Z = 3x_1^2 + x_2^2 + x_3^2$

    Subject to $x_1 + x_2 + x_3 = 2$ and $x_1, x_2, x_3 \geq 0$.

18. Max $Z = 4x_1 + 6x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2$

    Subject to $x_1 + 2x_2 = 2$ and $x_1, x_2 \geq 0$.

19. Max $Z = 7x_1 - 0.3x_1^2 + 8x_2 - 0.4x_2^2$

    Subject to $4x_1 + 5x_2 = 100$ and $x_1, x_2 \geq 0$.

20. Find the dimension of the rectungular parallelepiped with the largest volume whose sides are parallel to the cordinates planes, to be inscribed in the elipsoid

$$g(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$$

2.4 CONSTRAINED MULTIVARIABLE OPTIMIZATION WITH INEQUALITY CONSTRAINTS

In this section the necessary and sufficient conditions for a local optimum of the general non – linear programming problem, with both equality and inequality constraints will be derived. The Kuhn – Tucker conditions (necessary as well as sufficient) will be used to derive optimality conditions. Consider the following general non – linear LP problem:

2.4.1 Kuhn – Tucker Necessary Condition

Optimize Z = f(x) subject to the constraints $g_i(x) \leq 0 for\ i = 1, 2, \ldots, m\ where\ x = (x_1, x_2, \ldots, x_n)^T\ and\ g_i(x) = h_i(x) - b_i$.

Add non – negative slack variables s_i (i = 1, 2, …, m) in each of the constraints to convert them to equality constraints. The problem can then be related as:

Optimize Z = g (x) subject to the constraints $g_i(x) + s_i^2 = 0$, i = 1, 2, . . ., m

The s_i^2 has only been added to ensure non-negative value (feasibility requirement) of s_i and to avoid adding $s_i \geq 0$ as an additional side constraint.

The new problem is the constrained multivariable optimization problem with equality constraints with

n + m variables. Thus, it can be solved by using the Lagrangian multiplier method.

For this, let us form the Lagrangian function as:

$$L(x, s, \lambda) = f(x) - \sum_{i=1}^{m} \lambda_i [g_i(x) + s_i^2]$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)^T$ is the vector of Lagrange multiplier.

The necessary conditions for an extreme point to be local optimum (max or min) can be obtained by: solving the following equations:

$$\frac{\partial L}{\partial x_j} = \frac{\partial f(x)}{\partial x_j} - \sum_{i=1}^{m} \lambda_i \frac{\partial g_i(x)}{\partial x_j} = 0, \quad j = 1, 2, \ldots, n$$

$$\frac{\partial L}{\partial \lambda_i} = -[g_i(x) + s_i^2] = 0, \quad i = 1, 2, \ldots, m$$

$$\frac{\partial L}{\partial s_i} = -2s_i \lambda_i = 0, \quad i = 1, 2, \ldots, m$$

The equation $\frac{\partial L}{\partial \lambda_i} = 0$ gives us back the original set of constraints: $g_i(x) + s_i^2 = 0$. If a constraint is satisfied with equality sign, $g_i(x) = 0$ at the optimum point x, then it is called an active (binding or light) constraint, otherwise it is known as an inactive (slack) constraint.

The equation $\frac{\partial L}{\partial s_i}$= 0, provides us the set of rules for finding the unconstrained optimum. The condition $\lambda_i s_i$= 0 implies that either λ_i = 0 (or) s_i = 0. If s_i = 0 and λ_i > 0, then equation $\frac{\partial L}{\partial \lambda_i} = 0$ gives $g_i(x) = 0$. This means either $\lambda_i = 0$ (or) $g_i(x) = 0$, and therefore we may also write $\lambda_i g_i(x) = 0$.

Since s_i^2 has been taken to be a non-negative (≥ 0) slack variable, therefore $g_i(x)$ ≥ 0. Hence, the equation $\lambda_i g_i(x) = 0$ implies that when $g_i(x)$ < 0, λ_i = 0 and when $g_i(x)$ = 0, λ_i> 0. However λ_i is unrestricted in sign corresponding to $g_i(x)$ = 0.

But if λ_i = 0 and s_i^2 > 0, then the ith constraint is inactive (i.e., this constraint will not change the optimum value of Z* because I = $\frac{\partial Z}{\partial b_i}$ = 0) and hence can be discarded.

Thus the Kuhn-Tucker necessary conditions (when active constraints are known) to be satisfied at a local optimum (max or min) point can be stated as follows:

$$\frac{\partial f}{\partial x_i} - \sum_{i=1}^{m} \lambda_i \frac{\partial g_i}{\partial x_j} = 0\,;j = 1,2,\dots,n$$

$$\lambda_i g_i(x) = 0$$

$$g_i(x) \le 0\,, \lambda_i \ge 0\,; i = 1,2,\dots,m$$

Remark: If the given problem is of minimization or if the constraints are of the form $g_i(x) \ge 0$, then $\lambda_i \le 0$. On the other hand if the problem is of maximization with constraints of the form $g_i(x) \le 0$, then $\lambda_i \ge 0$.

2.4.2 Kuhn – Tucker Sufficient Condition

Theorem 2.5: The Kuhn – Tucker necessary conditions for the problem Maximize Z = f(x) subject to the constraints $g_i(x) \le 0\ for\ i = 1,2,\dots,m$ is also the sufficient condition if f(x) is concave and all $g_i(x)$ are convex functions of x.

Proof: The Lagrangian Function of the problem Maximize Z = f(x) subject to the constraints $g_i(x) \le 0\,; i = 1,2,\dots,m$

can be written as $L(x\,,s\,,\lambda) = f(x) - \sum_{i=1}^{m} \lambda_i[g_i(x) + s_i^2]$

If λ_i ≥ 0, then $\lambda_i g_i(x)$ is convex and – $\lambda_i g_i(x)$ is concave. Further, since $\lambda_i s_i$ = 0, we get $g_i(x) + s_i^2 = 0$. Thus, it follows that $L(x\,,s\,,\lambda)$ is a concave function. We have derived that a necessary condition for f(x) to be a relative maximum at an extreme point is that $L(x\,,s\,,\lambda)$ also have the same extreme point. However, if $L(x\,,s\,,\lambda)$ is concave, its first derivative must be zero only at one point, and obviously this point must be an absolute maximum for f(x).

Example 2.13 Find the optimum value of the objective function when separately subject to the following three sets of constraints:

Maximize Z = $10x_1 - x_1^2 + 10x_2 - x_2^2$ subject to the constraints

(a) $x_1 + x_2 \leq 14\,, -x_1 + x_2 \leq 6 \;and\; x_1\,, x_2 \geq 0$

(b) $x_1 + x_2 \leq 8\,, -x_1 + x_2 \leq 5 \;and\; x_1\,, x_2 \geq 0$

(c) $x_1 + x_2 \leq 9\,, x_1 - x_2 \geq 6 \;and\; x_1\,, x_2 \geq 0$

Solution: (a) Here the constraints are:

$$g_1(x) = x_1 + x_2 + s_1^2 - 14 = 0\,;\; g_2(x) = -x_1 + x_2 + s_2^2 - 6 = 0$$

The Lagrangian function is formulated as:

$$L(x,s,\lambda) = (10x_1 - x_1^2 + 10x_2 - x_2^2) - \lambda_1(x_1 + x_2 + s_1^2 - 14) - \lambda_2(-x_1 + x_2 + s_2^2 - 6)$$

The Kuhn – Tucker necessary conditions for a maximization problem are:

$$\frac{\partial L}{\partial x_1} = 10 - 2x_1 - \lambda_1 + \lambda_2 = 0\,;\; \frac{\partial L}{\partial x_2} = 10 - 2x_2 - \lambda_1 - \lambda_2 = 0$$

$$\frac{\partial L}{\partial \lambda_1} = -(x_1 + x_2 + s_1^2 - 14) = 0\,;\; \frac{\partial L}{\partial \lambda_1} = -(-x_1 + x_2 + s_2^2 - 6) = 0$$

$$\frac{\partial L}{\partial s_1} = -2\lambda_1 s_1 = 0\,;\; \frac{\partial L}{\partial s_2} = -2\lambda_2 s_2 = 0$$

The unconstrained solution (i.e., $\lambda_1 = \lambda_1 = 0$) obtained by solving the first four equations is: $x_1 = 5, x_2 = 5, s_1^2 = 4, s_2^2 = 6$ and Max Z = 50

Since both s_1^2 and s_2^2 are positive, the solution is feasible. As the solution, so obtained, is unconstrained, therefore in order to find whether or not the solution is maximum, we test the Hessian matrix for the given objective function as:

$$H = \begin{bmatrix} \frac{\partial^2 Z}{\partial x_1^2} & \frac{\partial^2 Z}{\partial x_1 \partial x_2} \\ \frac{\partial^2 Z}{\partial x_2 \partial x_1} & \frac{\partial^2 Z}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \text{ and } \det A_1 = \left|\frac{\partial^2 Z}{\partial x_1^2}\right| = -2\,; \det A_2 = |H| = 4$$

Since signs of the principal minors of H are alternating, matrix H is negative definite and the point x = (4, 4) gives the local maximum of the objective function Z.

(b) Here the constraints are:

$$g_1(x) = x_1 + x_2 + s_1^2 - 8 = 0\,;\; g_2(x) = -x_1 + x_2 + s_2^2 - 5 = 0$$

The Lagrangian function is formulated as:

$$L(x, s, \lambda) = (10x_1 - x_1^2 + 10x_2 - x_2^2) - \lambda_1(x_1 + x_2 + s_1^2 - 8) - \lambda_2(-x_1 + x_2 + s_2^2 - 5)$$

The Kuhn – Tucker necessary conditions for a maximization problem are:

$$\frac{\partial L}{\partial x_1} = 10 - 2x_1 - \lambda_1 + \lambda_2 = 0\,; \frac{\partial L}{\partial x_2} = 10 - 2x_2 - \lambda_1 - \lambda_2 = 0$$

$$\frac{\partial L}{\partial \lambda_1} = -(x_1 + x_2 + s_1^2 - 8) = 0\,; \frac{\partial L}{\partial \lambda_1} = -(-x_1 + x_2 + s_2^2 - 5) = 0$$

$$\frac{\partial L}{\partial s_1} = -2\lambda_1 s_1 = 0\,; \frac{\partial L}{\partial s_2} = -2\lambda_2 s_2 = 0$$

The unconstrained solution (i.e., $\lambda_1 = \lambda_1 = 0$) obtained by solving the first four equations is:

$x_1 = 5, x_2 = 5, s_1^2 = -2, s_2^2 = 5$ and Max Z = 50.

Since $s_1^2 = -2$, the solution is infeasible. By agaon solving these equations for $s_1 = \lambda_2 = 0$ (violated first constraint), we get $x_1 = 4, x_2 = 4, s_2^2 = 5, \lambda_1 = 2$, Max Z = 48. This solution satisfies both the constraints and conditions $\lambda_1 s_1 = \lambda_2 s_2 = 0$ are also satisfied. Therefore the point x = (4, 4) gives the maximum of objective function Z.

(c) Here the constraints are:

$$g_1(x) = x_1 + x_2 + s_1^2 - 9 = 0\,;\; g_2(x) = -x_1 + x_2 + s_2^2 + 6 = 0$$

The Lagrangian function is formulated as:

$$L(x, s, \lambda) = (10x_1 - x_1^2 + 10x_2 - x_2^2) - \lambda_1(x_1 + x_2 + s_1^2 - 9) - \lambda_2(-x_1 + x_2 + s_2^2 - 6)$$

The Kuhn – Tucker necessary conditions for a maximization problem are:

$$\frac{\partial L}{\partial x_1} = 10 - 2x_1 - \lambda_1 + \lambda_2 = 0\,; \frac{\partial L}{\partial x_2} = 10 - 2x_2 - \lambda_1 - \lambda_2 = 0$$

$$\frac{\partial L}{\partial \lambda_1} = -(x_1 + x_2 + s_1^2 - 9) = 0\,; \frac{\partial L}{\partial \lambda_1} = -(-x_1 + x_2 + s_2^2 - 6) = 0$$

$$\frac{\partial L}{\partial s_1} = -2\lambda_1 s_1 = 0\,; \frac{\partial L}{\partial s_2} = -2\lambda_2 s_2 = 0$$

The unconstrained solution (i.e., $\lambda_1 = \lambda_1 = 0$) obtained by solving the first four equations is:

$x_1 = 8, x_2 = 2, s_1^2 = -1, s_2^2 = -6$ and Max Z = 50.

Since both s_1^2, s_1^2 are negative, the solution is infeasible. By again solving these equations for

$s_2 = \lambda_1 = 0$ (violated second constraint), we get

$x_1 = 2, x_2 = 8, s_1^2 = -1, \lambda_2 = 6$, Max Z = 32.

This solution is also infeasible as s_1^2 is negative. By again solving these equations for $s_1 = s_2 = 0$ (i.e., $\lambda_1 = \lambda_1 \neq 0$) we get: $x_1 = 7.5, x_2 = 1.5, \lambda_1 = 1, \lambda_2 = 6$ and Max Z = 31.50.

Since this solution does not violate any of the conditions, therefore the point

x = (7.5, 1.5) gives the maximum of objective function Z.

Example 2.14 Determine x_1 and x_2 so as to

Maximize Z = $12x_1 + 21x_2 + 2x_1x_2 - 2x_1^2 - 2x_2^2$ subject to the constraints

(i) $x_2 \leq 8$ (ii) $x_1 + x_2 \leq 10$ and $x_1, x_2 \geq 0$

Solution: Here $f(x_1, x_2) = 12x_1 + 21x_2 + 2x_1x_2 - 2x_1^2 - 2x_2^2$

$$g_1(x_1, x_2) = x_2 - 8 \leq 0\,;\; g_1(x_1, x_2) = x_1 + x_2 - 10 \leq 0$$

The Lagrangian function is formulated as:

$$L(x, s, \lambda) = f(x) - \lambda_1[g_1(x) + s_1^2] - \lambda_2[g_2(x) + s_2^2]$$

The Kuhn – Tucker necessary conditions for a maximization problem are:

(i) $\frac{\partial f}{\partial x_j} - \sum_{i=1}^{2} \lambda_i \frac{\partial g_i}{\partial x_j}\,; j = 1, 2$ (ii) $\lambda_i g_i(x) = 0\,; i = 1, 2$

$12 + 2x_2 - 4x_1 - \lambda_2 = 0$ $\lambda_1(x_2 - 8) = 0$

$21 + 2x_1 - 4x_2 - \lambda_1 - \lambda_2 = 0$ $\lambda_2(x_1 + x_2 - 10) = 0$

(iii) $g_i(x) \leq 0$ (iv) $\lambda_i \geq 0\,; i = 1, 2$

$x_2 - 8 \leq 0$

$x_1 + x_2 - 10 \leq 0$

There may arise four cases:

Case 1: If $\lambda_1 = 0, \lambda_2 = 0$, then from condition (i), we have:

$$12 + 2x_2 - 4x_1 = 0 \; and \; 21 + 2x_1 - 4x_2 = 0$$

Solving these equations, we get $x_1 = \frac{15}{2}, x_2 = 9$. However, this solution violates condition (iii) and therefore it should be discarded.

Case 2: If $\lambda_1 \neq 0, \lambda_2 \neq 0$, then from condition (ii), we have:

$$x_2 - 8 = 0 \rightarrow\rightarrow x_2 = 8$$

$$x_1 + x_2 - 10 = 0 \rightarrow\rightarrow x_1 = 2$$

Substituting these values in condition (i), we get $\lambda_1 = -27$ and $\lambda_2 = 20$. However, the solution violates the condition (iv) and therefore it should be discarded.

Case 3: If $\lambda_1 \neq 0, \lambda_2 = 0$, then from conditions (ii) and (i), we have:

$$x_1 + x_2 = 10$$
$$2x_2 - 4x_1 = -12$$
$$2x_1 - 4x_2 = 12 + \lambda_1$$

Solving these equations, we get $x_1 = 2, x_2 = 8$ and $\lambda_1 = -16$. However, this solution violated the condition (iv) and therefore it should be discarded.

Case 4: If $\lambda_1 = 0, \lambda_2 \neq 0$, then from conditions (i) and (ii), we have:

$$2x_2 - 4x_1 = -12 + \lambda_2$$
$$2x_1 - 4x_2 = 21 + \lambda_2$$
$$x_1 + x_2 = 10$$

Solving these equation, we get $x_1 = \frac{17}{4}, x_2 = \frac{23}{4}, \lambda_2 = \frac{13}{4}$. This solution does not violate any of the Kuhn – Tucker conditions and therefore must be accepted.

Hence the optimum solution of the given problem is $x_1 = \frac{17}{4}, x_2 = \frac{23}{4}, \lambda_1 = 0, \lambda_2 = \frac{13}{4}$ and Max Z = $\frac{1734}{16}$.

Let Us Sum Up

We have studied about constrained multi-variable optimization with inequality constraints from the Kuhn-tucker sufficient condition.

Check Your Progress

Use the Kuhn – Tucker condition to solve the following non – linear programming problems:

21.. Max $Z = 2x_1^2 + 12x_1x_2 - 7x_2^2$

Subject to $2x_1 + 5x_2 \leq 98$, and $x_1, x_2 \geq 0$.

22. Max $Z = -x_1^2 - x_2^2 - x_3^2 + 4x_1 + 6x_2$

Subject to (i) $x_1 + x_2 \leq 2, (ii) 2x_1 + 3x_2 \leq 12$ and $x_1, x_2 \geq 0$.

23. Max $Z = 8x_1 + 10x_2 - x_1^2 - x_2^2$

Subject to $3x_1 + 2x_2 \leq 6$, and $x_1, x_2 \geq 0$.

24. Max $Z = 7x_1^2 - 6x_1 + 5x_2^2$

Subject to (i) $x_1 + 2x_2 \leq 10, (ii) x_1 - 3x_2 \leq 9$ and $x_1, x_2 \geq 0$.

25. Define a convex programming problem. What is the Lagrangian function associated with it? Solve the non – linear programming problems:

Min $Z = -\log x_1 - \log x_2$

Subject to $x_1 + x_2 \leq 2$, and $x_1, x_2 \geq 0$.

2.5 NON-LINEAR PROGRAMMING METHODS

Linear programming is usefull for solving decision problems that involve linear relationship among decision variables. Any non-linear change in the input variable values either in objective function or constraints, restrict the use of usual simplex method to solve the decision problem. Hence, decision-makers use non-linear programming methods to solve such decision problems.

The Lagrange multiplier method to determine the optimum value of a function of two or more variables, subject to one inequality constraint, can be modified to optimize an objective function of two or more variables, subject to more than one inequality (or equality) constraint. In general, conditions necessary for an optimum value of a function subject to inequality constraints are known as Kuhn-Tucker conditions, as discussed in Chapter 23. For ready reference, the Kuhn-Tucker necessary conditions to achieve relative maximum for the LP problem

Maximize z = f(x) subject to the constraints $g_i(x) = 0$; I = 1, 2, …, m and x ≥ 0 for all i can be summarized as follows:

(i) $\frac{\partial f(x)}{\partial x_j} - \sum_{i=1}^{m} \lambda_i \frac{\partial g_i(x)}{\partial x_j} = 0$; j = 1, 2, …, n (ii) $\lambda_i g_i(x) = 0$; i = 1, 2, …, m

(iii) $g_i(x) \leq 0$ (iv) $\lambda_i \geq 0$

These conditions are also applicable to the minimization LP problems, with the exception that λ ≤ 0. The λ's are unrestricted in sign, corresponding to equality constraints in both the maximization and the minimization LP problems.

In deriving conditions (i) to (iv), the non-negativity conditions x ≥ 0 were not taken into consideration. Now if non-negativity conditions are also considered as one of the constraints, then Kuhn-Tucker conditions for the following maximization LP problem may to be derived:

Maximize Z = f (x) subject to the constraints $g_i(x) \leq 0$ and –x ≤ 0, i = 1, 2, …, m where x = [x_1, x_2, …, x_n].

In this LPP problem there are m + n inequality constraints. Introducing m + n squared slack variables s_i^2(i = 1, 2, …, m, m + 1, …, m + n) in the respective inequalities to convert them into the following equations:

$$g_i(x) + s_i^2 = 0 \text{ ; i = 1, 2, …, m}$$
$$-x_i + s_{m+j}^2 = 0 \text{ ; j = 1, 2, …, n}$$

The Kuhn-Tucker necessary conditions for the maximum of f (x) can be obtained as follows:

Step 1: Formulating the Lagrangian function as

$$L(x\,,s\,,\lambda) = f(x) - \sum_{i=1}^{m} \lambda_i[g_i(x) + s_i^2] - \sum_{j=1}^{n} \lambda_{m+j}[-x_j + s_{m+j}^2]$$

Step 2: Differentiate $\mathbf{L(x\,,s\,,\lambda)}$ partially with respect to x, s and $\boldsymbol{\lambda}$ and equate them with zero

$$\frac{\partial L}{\partial x_j} = \frac{\partial f(\mathbf{x})}{\partial x_j} - \sum_{i=1}^{m} \lambda_i \frac{\partial g_i(\mathbf{x})}{\partial x_j} + \lambda_{m+j} = 0, \quad j = 1, 2, \ldots, n$$

$$\frac{\partial L}{\partial s_i} = -2\lambda_i s_i, \quad i = 1, 2, \ldots, m$$

$$\frac{\partial L}{\partial s_{m+j}} = -2\lambda_{m+j}\, s_{m+j} = 0$$

$$\frac{\partial L}{\partial \lambda_i} = -\{g_i(\mathbf{x}) + s_i^2\} = 0$$

$$\frac{\partial L}{\partial \lambda_{m+j}} = -\{-x_j + s_{m+j}^2\} = 0$$

Step 3: Simplify these equations to get the following Kuhn – Tucker conditions

These Kuhn-Tucker necessary conditions also become sufficient conditions if f (x) is concave and gi (x) is convex in x. Similarly for the minimization case, f (x) and gi (x) must be convex and concave in x, respectively.

Example 2.15 An engineering company has received a rush order for a maximum number of two types of items that can be produced and transported during a two-week-period. The profit in thousand rupees on this order is related to the number of each type of item manufactured by the company and is given by $12x_1 + 10x_2 - x_1^2 - x_2^2 + 61$ where x_1 is the number of units (in thousands) of type I item and x_2 is the number of units (in thousands) of type II item.

Because of other commitments over the next two weeks, the company has only 60 hours available in its shifting and packing department. It is estimated that every thousand units of type I and type II items will require 20 hours and 30 hours, respectively, in the shifting and packing departments. Given the above information, how many units of each type of item should the company produce in order to maximize profit ?

Model Formulation: The mathematical model of the problem can be stated as follows:

Maximize *Z* = $12x_1 + 10x_2 - x_1^2 - x_2^2 + 61$ subject to the constraint $20x_1 + 30x_2 \leq 60$ and $x_1, x_2 \geq 0$

In this model, the objective function is non-linear while the constraint is linear. Thus, it is a non-linear programming problem.

Example 2.16 A company sells two types of items *A* and *B*. Item *A* sells for Rs. 25 per unit. No quantity discount is given. The sales revenue for item *B* decreases as the number of its units sold increases. It is given by: Sales revenue = (30 – 0.30 x_2^2) where x_2 is the number of units sold of item B.

The marketing department has only 1200 hours available for distributing these items in the next year. Further, the company estimates the sales time function is non-linear and is given by: Sales time = $x_1 + 0.2x_1^2 + 3x_2 + 0.35x_2^2$

The company can only procure 6000 units of item *A* and *B* for sales in the next year. Given the above information, how many units of item *A* and *B* should the company procure in order to maximize its total revenue?

Model Formulation: The mathematical model of the problem can be stated as follows:

Maximize *Z* = $25x_1 + 30x_2 - 0.30x_2^2$ subject to the constraints

(i) $x_1 + 0.2x_1^2 + 3x_2 + 0.35x_2^2 \leq 1200$

(ii) $x_1 + x_2 \leq 6000$ and $x_1, x_2 \geq 0$

In this model, the objective function and one of the constraints is non-linear. Therefore, this is a non-linear programming problem.

2.6 THE GENERAL NON-LINEAR PROGRAMMING PROBLEM

The general non-linear programming problem can be stated in the following form:

Optimize (max or min) $Z = f(x_1, x_2, \ldots, x_n)$ subject to the constraints

$g_i(x_1, x_2, \ldots, x_n)\{\le, =, \ge\}\ b_i\ ; i = 1, 2, \ldots, m$ and $x_i \ge 0$ for all j = 1, 2, …, n

where $f(x_1, x_2, \ldots, x_n)$ and $g_i(x_1, x_2, \ldots, x_n)$ are real valued function of *n* decision variables, and at least one of these is non-linear.

Several methods have been developed for solving non-linear programming problems. In this chapter we will discuss the methods for solving quadratic programming problems, separable programming problems, geometric programming problems and stochastic programming problems.

2.7 GRAPHICAL SOLUTION METHOD

As we know, the optimal solution of an LP problem is obtained at one of the extreme points of the feasible solution space. However, in case of non-linear programming, the optimal solution may not be obtained at the extreme point of its feasible region. This is illustrated through Examples 2.15 and 2.16.

Example 2.17 Solve graphically the following NLP problem:

Maximize $Z = 2x_1 + 3x_2$ subject to the constraints

(i) $x_1^2 + x_2^2 \le 20$ (ii) $x_1 . x_2 \le 8$ and $x_1, x_2 \ge 0$

Solution In the given NLP problem, the objective function is linear, and constraints are non-linear. Plot the given constraints on the graph by the usual method, as shown in Fig. 2.3.

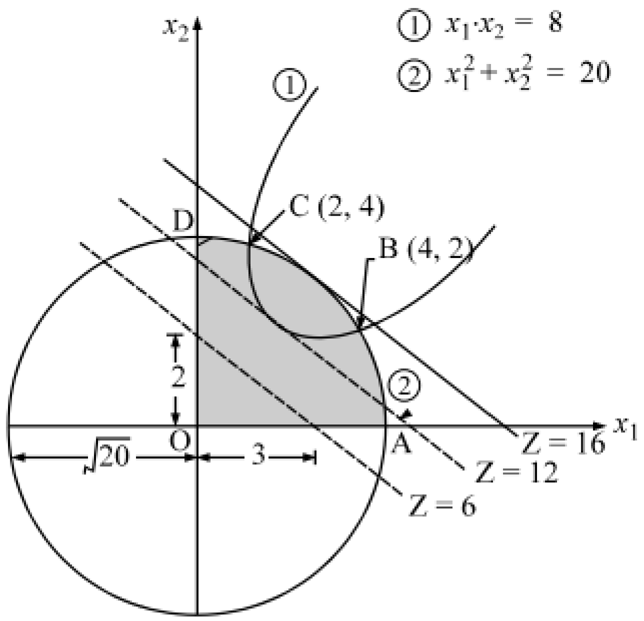


Figure 2.3: Graphical Solution

The constraint $x_1^2 + x_2^2 = 20$ represents a xircle whoes radius and centre are: a = $\sqrt{20}$; (h, k) = (0, 0) respectively and $x_1.x_2 = 8$ represents a rectangular hyperbola whoes asymptotes are represented by the x – axis and y – axis.

Solving the two equations: $x_1^2 + x_2^2 = 20$ and $x_1.x_2 = 8$, we get (x_1, x_2) = (4, 2) and (x_1, x_2) = (2, 4). These solution points, which also satisfy both the constraints, may be obtained within the shaded non – convex region OABCD, also called the feasible region.

Now we need to find a point (x_1, x_2) within the convex region OABCD where the value of the given objective function $Z = 2x_1 + 3x_2$ is maximum. Such a point can be located by the iso-profit function approach. i.e., draw parallel objective function $2x_1 + 3x_2 = k$ lines for different constant values of *k*, and stop the process when a line touches the extreme boundary point of the feasible region for some value of *k*. Starting with *k* = 6 and so on we find that the iso-profit line with *k* = 16 touches the extreme boundary point *C* (2, 4) where the value of *Z* is maximum. Hence the optimal solution is: x_1 = 2, x_2 = 4 and Max *Z* = 16.

Example 2.18 Graphically solve the following NLP problem:

Maximize Z = $8x_1 - x_1^2 + 8x_2 - x_2^2$ subject to the constraints

(i) $x_1 + x_2 \leq 12$ (ii) $x_1 - x_2 \geq 4$ and $x_1, x_2 \geq 0$

Solution: In this NLP problem, the objective function is non-linear whereas the constraints are linear. Plot the given constraints on the graph by the usual method, as shown in the Fig. 2.4.

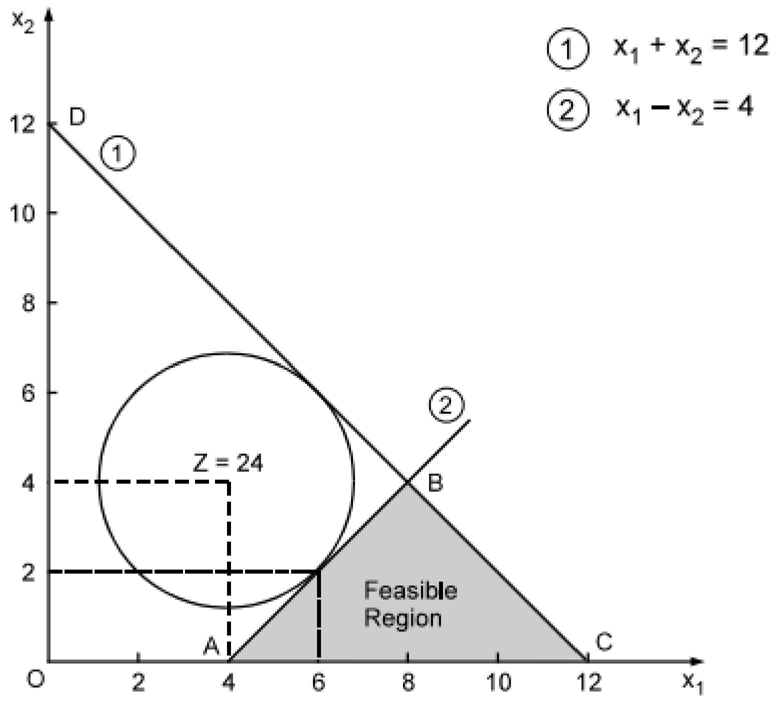


Figure 2.4: Graphical Solution

The feasible region is shown by the shaded region in Fig. 2.4. Thus, in the feasible region the optimal point (x_1, x_2) must be that at which a side of the convex region is tangent to the circle,

$$Z = 8x_1 - x_1^2 + 8x_2 - x_2^2.$$

The gradient of the tangent to the circle can be obtained by differentiating the equation of the circle, Z with respect to x_1 as follows:

$$Z = 8x_1 - x_1^2 + 8x_2 - x_2^2$$

$$8 - 2x_1 + 8\frac{dx_2}{dx_1} - 2x_2\frac{dx_2}{dx_1} = 0 \text{ (or) } \frac{dx_2}{dx_1} = \frac{2x_1 - 8}{8 - 2x_2} \quad \rightarrow (22)$$

The gradient of the line $x_1 + x_2 = 12$ *and* $x_1 - x_2 = 4$ is:

$$\{dx_1 + dx_2 = 0 \text{ (or) } \frac{dx_2}{dx_1} = -1 \text{ and } dx_1 - dx_2 = 0 \text{ (or) } \frac{dx_2}{dx_1} = 0\} \quad \rightarrow (23)$$

respectively.

If the line $x_1 + x_2 = 12$ is the tangent to the circle, then substituting $\frac{dx_2}{dx_1} = -1$ from (23) in (22), we get: $x_1 = x_2$ and hence for $x_1 + x_2 = 12$ gives $(x_1, x_2) = (6, 6)$. This means the tangent of the line

$x_1 + x_2 = 12$ at $(6, 6)$. But this does not satisfy all the constraints.

Similarly, if the line $x_1 - x_2 = 4$ is the tangent to the circle, then substituting $\frac{dx_2}{dx_1} = 1$ from (23) in (21), we get: $x_1 + x_2 = 8$, and hence for $x_1 + x_2 = 8$, the equation $x_1 - x_2 = 4$ gives $(x_1, x_2) = (6, 2)$. This means the tangent of the circle to the line $x_1 - x_2 = 4$ is at (6, 2). This point lies in the feasible region and also satisfies both the constraints. Hence, the optimal solution is: $x_1 = 6$, $x_2 = 2$ and

Max Z = 24.

Example 2.19 Solve graphically the following NLP problem:

Minimixe Z = $x_1^2 + x_2^2$ subject to the constraints

(i) $x_1 + x_2 \geq 8$ (ii) $x_1 + 2x_2 \geq 10$ (iii) $2x_1 + x_2 \geq 10$ and $x_1, x_2 \geq 0$

Solution: Since in the given NLP problem all constraints are linear, plotting them on the graphs as usual. The shaded solution space bounded by convex region ABCD is shown fig. 2.4. The objective function is non – linear and represents a circle. If r is the radius of the circle, Z = (r) = $x_1^2 + x_2^2$. Then the objective is to determine the minimum value of r, so that the circle woth center (0, 0) and radius, r just touches the solution space. As shown in fig. 2.4, the solution point (4, 4) lies on the line $x_1 + x_2 = 8$, and the line is tangent to the circle at this point.

Since the circle touches one of the sides of the convex region, one of the side the convex solution space would be tangent to the circle. Thus the solution can also be obtained by differentiating the equation: Z = $x_1^2 + x_2^2$ with respect to x_1, i.e.,

$2x_1 dx_1 + 2x_2 dx_2 = 0$ (or) $\frac{dx_1}{dx_2} = -\frac{x_1}{x_2}$.

Differentiate the constraint equations which form the sides of the convex space as follows: $dx_1 + dx_2 = 0$ (or) $\frac{dx_2}{dx_1} = -1$,

$dx_1 + dx_2 = 0$ (or) $\frac{dx_2}{dx_1} = -\frac{1}{2}$, and

$dx_1 + dx_2 = 0$ (or) $\frac{dx_2}{dx_1} = -2$.

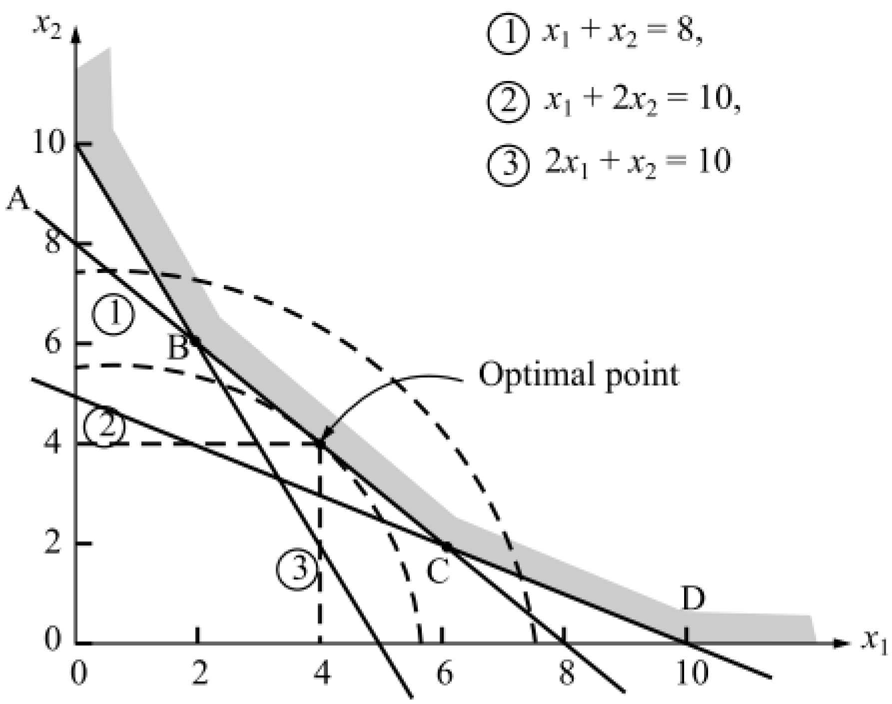


Figure 2.5: Graphical solution

Three alternative solutions which can now be obtained are:

(i) Taking equations first and second and the constraint $x_1 + x_2 = 8$, we have $\frac{dx_2}{dx_1} = -\frac{x_1}{x_2} = -1$, which gives $x_1 = x_2 = 4$. This solution satisfies all the constraint equations and so is feasible.

(ii) Taking equations first and third and the constraint $x_1 + 2x_2 = 10$, we have $\frac{dx_2}{dx_1} = -\frac{x_1}{x_2} = -\frac{1}{2}$, which gives $x_1 = 2$ and $x_2 = 4$. This solution does not satisfy all the constraints and is discarded.

(iii) Taking equations first and third and the constraint $2x_1 + x_2 = 10$, we have $\frac{dx_2}{dx_1} = -\frac{x_1}{x_2} = -2$, which gives $x_1 = 4$ and $x_2 = 2$. This solution, does not satisfy all the constraints and is discarded.

Hence optimal solution is: $x_1 = 4$, $x_2 = 4$, and Min Z = 32.

Example 2.20 Solve graphically the followings NLP problem:

Maximize Z = $x_1 + 2x_2$, subject to the constraints

(i) $x_1x_2 - 2x_2 \geq 3$ (ii) $3x_1 + 2x_2 \leq 24$ and $x_1, x_2 \geq 0$.

Solution: In this NLP problem the objective function is linear, while one of the constraints is non-linear. Plotting the constraint: $x_1x_2 - 2x_2 \geq 3$ on the graph assuming that it is an equation: $x_1x_2 - 2x_2 = 3$. Thus, for $x_2 \geq 0$, the value of x_1 cannot be less than 2.

For different values of x_1, the corresponding values of x_2 which satisfy the equation: $x_1x_2 - 2x_2 = 3$ are given below:

x_1	2.1	2.2	2.4	2.6	3.0	3.5	4.0	5.0	6.0	7.0	8.0	12.0
x_2	30	15	7.5	5	3	2	1.5	1	0.75	0.6	0.5	0.3

When these points are plotted as usual, the graph of the line $x_1x_2 - 2x_2 = 3$, is shown in Fig. 2.6. Also plotting the constraint, $3x_1 + 2x_2 = 24$, on the graph. The different values of x_2 and x_2 are, x_1 = 0, x_2 = 12 and x_1 = 8, x_2 = 0. The solution space bounded by two lines is shown by shaded area in Fig. 2.6.

The objective function line, Z = $x_1 + x_2$ inclined at 45°, when moved away from the origin. The farthest point through which it passes gives optimal solution: x_2 = 8.45, x_1 = 2.35 and Max Z = 10.81

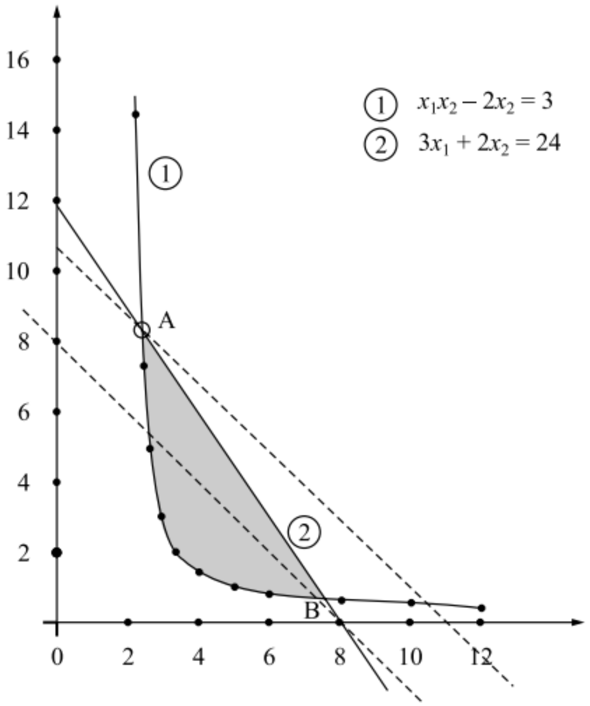


Figure 2.6: Graphical solution

Let us sum up

We have learned about general non-linear programming problem and to solve it by using graphical method.

Check Your Progress

26. Solve the following non – linear Programmimg problems graphically.

(a) Max $Z = x_1$

Subject to $(1 - x_1)^2 - x_2 \geq 0$

And $x_1, x_2 \geq 0$.

(b) Max $Z = x_1$

Subject to $(3 - x_1)^3 - (x_2 - 2) \geq 0$

$(3 - x_1)^2 - (x_2 - 2) \geq 0$

And $x_1, x_2 \geq 0$.

Also show that the Khun – Tucker necessary condition for maxima do not hold. What do you conclude?

27.. (a) Min $Z = x_1^2 + x_2^2$

Subject to $x_1 + x_2 \geq 4$

$2x_1 + x_2 \geq 5$

And $x_1, x_2 \geq 0$.

(b) Min $Z = (x_1 - 1)^2 + (x_2 - 2)^2$

Subject to $x_1 \leq 2$

$x_2 \leq 1$

And $x_1, x_2 \geq 0.$

28. (a) Max $Z = 100x_1 - x_1^2 + 100x_2 - x_2$

Subject to $x_1 + x_2 \geq 80$

$x_1 + 2x_2 \leq 100$

And $x_1, x_2 \geq 0.$

(b) Min $Z = (x_1 - 2)^2 + (x_2 - 1)^2$

Subject to $-x_1 + x_2 \geq 0$

$x_1 + x_2 \leq 2$

And $x_1, x_2 \geq 0.$

29.. A company manufactures two products, *A* and *B*. It takes 30 minutes to process one unit of product *A* and 15 minutes to process each unit of *B*. The maximum machine time available is 35 hours per week. Products *A* and *B* require 2 kg and 3 kg of raw material per unit, respectively. The available quantity of raw material is envisaged to be 180 kg per week.

The products *A* and *B*, which have unlimited market potential, sell for Rs 200 and Rs 500 per unit, respectively. If the manufacturing costs for products *A* and *B* are $2x^2$ and $3y^2$, respectively, find how much of each product should be produced per week, where x and y represent the quantity of products *A* and *B* to be produced, respectively.

2.8 QUADRATIC PROGRAMMING

Among several non-linear programming methods available for solving NLP problems, we shall discuss in this section, an NLP problem with non-linear objective function and linear constraints. Such an NLP problem is called quadratic programming problem. The general mathematical model of quadratic programming problem is as follows:

Optimize (Max or Min) Z = $\left\{\sum_{j=1}^{n} c_j x_j + \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} x_j d_{jk} x_k\right\}$ subject to the constraints $\sum_{j=1}^{n} a_{ij}x_j \le b_i$ and $x_j \ge 0$ for all i and j.

In matrix notations, QP problem is written as:

Optimize (Max or Min) Z = cx + $\frac{1}{2}$ x^T Dx subject to the constraints Ax ≤ b and x ≥ 0

where x = $(x_1, x_2, \dots, x_n)^T$; c = $(x_1, x_2, \dots, x_n)$; b = $(x_1, x_2, \dots, x_n)^T$

D = $\left[d_{jk}\right]$ is an n x n symmetric matrix ; A = $\left[a_{ij}\right]$ is an m x n matrix.

If the objective function in QP problem is of minimization, then matrix D is symmetric and positive definite (i.e., the quadratic term x^T Dx in x is positive for all values of x except at x = 0) and objective function is strictly convex in x. But, if the objective function is of maximization, then matrix D is symmetric and negative-definite (i.e., x^T Dx < 0 for all values of x except for x = 0) and objective function is strictly concave in x. If matrix, D is null, then the QP problem reduces to the standard LP problem.

2.8.1 Kuhn – Tucker conditions

The necessary and sufficient Kuhn-Tucker conditions to get an optimal solution to the maximization QP problem subject to linear constraints can be derived as follows:

Step 1: Introducing slack variables s_i^2 and r_j^2 to constraints, the QP problem becomes:

Max f(x) = $\sum_{j=1}^{n} c_j x_j - \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} x_j d_{jk} x_k$ subject to the constraints

$$\sum_{j=1}^{n} a_{ij}x_j + s_i^2 = b_i \; ; i = 1, 2, \dots, m$$

$$-x_j + r_j^2 = 0 \; ; j = 1, 2, \dots, n$$

Step 2: Forming the Lagrange function as follows:

$$L(x, s, r, \lambda, \mu) = f(x) - \sum_{j=1}^{n} \lambda_i\{a_{ij}x_j + s_i^2 - b_i\} - \sum_{j=1}^{n} \mu_j\{-x_j + r_j^2\}$$

Step 3: Differentiate $L(x, s, r, \lambda, \mu)$ partially with respect to the components of $x, s, r, \lambda \; and \; \mu$. Then equate these derivatives with zero in order to get the required Kuhn – Tucker necessary condition. i.e.,

(i) $c - \frac{1}{2}(2x^T D) - \lambda A + \mu = 0$ (or) $c_j - \sum_{k=1}^{n} x_k d_{jk} - \sum_{i=1}^{m} \lambda_i a_{ij} + \mu_j = 0$;

for all $j = 1,2, \dots, n$

(ii) $-2\lambda s = 0$ (or) $\lambda_i s_i^2 = 0$ (or) $\lambda_i \{\sum_{j=1}^{n} a_{ij} x_j - b_j\} = 0$ for all $i = 1,2, \dots, m$

(iii) $-2\mu r = 0$ (or) $\mu_j r_j = 0$ for all $j = 1,2, \dots, n$

(iv) $Ax + s^2 - b = 0$ (or) $\sum_{j=1}^{n} a_{ij} x_j \leq b_i$ for all $i = 1,2, \dots, m$

(v) $-x + r^2 = 0$ (or) $x_j \geq 0$ for all $j = 1,2, \dots, n$

(vi) $\lambda_i , \mu_j , x_j , s_i , r_j \geq 0$

These conditions, except (ii) and (iii), are linear constraints involving 2 (n + m) variables. The condition $\mu_j x_j = \lambda_i s_i = 0$ implies that both x_j and μ_i as well as s_i and λ_i cannot be basic variables at a time in a non – degenerate basic feasible solution. The conditions $\mu_j x_j = 0$ and $\lambda_i s_i = 0$ are also called complementary slackness conditions.

2.8.2. Wolfe's Modified Simplex Method

The Wolf 's method for solving a quadratic programming problem can be summarized in the following steps:

Step 1: Introduce artificial variables A_j (j = 1, 2, …, n) in the Kuhn – Tucker condition (i). Then we have

$$c_j - \sum_{k=1}^{n} x_k d_{jk} - \sum_{i=1}^{m} \lambda_i a_{ij} + \mu_j + A_j = 0$$

For a starting basic feasible solution we shall have $x_j = 0 , \mu_j = 0 , A_j = -c_j , s_i^2 = b_i$. However, this solution would be desirable if and only if $A_j = 0$ for all j.

Step 2: Apply Phase I of the simplex method to check the feasibility of the constraints Ax ≤ b. If there is no feasible solution, then terminate the solution procedure, otherwise get an initial basic feasible solution for Phase II. To obtain the desired feasible solution solve the following problem:

Minimize Z = $\sum_{j=1}^{n} A_j$ subject to the constraints

$$\sum_{k=1}^{n} x_k d_{jk} + \sum_{i=1}^{m} \lambda_i a_{ij} - \mu_j + A_j = -c_j \; ; j = 1, 2, \dots, n$$

$$\sum_{j=1}^{n} a_{ij}x_j + s_i^2 = b_i \,;\, i = 1, 2, \dots, m$$

and $\lambda_i, x_j, \mu_j, s_i, A_j \geq 0$ for all i and j

$$\left.\begin{matrix} \lambda_i s_i = 0 \\ \mu_j x_j = 0 \end{matrix}\right\} \text{Complement slackness conditions}$$

Thus, while deciding for a variable to enter into the basis at each iteration, the complementary slackness conditions must be satisfied.

This problem has 2(m + n) variables and (m + n) linear constraints, together with (m + n) complementary slackness conditions.

Step 3: Apply Phase II of the simplex method to get an optimal solution to the problem given in Step 2. The solution, so obtained, will also be an optimal solution of the quadratic programming problem.

Example 2.21 Use Wolfe's method to solve the quadratic programming problem:
Maximize Z = $4x_1 + 6x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2$ subject to the constraint $x_1 + 2x_2 \leq 2$ and $x_1, x_2 \geq 0$.

Solution: Consider non-negativity conditions $x_1, x_2 \geq 0$ as inequality constraints. Add slack variables to all inequality constraints in order to express them as equations. The standard form of QP problem becomes:

Maximize Z = $4x_1 + 6x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2$ subject to the constraints

(i) $x_1 + 2x_2 + s_1^2 = 2$ (ii) $-x_1 + r_1^2 = 0$ (iii) $-x_2 + r_2^2 = 0$ and

$x_1, x_2, s_1, r_1, r_2 \geq 0$

To obtain the necessary conditions, we construct the Lagrange function as follows:

$$L(x_1, x_2, s_1, \lambda_1, \mu_1, \mu_2, r_1, r_2) = (4x_1 + 6x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2)$$
$$-\lambda_1(x_1 + 2x_2 + s_1^2 - 2) - \mu_1(-x_1 + r_1^2) - \mu_2(-x_2 + r_2^2)$$

The necessary and sufficient conditions for the maximum of L and hence of Z are:

$$\frac{\partial L}{\partial x_1} = 4 - 4x_1 - 2x_2 - \lambda_1 + \mu_1 = 0 \,;\, \frac{\partial L}{\partial x_2} = 6 - 2x_1 - 4x_2 - 2\lambda_1 + \mu_2 = 0$$

$$\frac{\partial L}{\partial \lambda_1} = x_1 + 2x_2 + s_1^2 - 2 = 0 \,;\, \frac{\partial L}{\partial s_1} = 2\lambda_1 s_1 = 0$$

$$\frac{\partial L}{\partial \mu_1} = -x_1 + r_1^2 = 0 \,;\, \frac{\partial L}{\partial \mu_2} = -x_2 + r_2^2 = 0$$

$$\frac{\partial L}{\partial r_1} = 2\mu_1 r_1 = 0 \,;\, \frac{\partial L}{\partial r_2} = 2\mu_2 r_2 = 0$$

After simplifying these conditions, we get:

(i) $4x_1 + 2x_2 + \lambda_1 - \mu_1 = 4$ (ii) $2x_1 + 4x_2 + 2\lambda_1 - \mu_2 = 6$

(iii) $x_1 + 2x_2 + s_1^2 = 2$

$\begin{Bmatrix} \lambda_1 s_1 = 0 \\ \mu_1 x_1 = \mu_2 x_2 = 0 \end{Bmatrix}$ (Complementray conditions) and $x_1, x_2, \lambda_1, \mu_1, \mu_2, s_1 \geq 0$

Introducing artificial variables A_1 and A_2 in the first two constraints respectively. Then the modified LP problem becomes:

Minimize Z* = $A_1 + A_2$ subject to the constraints

(i) $4x_1 + 2x_2 + \lambda_1 - \mu_1 + A_1 = 4$ (ii) $2x_1 + 4x_2 + 2\lambda_1 - \mu_2 + A_2 = 6$

(iii) $x_1 + 2x_2 + s_1^2 = 2$

and $x_1, x_2, \lambda_1, \mu_1, \mu_2, s_1, A_1, A_2 \geq 0$

		$c_j \rightarrow$	0	0	0	0	0	0	1	1	
c_B	Basic Variables B	Solution Values $b (= x_B)$	x_1	x_2	λ_1	μ_1	μ_2	s_1	A_1	A_2	
1	A_1	4	(4)	2	1	– 1	0	0	1	0	→
1	A_2	6	2	4	2	0	– 1	0	0	1	
0	s_1	2	1	2	0	0	0	1	0	0	
$Z^* = 10$		$c_j - z_j$	– 6 ↑	– 6	– 3	1	1	0	0	0	

The initial basic feasible solution to this LP problem is shown in Table 2.2

Table 2.2: Initial Solution

Iteration 1: In Table 2.2, the largest negative values among $c_j - z_j$ values is – 6 corresponding to x_1 and x_2 columns. This means either of these two variables can be entered into the basis. Since $\mu_1 = 0$ (not in the basis), x_1 is considered to enter into the basis. It will replace A_1 in the basis. The new solution is shown in Table 2.3.

		$c_j \rightarrow$	0	0	0	0	0	0	1	
c_B	Basic Variables B	Solution Values $b\ (= x_B)$	x_1	x_2	λ_1	μ_1	μ_2	s_1	A_2	
0	x_1	1	1	1/2	1/4	– 1/4	0	0	0	
1	A_2	4	0	3	3/2	1/2	– 1	0	1	
0	s_1	1	0	(3/2)	– 1/4	1/4	0	1	0	→
$Z^* = 4$		$c_j - z_j$	0	– 3 ↑	– 3/2	– 1/2	1	0	0	

Table 2.3: First Iteration

Iteration 2: In Table 2.3 $\mu_2 = 0$, (not in the basis), therefore x_2 can be introduced into the basis to replace s_1, in the basis. The new solution is shown in Table 2.4.

		$c_j \rightarrow$	0	0	0	0	0	0	1	
c_B	Basic Variables B	Solution Values $b\ (= x_B)$	x_1	x_2	λ_1	μ_1	μ_2	s_1	A_2	
0	x_1	2/3	1	0	1/3	– 1/3	0	– 1/3	0	
1	A_2	2	0	0	(2)	0	– 1	– 2	1	→
0	x_2	2/3	0	1	– 1/6	1/6	0	2/3	0	
$Z^* = 2$		$c_j - z_j$	0	0	– 2 ↑	0	1	2	0	

Table 2.4: Second Iteration

Iteration 3: In table 2.4, $s_1 = 0$ (not in the basis), therefore λ_1 can be entered into the

		$c_j \rightarrow$	0	0	0	0	0	0
c_B	Basic Variables B	Solution Values $b\ (= x_B)$	x_1	x_2	λ_1	μ_1	μ_2	s_1
0	x_1	1/3	1	0	0	– 1/3	1/6	0
0	λ_1	1	0	0	1	0	– 1/2	– 1
0	x_2	5/6	0	1	0	1/6	– 1/12	1/2
$Z^* = 0$		$c_j - z_j$	0	0	0	0	0	0

basis to replace A_2. The new solution is shown in Table 2.5.

Table 2.5: Third Iteration

In table 2.5, since all $c_j - z_j = 0$, an optimal solution for Phase I is reached. The optimal solution is: $x_1 = \frac{1}{3}, x_2 = \frac{5}{6}, \lambda_1 = 1, \lambda_2 = 0, \mu_1 = \mu_2 = 0, s_1 = 0$

This solution also satisfies the complementary conditions: $\lambda_1 s_1 = 0$; $\mu_1 x_1 = \mu_2 x_2 = 0$

and the restriction on the signs of Lagrange multipliers, λ_1, μ_1 and μ_2.

Further, as Z* = 0, this implies that the current solution is also feasible. Thus, the maximum value of the given quadratic programming problem is:

Max Z = $4x_1 + 6x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2$

$$= 4\left(\frac{1}{3}\right) + 6\left(\frac{5}{6}\right) - 2\left(\frac{1}{3}\right)^2 - 2\left(\frac{1}{3}\right)\left(\frac{5}{6}\right) - 2\left(\frac{5}{6}\right)^2 = \frac{25}{6}.$$

Example 2.22 Use Wolfe's method to solve the quadratic programming problem:

Maximize Z = $2x_1 + x_2 - x_1^2$ subject to the constraint

(i) $2x_1 + 3x_2 \leq 6$ (ii) $2x_1 + x_2 \leq 4$ and $x_1, x_2 \geq 0$.

Solution: Consider non-negativity conditions $x_1, x_2 \geq 0$ as inequality constraints. Add slack variables to all inequality constraints in order to express them as equations. The standard form of QP problem becomes:

Maximize Z = $2x_1 + x_2 - x_1^2$ subject to the constraints

(i) $2x_1 + 3x_2 + s_1^2 = 6$ (ii) $2x_1 + x_2 + s_2^2 = 4$

(iii) $-x_1 + r_1^2 = 0$ (iv) $-x_2 + r_2^2 = 0$

To obtain the necessary conditions, we construct the Lagrange function as follows:

$$L(x, s, \lambda, r, \mu) = (2x_1 + x_2 - x_1^2) - \lambda_1(2x_1 + 3x_2 + s_1^2 - 6) - \lambda_2(2x_1 + x_2 + s_2^2 - 4) - \mu_1(-x_1 + r_1^2) - \mu_2(-x_2 + r_2^2)$$

The necessary and sufficient conditions for the maximum of L and hence of Z are:

$$\frac{\partial L}{\partial x_1} = -2 - 2x_1 - 2\lambda_1 - 2\lambda_2 + \mu_1 = 0\ ;\ \frac{\partial L}{\partial x_2} = 1 - 3\lambda_1 - \lambda_2 + \mu_2 = 0$$

$$\frac{\partial L}{\partial \lambda_1} = 2x_1 + 3x_2 + s_1^2 - 6 = 0\ ;\ \frac{\partial L}{\partial \lambda_2} = 2x_1 + x_2 + s_2^2 - 4 = 0$$

$$\frac{\partial L}{\partial \mu_1} = -x_1 + r_1^2 = 0\ ;\ \frac{\partial L}{\partial \mu_2} = -x_2 + r_2^2 = 0$$

$$\frac{\partial L}{\partial r_1} = -2\mu_1 r_1 = 0\ ;\ \frac{\partial L}{\partial r_2} = -2\mu_2 r_2 = 0$$

$$\frac{\partial L}{\partial s_1} = -2\lambda_1 s_1 = 0\ ;\ \frac{\partial L}{\partial s_2} = -2\lambda_2 s_2 = 0$$

After simplifying these conditions, we get:

(i) $2x_1 + 2\lambda_1 + 2\lambda_2 - \mu_1 = 2$ (ii) $3\lambda_1 + \lambda_2 - \mu_2 = 1$

(iii) $2x_1 + 3x_2 + s_1^2 = 6$ (iv) $2x_1 + x_2 + s_2^2 = 4$

$\begin{Bmatrix} \lambda_1 s_1 = \lambda_1 s_1 = 0 \\ \mu_1 x_1 = \mu_2 x_2 = 0 \end{Bmatrix}$ (Complementray conditions) and $x_1, x_2, A_1, A_2, \mu_1, \mu_2, s_1, s_2 \geq 0$

Introducing artificial variables A_1 and A_2 in the first two constraints respectively. Then the modified LP problem becomes:

Minimize Z* = $A_1 + A_2$ subject to the constraints

(i) $2x_1 + 2\lambda_1 + 2\lambda_2 - \mu_1 + A_1 = 2$ (ii) $3\lambda_1 + \lambda_2 - \mu_2 + A_2 = 1$

(iii) $2x_1 + 3x_2 + s_1^2 = 6$ (iv) $2x_1 + x_2 + s_2^2 = 4$

and $x_1, x_2, \lambda_1, \mu_1, \mu_2, s_1, A_1, A_2 \geq 0$

		$c_j \rightarrow$	0	0	0	0	0	0	0	0	1	1	
c_B	Basic Variables B	Solution Values $b(= x_B)$	x_1	x_2	λ_1	λ_2	μ_1	μ_2	s_1	s_2	A_1	A_2	
1	A_1	2	(2)	0	2	2	−1	0	0	0	1	0	→
1	A_2	1	0	0	3	1	0	−1	0	0	0	1	
0	s_1	6	2	3	0	0	0	0	1	0	0	0	
0	s_2	4	2	1	0	0	0	0	0	1	0	0	
$Z^* = 3$		$c_j - z_j$	−2 ↑	0	−5	−3	1	1	0	0	0	0	

The initial basic feasible solution to this LP problem is shown in Table 2.6

Table 2.6: Initial Solution

Iteration 1: In Table 2.6, the largest negative values among $c_j - z_j$ values is – 5, but we cannot enter λ_1 (or λ_2) in the basis of the complementary conditions $\lambda_1 s_1 = \lambda_2 s_2 = 0$. Since $\mu_1 = 0$, x_1 can be entered into the basis with A_1 as the leaving variable. The new solution is shown in Table 2.7.

		$c_j \rightarrow$	0	0	0	0	0	0	0	0	1	
c_B	Basic Variables B	Solution Values $b(= x_B)$	x_1	x_2	λ_1	λ_2	μ_1	μ_2	s_1	s_2	A_2	
0	x_1	1	1	0	1	1	1/2	0	0	0	0	
1	A_2	1	0	0	3	1	0	−1	0	0	1	
0	s_1	4	0	(3)	−2	−2	1	0	1	0	0	→
0	s_2	2	0	1	−2	−2	1	0	0	1	0	
$Z^* = 1$		$c_j - z_j$	0	0 ↑	−3	−1	0	1	0	0	0	

Table 2.7: First Iteration

Iteration 2: Again, we cannot enter λ_1, λ_2 and μ_1 in the basis in Table 2.7 because s_1, s_2 and x_1, respectively, are already in the basis. Entering x_2 into the basis with s_1 as the leaving variable because $\mu_2 = 0$. The new solution is shown in Table 2.8.

		$c_j \rightarrow$	*0*	*0*	*0*	*0*	*0*	*0*	*0*	*0*	*1*	
c_B	*Basic Variables* **B**	*Solution Values* $b(= x_B)$	x_1	x_2	λ_1	λ_2	μ_1	μ_2	s_1	s_2	A_2	
0	x_1	1	1	0	1	1	– 1/2	0	0	0	0	
1	A_2	1	0	0	(3)	1	0	– 1	0	0	1	$\rightarrow$
0	x_2	4/3	0	1	– 2/4	– 2/3	1/3	0	1/3	0	0	
0	s_2	2/3	0	0	– 4/3	– 4/3	2/3	0	– 1/3	1	0	
$Z^* = 1$		$c_j - z_j$	0	0	– 3 $\uparrow$	– 1	0	1	0	0	1	

Table 2.8: Second Iteration

Iteration 3: Since $s_1 = 0$, λ_1 can be entered into the basis in Table 2.8, with A_2 as the leaving variable. The new solution is shown in Table 2.9.

		$c_j \rightarrow$	*0*	*0*	*0*	*0*	*0*	*0*	*0*	*0*
c_B	*Basic Variables* **B**	*Solution Values* $b\ (= x_B)$	x_1	x_2	λ_1	λ_2	μ_1	μ_2	s_1	s_2
0	x_1	2/3	1	0	0	2/3	– 1/2	1/3	0	0
0	λ_1	1/3	0	0	1	1/3	0	– 1/3	0	0
0	x_2	14/9	0	1	0	– 4/9	1/3	– 2/9	1/3	0
0	s_2	10/9	0	0	0	– 8/9	2/3	– 4/9	– 1/3	1
$Z^* = 0$		$c_j - z_j$	0	0	0	0	0	0	0	0

Table 2.9: Third Iteration

In table 2.9, since all $c_j - z_j = 0$, an optimal solution for Phase I is reached. The optimal solution is: $x_1 = \frac{2}{3}, x_2 = \frac{14}{9}, \lambda_1 = \frac{1}{3}, \lambda_2 = 0, \mu_1 = \mu_2 = 0, s_1 = 0, s_2 = \frac{10}{9}$

This solution also satisfies the complementary conditions: $\lambda_1 s_1 = \lambda_2 s_2 = 0$; $\mu_1 x_1 = \mu_2 x_2 = 0$ and the restriction on the signs of Lagrange multipliers, $\lambda_1, \lambda_2, \mu_1$ *and* μ_2.

Further, as Z* = 0, this implies that the current solution is also feasible. Thus, the maximum value of the given quadratic programming problem is:

Max Z = $2x_1 + x_2 - x_1^2 = 2\left(\frac{2}{3}\right) + \left(\frac{14}{9}\right) - \left(\frac{2}{3}\right)^2 = \frac{22}{9}$.

2.8.3. Beale's Method

In this method, instead of Kuhn-Tucker conditions, results based on calculus are used for solving a given quadratic programming problem. Let the general quadratic programming (QP) problem be of the form

Minimize Z = cx + $\frac{1}{2}$ x^T Dx → (24)

subject to the constraints Ax = b → (25)

and x ≥ 0 → (26)

where, x ∈ E^n, b ∈ E^m, c ∈ E^n, D is a symmetric n × n matrix and A is an m × n matrix.

Beale's method starts with the partitioning of n variables in QP problem into basic and non-basic variables at each iteration of the solution process, and expressing the basic variables as well as objective function in terms of non-basic variables. Let B be any m × m non – singular matrix that contains columns of A corresponding to the basic variables, x_B ∈ E^m. Let N be an m × (n – m) matrix that contains columns of A corresponding to non-basic variables, x_N ∈ E^{n-m}. Equation (25) can then be written as:

$$[B\ ,N]\begin{bmatrix}x_B\\x_N\end{bmatrix} = b\ (or)\ Bx_B + Nx_N = b$$

(or) $x_{B_i} = y_{i0} - \sum_{j=1}^{n-m} y_{ij}x_{N_j}\ ; i = 1, 2, \dots, m$ → (27)

where $y_{i0} = (y_{10}, y_{20}, \dots, y_{m0})^T = B^{-1}b$ and $y_{ij} = B^{-1}N$

For the current basic feasible solution $x_{N_j} = 0$ (j = 1, 2, …, n – m), we have $x_{B_i} = y_{i0}$, (i = 1, 2, …, m). Assuming that $y_{i0} \geq 0$,

The objective function (24) in terms of x_B and x_N can be written as:

$$Z = [c_B\ ,c_N]\begin{bmatrix}x_B\\x_N\end{bmatrix} + \frac{1}{2}[x_B^T\ ,x_N^T]\begin{bmatrix}d_{11} & d_{12}\\d_{21} & d_{22}\end{bmatrix}\begin{bmatrix}x_B\\x_N\end{bmatrix}$$

Expressing Z in terms of the remaining (n – m) non – basic variables x_N only, and simplifying, we get:

$Z = Z_0 + \alpha x_N + x_N^T G x_N$ → (28)

where Z_0 = value of objective function Z when $x_N = 0$ and $x_{B_i} = y_{i0}$

G = symmetric matrix of order (n – m) x (n – m)

$\alpha = \alpha_1, \alpha_2, \dots, \alpha_{n-m}$ (Constant).

The Procedure:

Step 1: Evaluate the partial derivatives of Z with respect to non-basic variables, $x_N = 0$. Thus, from equation (28) we get:

$$\frac{\partial Z}{\partial x_{N_j}} = \alpha_j + 2\sum_{k=1}^{n-m} g_{jk}x_{nk} \; ; j = 1, 2, \dots, n-m \qquad \rightarrow (29)$$

Step 2: See the nature of $\left|\frac{\partial Z}{\partial x_{N_j}}\right|_{x_N=0} = \alpha_j \, ; k = 1, 2, \dots, n-m$

(a) If $\alpha_j < 0$, for all j, then the current solution is also an optimal solution

(b) But if at least one $\alpha_j > 0$, then one of the non-basic variables, which is currently at zero level, corresponding to the largest positive value of α_j, will be selected to enter the basis.

Step 3: $\left|\frac{\partial Z}{\partial x_{N_j}}\right|_{x_N=0} = \alpha_r$ (largest), then choose non-basic variable x_r for entering the basis. For this it will be profitable to go on increasing its value from zero till a point where either:

(a) any one of the present basic variables becomes negative, or

(b) $\frac{\partial Z}{\partial x_{N_j}}$ reduces to zero and is about to become negative.

Step 4: For maintaining the feasibility of the solution we must consider only that value of non-basic variable x_r, say β_1, which has only a positive coefficient. In this case, the first basic variable selected to leave the basis should satisfy the usual minimum ratio rule of the simplex method and will be given by:

$$\beta_1 = \left\{\begin{matrix} Min\left\{\frac{y_{i0}}{y_{ij}} \; ; \; y_{ij} > 0\right\} & \\ \infty & ; \; y_{ij} \le 0 \, ; j = 1, 2, \dots, n-m \end{matrix}\right\} \qquad \rightarrow (30)$$

where $y_{i0} = x_{B_i}$

Since it is not desirable to increase the value of the non-basic variable x_r beyond the point where $\frac{\partial Z}{\partial x_{N_j}}$ becomes zero, the critical value of x_r say β_2, at which $\frac{\partial Z}{\partial x_{N_j}}$ becomes zero is given by:

$$\beta_2 = \left\{\begin{matrix} \frac{|\alpha_j|}{2g_{jj}} & ; \; g_{jj} > 0 \\ \infty & ; \; g_{jj} \le 0 \end{matrix}\right\}$$

where g_{jj} is the element of matrix G.

Hence the value of non-basic variable xr must be determined by taking the minimum between β_1 and β_2, that is, $x_r = Min\,\{\beta_1, \beta_2\}$. However, if $\beta_1 = \beta_2 = \infty$, the value of x_r can be increased indefinitely without violating either the conditions (a) or (b) of Step 3 and the condition that QP problem must have an unbounded solution. Moreover,

(i) If the entering variable x_r is increased up to only β_1 and at least one basic variable is reduced to zero, then a new basic feasible solution can be obtained by the usual simplex method. But if by entering x_r into the basis two or more basic variables are reduced to zero, then the new solution, so obtained, will be degenerate and thus cycling can occur.

(ii) If the entering variable is increased up to β_2 ($< \infty$), then we may have more than m variables at positive level at any iteration. This stage comes when the new (non-basic) feasible solution occurs where $\frac{\partial Z}{\partial x_{N_j}}=0$. At this stage we define a new variable (unrestricted) u_j as:

$$u_j = \frac{\partial Z}{\partial x_r} = \alpha_j + 2\sum_{k=1}^{n-m} g_{jk} x_{Nk}$$

The variable u_j is also called free variable. Clearly, we now have m + 1 non – zero variables and m + 1 constraints. These variables form a basic feasible solution to the new set of constraints:

$$Ax = b \text{ and } u_j - 2\sum_{k=1}^{n-m} g_{jk} x_{Nk} = \alpha_j$$

The variable u_j is introduced in the set of constraints only for computational purposes and its value is zero at the next basic feasible solution. Now, the variables x_B and u_j are treated as basic variables. The new set of constraints is again expressed in terms of non-basic variables for obtaining the new basic feasible solution.

Step 5: Go to Step 1 and repeat the entire procedure of getting a new basic feasible solution until no further improvement in the objective function can be obtained by making any permitted changes in one of the non – basic variables. The permitted changes here include increase in all variables and decrease in free variables. In other words, the procedure terminates when:

$$\frac{\partial Z}{\partial x_{N_j}} \begin{Bmatrix} \leq 0, & if\ x_{N_j}\ is\ a\ restricted\ variable \\ = 0, & if\ x_{N_j}\ is\ a\ free\ variable \end{Bmatrix} \qquad \rightarrow (31)$$

The necessary conditions (31) for terminating the procedure are also sufficient for a global minimum if D is positive semi – definite or positive definite.

Remarks:

1) While evaluating $\frac{\partial Z}{\partial u_j}$, both increase and decrease must be checked, as u_j is unrestricted in sign.

2) If at any iteration a free variable becomes a basic variable and is non – zero, then drop the new constraint containing it. This should be done because it is a free variable, and therefore, will neither be chosen to leave the basis nor will appear in the selection of leaving variable.

Example 2.23 Use Beale's method to solve the following QP problem:

Minimize Z = $-4x_1 + x_1^2 - 2x_1x_2 + 2x_2^2$ subject to the constraints

(i) $2x_1 + x_2 \geq 6$ (ii) $x_1 - 4x_2 \geq 0$ and $x_1, x_2 \geq 0$

Solution: Introducing surplus variables s_1 and s_2, the constraint equations becomes:

(i) $2x_1 + x_2 - s_1 = 6$ and (ii) $x_1 - 4x_2 - s_2 = 0$.

Also, converting the minimization objective function into a maximization, we have

Maximize Z = $4x_1 - x_1^2 + 2x_1x_2 - 2x_2^2$

Making s_1 and s_2 basic variables in the initial solution and expressing these in terms of non – basic variables x_1 and x_2 as follows:

(i) $-6 + 2x_1 + x_2 = s_1$ (ii) $x_1 - 4x_2 = s_2$ and (iii) $Z = 4x_1 - x_1^2 + 2x_1x_2 - 2x_2^2$

Thus, $x_B = (s_1, s_2) = (-6, 0)$ and $x_N = (x_1, x_2) = (0, 0)$

At the current solution $\alpha_1 = 4$ and $\alpha_2 = 0$. Since both of these are positive, therefore we choose x_1 (due to most positive value of α_1) to enter into the basis. The critical value β_1 of x_1 is given by

$$Min \left\{\frac{-6}{|2|}, \frac{0}{|1|}\right\} = -3$$

The variable s_1 is eligible to leave the basis. Expressing the new basic variables x_1, s_2 and Z in terms of new non – basic variables x_1 and s_2 as follows:

(i) $x_1 = 3 - \frac{1}{2}x_2 + \frac{1}{2}s_1$ (ii) $s_2 = 3 - \frac{3}{2}x_2 + \frac{1}{2}s_1$ and

(iii) $Z = 4\left(3 - \frac{1}{2}x_2 + \frac{1}{2}s_1\right) - \left(3 - \frac{1}{2}x_2 + \frac{1}{2}s_1\right)^2 + 2\left(3 - \frac{1}{2}x_2 + \frac{1}{2}s_1\right)x_2 - 2x_2^2$

$$= 9 + x_2 - s_1 + \frac{3}{2}x_2s_1 - \frac{13}{4}x_2^2 - \frac{1}{4}s_1^2$$

Again, differentiating Z with respect to x_2 and s_1, we have

$$\left|\frac{\partial Z}{\partial x_2}\right|_{\substack{x_2=0\\ s_1=0}} = 1+\frac{3}{2}s_1-\frac{13}{2}x_2 = 1\,;\ \left|\frac{\partial Z}{\partial s_1}\right|_{\substack{x_2=0\\ s_1=0}} = -1+\frac{3}{2}x_2-\frac{1}{2}s_1 = -1$$

The variable x_2 is eligible to enter the basis. Again compute the ration

$$Min\left\{\frac{3}{|(-1/2)|},\frac{3}{|(-3/2)|}\right\} = 2$$

Since the minimum ration corresponds β_2, we introduce a non – basic free variable u_1, defined by

$$u_1 = \frac{1}{2}\frac{\partial Z}{\partial x_2} = \frac{1}{2}+\frac{3}{4}s_1-\frac{13}{4}x_2$$

Now we have $x_B = (x_1, s_2, x_2)$ and $x_N = (s_1, u_1)$. Expressing basic variables and Z in terms of non – basic variables, we have:

(i) $x_1 = \frac{38}{13}-\frac{3}{26}s_1+\frac{2}{13}u_1$ (ii) $x_1 = \frac{2}{13}+\frac{3}{13}s_1-\frac{4}{13}u_1$

(iii) $s_2 = \frac{30}{13}-\frac{27}{26}s_1+\frac{18}{13}u_1$

(iv) $Z = 9+\frac{1}{13}(2+3s_1-4u_1)-s_1+\frac{3}{26}s_1(2+3s_1-4u_1)-\frac{1}{52}(2+3s_1-u_1)^2-\frac{1}{4}s_1^2$

Again, $\left|\frac{\partial Z}{\partial s_1}\right|_{\substack{s_1=0\\ u_1=0}} = \frac{3}{13}-1+\frac{3}{26}(2-4u_1)+\frac{18}{26}s_1-\frac{6}{52}(2+3s_1-4u_1)-\frac{1}{2}s_1 = -\frac{9}{13}$

$$\left|\frac{\partial Z}{\partial s_1}\right|_{\substack{s_1=0\\ u_1=0}} = -\frac{4}{13}-\frac{12}{26}s_1+\frac{8}{52}(2+3s_1-4u_1) = 0$$

Since both $\alpha_j < 0$, the optimal value of Z is obtained by setting $u_1 = 0, s_1 = 0$ in the current value of the objective function:

$$Z^* = 9+\frac{2}{13}-\frac{2}{52} = \frac{474}{52}$$

Hence, the optimum solution to the given QP problem is $x_1 = \frac{38}{13}$ and $x_2 = \frac{2}{13}$ with Min Z = 9.115.

Example 2.24 The operations Research team of the ABC Company has come up with the mathematical data (daily basis) needed for two products which the firm manufactures. It also has determined that this is a non-linear programming problem, having linear constraints and objective function which is the sum of a linear and a quadratic form. The pertinent data, gathered by the OR team are:

Maximize (Contribution) = 12x + 21y + 2xy – 2x^2 – 2y^2 subject to the constraints

(i) 8 – y ≥ 0 (ii) 10 – x – y ≥ 0 and x, y ≥ 0

Find the maximum contribution and number of units that can be expected for these two products which are a part of the firm's total output. (x and y represent the number of units of the two products.)

Solution: The problem can be written as

Maximize Z = 12x + 21y + 2xy – 2x^2 – 2y^2 subject to the constraints

(i) 8 – y ≥ 0 (ii) 10 – x – y ≥ 0 and x, y ≥ 0

Introducing slack variables s_1, s_2 and treating x, y as the basic variables, we express the basic variables and Z in terms of non-basic variables s1, s2 as follows:

(i) y = 8 – s_1; (ii) x = 2 – s_1 – s_2 and

(iii) Z = 12(2 – s_1 – s_2) + 21(8 – s_1) + 2(1 – s_1)(2 – s_1 – s_2) – (2 – s_1 – s_2)^2 – 2(8 – s_1)^2

= 224 – 53s_1 – 28s_2 + 2s_1 s_2 + 2s_1^2 – 2(2s_1 – s_2)^2 – 2(8 – s_1)^2

Thus, $\frac{\partial Z}{\partial s_1}$ = –53 + 2s_2 + 4(2 – s_1 – s_2) + 4(8 – s_1) ; $\left|\frac{\partial Z}{\partial s_1}\right|_{\substack{s_1=0 \\ s_2=0}} = -13$

$\frac{\partial Z}{\partial s_2}$ = –28 + 2s_1 + 4(2 – s_1 – s_2) ; $\left|\frac{\partial Z}{\partial s_2}\right|_{\substack{s_1=0 \\ s_2=0}} = -20$

Since both the partial derivatives are negative, the current solution is optimum. Thus the optimum solution is: x = 2, y = 8, with Max Z = 88. Hence, in order to have a maximum contribution of Rs. 88, the ABC company must expect 2 and 8 units of the two products, respectively.

Let us sum up

We have learned about Quadratic programming problem, to find it solution by using Kuhn-Tucker conditions, Wolfe's Modified simplex method and Beale's method.

Check your progress

Use Wolfe's method for solving the following quadratic programming problems:

30. Max Z = $2x_1 + 3x_2 - 2{x_1}^2$

Subject to, $x_1 + 4x_2 \leq 4$;

$x_1 + x_2 \leq 2$;

$x_1,\ x_2 \geq 0$

31. Max Z = $8x_1 + 10x_2 - {x_1}^2 - {x_2}^2$

Subject to, $3x_1 + 2x_2 \leq 6$;

$x_1,\ x_2 \geq 0$

Use Beale's method to solve the following quadratic programming problems:

32. Max Z = $2x_1 + 3x_2 - {x_1}^2$

Subject to, $x_1 + 2x_2 \leq 4$;

$x_1,\ x_2 \geq 0$

Unit Summary

The classical optimization methods are used to obtain an optimal solution of certain types of problems that involve continuous and differentiable function. These methods are analytical in nature and make use of differential calculus in order to find points of maxima and minima for

(a) an constrained single and multiple variable continuous function, and

(b) constrained multivariable functions with equality and inequality constraints.

In this unit conditions for local as well as global minimum and maximum value of an unconstrained objective function have been derived followed by numerical exercises. Direct substitution method, Lagrange's multipliers method and Kuhn-Tucker method have also been discussed to find optimal value of an objective function with equality and inequality constraints, respectively

Linear programming requires the objective function and constraints to be linear. However, if either of these are not linear, then non-linear programming methods are used to find optimal value of the objective function with or without constraints. In the more general procedure, conditions necessary for an optimum value of a function subject to inequality constraints, are known as Kuhn-Tucker conditions. Beale's and Wolf's methods have also been demonstrated to solve quadratic programming problems.

In case the objective function and constraints are separable, the separable programming technique is used for solving a NL programming problem. Sometimes, functions that are not separable can be made separable by using the approximation method.

Geometric programming is used to solve NL programming problems that involve special type of functions called polynomials. The GP approach first finds the optimal value of the objective function by solving its dual problem and then determines the solution to the given NLP problem from the optimal solution of the dual.

Glossary

- NLPP-Non-linear programming problem
- QPP-Quadratic programming problem

Self – Assessment Questions

1.State and prove Kuhn-Tucker necessary and sufficient conditions in non-linear programming.

2.Obtain the Kuhn-Tucker conditions for a solution of the problem:

Max Z = $cx + \frac{1}{2}x^T dx$

Subject to, $Ax = b$;

and $x \geq 0$

3. Explain what is meant by Kuhn-Tucker conditions.

4. What is meant by quadratic programming? How does a quadratic programming problem differ from a linear programming problem? Give an example.

5. Briefly mention Wolfe's algorithm for solving a quadratic programming problem:

Max Z = $cx + \frac{1}{2}x^T qx$

Subject to, $Ax \leq b$;

and $x \geq 0$

Activities

1. Derive the Kuhn-Tucker necessary conditions for an optimal solution to a quadratic programming problem.

2. What is quadratic programming? Explain Wolfe's method of solving it.

3. Discuss the economic interpretation of Lagrangian multipliers, the duality theory, and derive the Kuhn-Tucker conditions for the non-linear programming problem: Max Z = f (x) subject to the constraints $g_j(x) \leq b_i$; i = 1, 2, . . ., m.

4. Is it correct to say that in a quadratic programming problem the objective function and the constraints both should be quadratic? If not, give your own comments.

Suggested Readings

1. J. K. Sharma, *Operations Research*, Theory and Applications, Third Edition (2007) Macmillan India Ltd
2. Hamdy A. Taha, *Operations Research,* (seventh edition) Prentice - Hall of India Private Limited, New Delhi, 1997.
3. F.S. Hillier & J.Lieberman *Introduction to Operation Research* (7th Edition) Tata-McGraw Hill ompany, New Delhi, 2001.
4. Beightler. C, D.Phillips, B. Wilde ,*Foundations of Optimization* (2nd Edition) Prentice Hall Pvt Ltd., New York, 1979
5. S.S. Rao - *Optimization Theory and Applications*, Wiley Eastern Ltd. New Delhi. 1990

UNIT – III

THEORY OF SIMPLEX METHOD

THEORY OF SIMPLEX METHOD

Objectives:

After studying this unit, students should be able to learn both canonical and standard forms of LP model and their characteristics. To know the importance and interpretation of slack and surplus variables. To identify an alternative optimal solution and an unbounded solution of any LP model. To resolve certain complications, viz. unrestricted variables, degeneracy, etc., that may arise in applying the simplex method.

3.1 INTRODUCTION

The graphical method is applicable only to solve two-variable LP problems. Finding and evaluating all basic feasible solutions of an LP problem with more than two variables and many constraints is very difficult. Thus, an efficient computational procedure is required to solve the general mathematical model of LP problem. In this chapter we shall discuss an iterative method (or procedure) called the simplex method developed by G.B. Dantzig in 1947 for solving an LP problem with more than two variables. The simplex method is referred as an iterative procedure because it is based on the procedure of moving from one extreme (corner) point to another of the solution space (or feasible region) that is formed by the constraints and non-negativity conditions of the linear programming problem. Since the number of extreme points (corners or vertices) of a solution space is finite, the method leads to find an extreme point in a finite number of steps where either LP problem has optimal solution or there exists an unbounded solution.

3.2 CANONICAL AND STANDARD FORM OF LP PROBLEM

Canonical form: If the general mathematical model of an LP problem is expressed as: Maximize Z = $\sum_{j=1}^{n} c_j x_j$ subject to the constraints

$\sum_{j=1}^{n} a_{ij} x_j \leq b_j \; ; i = 1, 2, \ldots, m$ and $x_j \geq 0 \; ; j = 1, 2, \ldots, n$

Then it is called the canonical form of an LP problem. The characteristics of this form are:

(i) The objective function should be of maximization type. If not, then it should be changed to the same by applying the method discussed earlier.

(ii) All constraints should be of ≤ type except for the non-negativity conditions. An inequality of ≥ type can be changed to an inequality of ≤ type by multiplying it with –1 on both sides.

(iii) All variables must have non-negative values. If any variable, say x_j , is unrestricted in sign (i.e., positive, negative or zero), then it can be replaced by: $x_j = x_j' - x_j''$ where x_j' *and* x_j'' are both non-negative.

(iv) The right-hand side of each constraint should be positive.

Standard form:

If the general formulation of an LP problem is expressed as:

Maximize Z = $\sum_{j=1}^{n} c_j x_j$

subject to the constraints $\sum_{j=1}^{n} a_{ij} x_j = b_j$; $i = 1, 2, \ldots, m$ and $x_j \geq 0$; $j = 1, 2, \ldots, n$

Then it is called the standard form of the LP problem. The characteristics of this form are:

(i) All the constraints should be expressed as equations.

(ii) The right-hand side of each constraint should be made non-negative. If it is not so, this should be done by multiplying both sides of the resulting constraints by –1.

(iii) The objective function should be of maximization type.

The standard form can also be written in matrix notation as follows:

Maximize Z = cx → (1)

subject to the constraints Ax = b, → (2)

and x ≥ 0 → (3)

where $c = (c_1, c_2, \ldots, c_n)$ is the row vector; $x = (x_1, x_2, \ldots, x_n)^T$ and $b = (b_1, b_2, \ldots, b_m)^T$ are column vectors and A is m × n coefficients matrix of rank m.

The LP problem can also be represented in terms of column vectors, $a_1, a_2, \ldots, a_n$ of matrix A as follows:

Maximize Z = $\sum_{j=1}^{n} c_j x_j$ subject to the constraints

$$\sum_{j=1}^{n} a_j x_j = b \text{ and } x_j \geq 0 \; ; j = 1, 2, \ldots, n$$

Remarks:

1) A minimization problem can also be in canonical form if all variables are non-negative and all the constraints are of ≥ type.

2) Any maximization LP problem can be converted into an equivalent minimization LP problem and vice versa by multiplying the given objective function by –1, without making any change in the constraints. For example, the objective function:

$$\text{Maximize } Z = \sum_{j=1}^{n} c_j x_j = Min\ Z^* = \sum_{j=1}^{n} (-c_j)\, x_j$$

3.3 SLACK AND SURPLUS VARIABLES

In the general LP problem each constraint may take one of the three possible forms, ≤ , =, or ≥. Inequality constraints of the LP problem are converted into equalities by adding additional non-negative variables called slack and surplus (negative slack) variables:

Case 1: The constraints with ≤ inequality sign, i.e.,

$\sum_{j=1}^{n} a_{ij} x_j \le b_i$ can be converted to the equality

$$\sum_{j=1}^{n} a_{ij} x_j + s_i = b_i\ ;\ i = 1, 2, \dots, m \qquad \to (4)$$

by adding non – negative variable s_i, called slack variables.

Case 2: The constraints with ≥ inequality sign, i.e.,

$\sum_{j=1}^{n} a_{ij} x_j \ge b_i$ can be converted to the equality

$$\sum_{j=1}^{n} a_{ij} x_j - s_i = b_i\ ;\ i = 1, 2, \dots, m \qquad \to (5)$$

by subtracting non – negative variable s_i, called surplus variables.

The general LP problem that involves mixed constraints can be stated as:

Optimize (Max or Min) Z = $\sum_{j=1}^{n} c_j x_j$ → (6)

subject to the constraints $\sum_{j=1}^{n} a_{ij} x_j \le b_i\ ; i = 1, 2, \dots, r$

$\sum_{j=1}^{n} a_{ij} x_j \ge b_i\ ; i = r + 2, r + 2, \dots, s$ → (7)

$\sum_{j=1}^{n} a_{ij} x_j = b_i\ ; i = s + 1, s + 2, \dots, m$

and $x_j \ge 0\ ; j = 1, 2, \dots, n$

Constraints with ≤ inequality sign can be converted to form (4) and those with ≥ inequality sign to form (5). Those having an equality sign remain unchanged. The general LP problem can be stated as:

Optimize Z = $\sum_{j=1}^{n} c_j x_j + \sum_{i=1}^{r} 0.s_i + \sum_{i=r+1}^{s} 0.s_i$ subject to the constraints

$\sum_{j=1}^{n} a_{ij} x_j + s_i = b_i\ ; i = 1, 2, \dots, r$

$\sum_{j=1}^{n} a_{ij} x_j - s_i = b_i\ ; i = r + 2, r + 2, \dots, s$

$\sum_{j=1}^{n} a_{ij}x_j = b_i$; $i = s+1, s+2, \dots, m$ and $x_j \geq 0$; $j = 1, 2, \dots, n$

Remark:

The coefficients of slack and surplus (negative slack) variables are zero in the objective function due to the reason that they represent unused capacity (or resource).

3.3.1 Basic Solution

Constraints, Ax = b of an LP problem are written as a system of m simultaneous linear equations in n (n > m) unknown, where A is an m × n matrix and rank (A) = m. Let B be an m × m non-singular submatrix of A obtained by reordering the column of A, and N an m × (n – m) matrix such that A = (B, N). Let x be partitioned as $[x_B, x_N]^T$ where $x_B^T \in E^m$ and $N \in E^{n-m}$ be the vector of variables associated with columns of matrix B and N, respectively. Then constraints, Ax = b can be rewritten as:

$[B, N]\begin{bmatrix} x_B \\ x_N \end{bmatrix} = b$ (or) $Bx_B + Nx_N = b$ (or) $x_B = B^{-1}b - B^{-1}Nx_N$

If all the (n – m) variables not associated with the columns of matrix B are set equal zero, i.e., $x_N = 0$, the solution to the resulting system of equations, i.e.,

$$\begin{bmatrix} x_B \\ x_N \end{bmatrix} = \begin{bmatrix} B^{-1} \\ 0 \end{bmatrix} \; ; \; x_B = (x_{B1}, x_{B2}, \dots, x_{Bm})$$

is called the basic solution to the given system of equations. The m variables that can be different from zero are called the basic variables. If all these variables also satisfy the non-negativity conditions, x ≥ 0, then the basic solution constituted by them is called a basic feasible solution. Again if these satisfy all the constraints Ax = b, then the solution is known as a feasible solution. The remaining n – m variables, i.e., the components of x_N are called non-basic variables. The matrix B is called the basis matrix (or simply the basis) having m linear independent columns selected from A. Let $\beta_1, \beta_2, \dots, \beta_m$ be the columns of basis matrix B that form a basis. Then we can write B = $[\beta_1, \beta_2, \dots, \beta_m]$. The column a_j of A can be expressed as a linear combination of columns of B as:

$$\begin{aligned} a_j &= y_{ij}\beta_1 + y_{2j}\beta_2 + \cdots + y_{mj}\beta_m \\ &= (\beta_1, \beta_2, \dots, \beta_m)\left(y_{1j}, y_{2j}, \dots, y_{mj}\right) = By_j \end{aligned}$$

or $y_j = B^{-1}a_j$

where $y_j = (y_{1j}, y_{2j}, \ldots, y_{mj})$ are the scalars. Obviously the vector y_j will change with the change in the columns of A that are part of basis matrix B.

If the value of the objective function Z can be increased or decreased with change in the values of basic variables, then such a solution is said to be unbounded.

3.3.2 Degenerate Solution

A basic feasible solution $x_B = B^{-1}b$ b is said to be degenerate if at least one component of x_B (basic variables) is zero. If all components of x_B are non-zero($x_B > 0$), then it is called a non-degenerate basic feasible solution. For a system of m equations with n variables (n > m) the total number of basic feasible solutions is less than or equal to the total number of combinations, ${}^nC_m = \frac{n!}{m!(n-m)!}$

3.3.3 Cost (or Price) Vector

Let the cost vector c, associated with the variables in objective function Z, of an LP problem be partitioned as$c = (x_B, x_N)$, where c_B and c_N are the coefficients of basic variables x_B and non-basic variables x_N , respectively. The objective function can then be written as:

$$Z = cx = [c_B, c_N]\begin{bmatrix} x_B \\ x_N \end{bmatrix} = c_B x_B + c_N x_N$$

Since for the basic feasible solution the value of all the non-basic variables become zero, i.e., $x_N = 0$, therefore the value of the objective function for basic feasible solution is given by Z = $c_B x_B$, where $c_B = (c_{B1}, c_{B2}, \ldots, c_{Bm})$. The vector, $c_B = (c_{B1}, c_{B2}, \ldots, c_{Bm})$ associated with the basic variable,
$c_B = (x_{B1}, x_{B2}, \ldots, x_{Bm})$ is called cost (or price) vector.

Example 3.1 Find all the basic feasible solutions to the system of linear equations ?
(i) $x_1 + 2x_2 + x_3 = 4$ (ii) $2x_1 + x_2 + 5x_3 = 5$. Are the solutions degenerate ?

Solution: The given system of equations can be written in the Ax = b form as follows:

$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ where $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 5 \end{bmatrix}$; $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$; $b = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

Since, rank (A) = 2, there are two linearly independent columns of A. Therefore, any of the following $3[^3C_2 = 3]$ submatrices of order 2 can be considered as the basis matrix B because following determinants of order 2 are not equal to zero.

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} ; \begin{bmatrix} 1 & 1 \\ 2 & 5 \end{bmatrix} \text{ and } \begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix}$$

Since each of these submatrices is non-singular, by putting variables not associated with the columns of B equal to zero, all possible basic feasible solution can be obtained. Let us consider the case where $x_2 = 0$, i.e., it is not associated with the columns of B. We then have:

$$\begin{bmatrix} 1 & 1 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$ (because $x_B = B^{-1}b$)

$$= \frac{1}{3} \begin{bmatrix} 5 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 15 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

(since $A^{-1} = \frac{adj\, A}{|A|}$)

(or) $x_1 = 3$ and $x_3 = -1$

We now set $x_1 = 0$ and solve the system of equations. The resulting matrix is non – singular. Then,

$$\begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$= \frac{1}{9} \begin{bmatrix} 5 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$= \frac{1}{9} \begin{bmatrix} 15 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 5/3 \\ 2/3 \end{bmatrix}$$

(or) $x_2 = \frac{5}{3}$ and $x_3 = \frac{2}{3}$

Next we set $x_3 = 0$ and solve the system of equations. Thus,

$$\begin{bmatrix}1 & 2\\2 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}4\\5\end{bmatrix}$$

$$\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}1 & 2\\2 & 1\end{bmatrix}^{-1}\begin{bmatrix}4\\5\end{bmatrix}$$

$$= -\frac{1}{3}\begin{bmatrix}1 & -2\\-2 & 1\end{bmatrix}\begin{bmatrix}4\\5\end{bmatrix}$$

$$= -\frac{1}{3}\begin{bmatrix}-6\\-3\end{bmatrix}$$

$$= \begin{bmatrix}2\\1\end{bmatrix}$$

(or) $x_1 = 2$ and $x_2 = 1$

The summary of the solution is given below:

Basic Vector	*Basic Variables*	*Non-basic Variable*
1. $\mathbf{x_B} = -\begin{bmatrix}3\\-1\end{bmatrix}$ $x_1 = 3,\ x_3 = -1$	x_1, x_3	x_2
2. $\mathbf{x_B} = \begin{bmatrix}5/3\\2/3\end{bmatrix}$ $x_2 = 5/3,\ x_3 = 2/3$	x_2, x_3	x_1
3. $\mathbf{x_B} = \begin{bmatrix}2\\1\end{bmatrix}$ $x_1 = 2,\ x_2 = 1$	x_1, x_2	x_3

Here it may be noted that all these solutions are non – degenerate.

Example 3.2 Compute all the basic feasible solutions to the system of linear equations. (i) $4x_1 + 2x_2 + x_3 = 7$ (ii) $-x_1 + 4x_2 + 2x_3 = 14$

Solution: By setting $x_1 = 0$, the resulting square matrix of coefficients of x_2 and x_3 is:

$$A = \begin{bmatrix}2 & 1\\4 & 2\end{bmatrix}$$

This matrix is singular because $|A| = 0$. It cannot be a basis matrix because the columns are not linearly independent.

Since $x_2 = 0$, the resulting square matrix of coefficients of x_1 and x_3 is:

$$A = \begin{bmatrix}4 & 1\\-1 & 2\end{bmatrix}$$

This matrix is a basis matrix, B because $|A| \neq 0$ and so

$$x_B = B^{-1}b = \frac{1}{9}\begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 7 \\ 14 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 0 \\ 63 \end{bmatrix} = \begin{bmatrix} 0 \\ 7 \end{bmatrix}$$

(or) $x_1 = 0$ and $x_3 = 7$.

Here it may be noted that x_1, although not a non – basic variable, still has a zero value in this solution.

Finally, setting $x_3 = 0$, we get a coefficient matrix of x_1 and x_2, which mat serve as a basis:

$$A = \begin{bmatrix} 4 & 2 \\ -1 & 4 \end{bmatrix}$$

Thus, $x_B = B^{-1}b = \frac{1}{18}\begin{bmatrix} 4 & -2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 7 \\ 14 \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 0 \\ 63 \end{bmatrix} = \begin{bmatrix} 0 \\ 63/18 \end{bmatrix}$

(or) $x_1 = 0$ and $x_2 = \frac{63}{18} = \frac{7}{2}$

and again a basic variable, x_1 has a zero value.

Example 3.3 Compute all the basic feasible solutions to the LP problem.

Maximize Z = $2x_1 + 3x_2 + 4x_3 + 7x_4$ subject to the constraints

(i) $2x_1 + 3x_2 - x_3 + 4x_4 = 8$ (ii) $x_1 - 2x_2 + 6x_3 - 7x_4 = -3$

and $x_1, x_2, x_3, x_4 \geq 0$

Solution: The maximum number of basic feasible solutions to the given LP problem are $^4C_2 = 4$.

(i) Putting $x_3 = x_4 = 0$ in the constraints, we get $2x_1 + 3x_2 = 8$ and $x_1 - 2x_2 = -3$. Solving these equations for x_1 and x_2 by using matrix inversion method, we get a basic feasible solution to the given LP problem:

(i) Basic variable: $x_1 = 1, x_2 = 2$ Non – basic variable: $x_3 = x_4 = 0$

(ii) Basic variable: $x_1 = \frac{22}{9}, x_4 = \frac{7}{9}$ Non – basic variable: $x_2 = x_3 = 0$

(iii) Basic variable: $x_2 = \frac{45}{16}, x_3 = \frac{7}{16}$ Non – basic variable: $x_1 = x_4 = 0$

(iv) Basic variable: $x_3 = \frac{44}{17}, x_4 = \frac{45}{17}$ Non – basic variable: $x_1 = x_2 = 0$

The value of objective function Z at each of these solutions is given by:

(i) Z = 2(1) + 3(2) + 4(0) + 7(0) = 7

(ii) Z = 2(22/9) + 3(0) + 4(0) + 7(7/9) = 93/9

(iii) Z = 2(0) + 3(45/16) + 4(7/16) + 7(0) = 19/2

(iv) Z = 2(0) + 3(0) + 4(44/17) + 7(45/17) = 144/5

The maximum value of objective function Z = 144/5 occurs at the basic feasible solution, $x_1 = x_2 = 0, x_3 = 44/17\,, x_4 = 45/17$.

Let us sum up

We have learned about both canonical and standard forms of LP model, notion of slack and surplus variable. Also find its basic solution.

Check Your Progress

33. Find the basic feasible solution for the system of equations given below:

(a) $$2x_1 + 6x_2 + 2x_3 + x_4 = 3$$
$$6x_1 + 4x_2 + 4x_3 + 6x_4 = 2$$
$$x_j \geq 0,\;\; j = 1, 2, 3, 4$$

(b) (i) $3x_1 + x_2 - x_3 = 8,$ (ii) $x_1 + x_2 + x_3 = 4$

$$x_1,\; x_2,\; x_3\; \geq 0$$

34. Show that the following system of linear equation has a degenerate solution:

(i) $2x_1 + x_2 - x_3 = 2$ (ii) $3x_1 + 2x_2 - x_3 = 3$

35. Compute all the non-degenerate basic feasible solution of the following equation:

(i) $x_1 + 2x_2 - x_3 + x_4 = 2$ (ii) $x_1 + 2x_2 + 0.5x_3 + x_5 = 2$

Is the solution (1, ½, 0, 0, 0) a basic solution?

3.4 REDUCTION OF FEASIBLE SOLUTION TO A BASIC FEASIBLE SOLUTION

Theorem 3.1: A collection of all feasible solutions (if they exist) of an LP problem constitute a convex set.

Proof: The LP problem in its standard form is written as:
Optimize (Max or Min) Z = cx subject to the constraints Ax = b, and x ≥ 0.
Let S be the set of all feasible solutions of the system Ax = b, x ≥ 0. Now if S contains only one element, then obviously S is a convex set and hence the statement of the theorem is true. However, if x', x'' ∈ S such that x' ≠ x'' then we have:
Ax' = b, x' ≥ 0, and Ax'' = b, x'' < 0
Let there exists a point x''' such that x''' = λx' + (1 – λ)x'', 0 ≤ λ ≤ 1. In order to show that S is convex, we have to show that x''' ∈ S. In other words, the point x''' must satisfy the system Ax = b, x ≥ 0. Thus,
Ax''' = A {λx' + (1 – λ)x'' } = λ Ax' + (1 – λ) Ax''
= λ b + (1 – λ) b = b
Since x', x'' ≥ 0 and 0 ≤ λ ≤ 1, we have x''' ≥ 0. Hence x''' ∈ S and the set S is convex.

Theorem 3.2 A necessary and sufficient condition for a vector x in a convex set S to be an extreme point is that x is a basic feasible solution satisfying the system Ax = b, x ≥ 0. In other words, a point is a basic feasible solution to Ax = b if and only if it is an extreme point of the convex set of the feasible solution.

Proof: Consider the following LP problem:
Minimize Z = cx subject to the constraints Ax = b and x ≥ 0 where A is an m x n matrix of rank m.
(a) Basic solution: Let x be an extreme point of the feasible region of the convex polyhedron. Let the first p (≤ n) components x_j (j = 1, 2, …, p) of x be positive. Then:

$$a_1x_1 + a_2x_2 + \cdots + a_px_p = b \qquad \rightarrow (6)$$

where a_j (j = 1, 2, …, p) are the columns of A.

to prove that x is a basic feasible solution, we should prove that the vectirs $a_1, a_2, \dots, a_p$ associated with the positive components of x are linearly independent. And we shall do so by contradiction. Suppose the vectors $a_1, a_2, \dots, a_p$ are not linearly independent, then there must exist scalars $\lambda_1, \lambda_2, \dots, \lambda_p$ not all zero, such that:

$$\lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_p a_p = 0 \qquad \rightarrow (7)$$

Let there exist two distinct feasible solutions x' and x'' such that x = x' + x'', which violate the assumption that x is an extreme point. Let x' and x'' be defined as:

$$x_j' = \begin{cases} x_j + \delta\lambda_j & j = 1, 2, \dots, p \\ 0 & j = p+1, p+2, \dots, n \end{cases}$$

$$x_j'' = \begin{cases} x_j + \delta\lambda_j & j = 1, 2, \dots, p \\ 0 & j = p+1, p+2, \dots, n \end{cases}$$

Since $x_j > 0$ (j = 1, 2, …, p), it is possible to select any arbitary $\delta > 0$ such that: $x_j + \delta\lambda_j \geq 0 \; and \; x_j - \delta\lambda_j \geq 0$. Furthermore,

$$Ax' = \sum_{j=1}^{p} a_j x_j' = \sum_{j=1}^{p} a_j \left(x_j \pm \delta\lambda_j\right) = \sum_{j=1}^{p} a_j x_j + \delta \sum_{j=1}^{p} a_j \lambda_j = b + 0 = b$$

Similarly, Ax'' = b. Thus, the two points x' and x'' satisfy the system of equations Ax = b. Hence x' and x'' are two different feasible solutions. It can also be seen that inequalities are true if:

$$0 < \delta < Min\left\{\frac{x_j}{|\lambda_j|} \; ; \; \lambda_j \neq 0 \; ; j = 1, 2, \dots, p\right\}$$

i.e., the first p-components of x' and x'' will always be positive. Now,

x' + x'' =$2\left(x_1, x_2, \dots, x_p, 0, 0, \dots, 0\right)$ (or) $x = \frac{1}{2}x' + \frac{1}{2}x''$

This shows that an extreme point x can be expressed as a linear combination of two distinct feasible points different from x, which cannot be true. Hence vectors $a_1, a_2, \dots, a_p$ are linearly independent. Since rank of A = m, therefore (m – p) additional column vectors from $a_{p+1}, a_{p+2}, \dots, a_n$ of A can be added with their corresponding variables, together with $a_1, a_2, \dots, a_p$, to form a linearly independent set of column vectors. If needed, after rearranging the columns, let the new column vectors be $a_{p+1}, a_{p+2}, \dots, a_m$. Let B = $\left(a_1, a_2, \dots, a_p, a_{p+1}, a_{p+2}, \dots, a_m\right)$ be the new basis matrix whose columns are linearly independent. Further, $x_N = 0$ and $x_B = \left(x_1, x_2, \dots, x_p, 0, 0, \dots, 0\right)^T$. Since Ax = b, x is a feasible solution.

(b) Extreme-point Correspondence: Let x be a basic feasible solution to the system of equations Ax = b, x ≥ 0, such that x = $(x_B, 0)^T$, where $x_B = B^{-1}b$, for a non-singular matrix B called basis matrix (or basis).

Let there exist two distinct points x' and x'' satisfying Ax = b (feasible solution) such that:

x = λx' + (1 – λ)x'' ; 0 < λ < 1 → (9)

Now to prove that x is an extreme point it is sufficient to show that x' = x'' = x. Let

$$x' = \begin{bmatrix} x'_B \\ x'_N \end{bmatrix} \; and \; x'' = \begin{bmatrix} x''_B \\ x''_N \end{bmatrix}$$

It may be notes that $x'_N \geq 0$ and $x''_N \geq 0$. Then substituting in equation (9), we get

$$\begin{bmatrix} x_B \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} x'_B \\ x'_N \end{bmatrix} + (1 + \lambda) \begin{bmatrix} x''_B \\ x''_N \end{bmatrix}$$

$$x_B = \lambda x'_B + (1 - \lambda) x''_B$$

→ (10)

$$0 = \lambda x'_N + (1 - \lambda) x''_N$$

→ (11)

Since 0 < λ < 1 and $x'_N, x''_N \geq 0$, from equation (11) we have $x'_N = x''_N = 0$. Again since x' and x'' satisfy Ax' = b and Ax'' = b, we have, $x'_B = x_B$.

Similarly $x''_B = x_B$. It follows that x = x' = x''. This is a contradiction for x' ≠ x''. Hence, x is an extreme point.

Theorem 3.3

(a) If the convex set of the feasible solutions of the system of equations Ax = b, x ≥ 0, is a convex polyhedron, then at least one of the extreme points gives an optimal solution.

(b) If the optimal solution occurs at more than one extreme point, then the value of the objective function will be the same for all convex combinations of these extreme points.

Proof: (a) Let the points $x_1, x_2, \ldots, x_p$ represent the extreme points of the feasible region of the convex polyhedron of the LP problem:

Maximize Z = cx subject to the constraints Ax = b, and x ≥ 0

Suppose x' be an extreme point among $x_1, x_2, \ldots, x_p$, at which the maximum value of the objective function Z occurs. Let it be Z* = cx^i. Let x' be any point of the feasible region and Z' be the value of objective function at x'. Then Z' = cx'. Since x' is not an extreme point, there exist scalars $\lambda_1, \lambda_2, \dots, \lambda_p$ not all zero such that:

$$x' = \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_p x_p$$

where $\sum_{j=1}^{p} \lambda_j = 1\ ;\ \lambda_j \geq 0\ ; j = 1, 2, \dots, p$

Substituting for x' in Z' = cx', we get: Z' = $c\{\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_p x_p\}\ cx^i$, i.e., $Z' \leq Z^*$

This result shows that at the optimum solution, the extreme point solution is better than any other feasible solution.

(b) Let $x_1, x_2, \dots, x_k\ (k \leq p)$ be the extreme points of the feasible region at which objective function has equal and optimum value. i.e.,

$$Z^* = cx_1 = cx_2 = \cdots = cx_k$$

Further, let $x = \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k$ where $\sum_{j=1}^{k} \lambda_j = 1\ ;\ \lambda_j \geq 0\ ; j = 1, 2, \dots, k$

Then, $cx = \{c\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k\} = \lambda_1 cx_1 + \lambda_2 cx_2 + \cdots + \lambda_k cx_k$

$= (\lambda_1 + \lambda_2 + \cdots + \lambda_k) Z^* \qquad = Z^*$

Hence, the theorem is proved.

Theorem 3.4

If a standard LP problem with constraints Ax = b, x ≥ 0, where A is an m × n matrix of rank m (£ n) has a feasible solution, then it also has a basic feasible solution.

Proof: Suppose that $x^T = (x_1, x_2, \dots, x_n)^T$ be a feasible solution to the system Ax = b. We rearrange the components of x such that the first p (≤ n) components x_j (j = 1, 2, …, p) of x are positive and the remaining n – p components are zero. The feasible solution can then be written as:

$$\sum_{j=1}^{p} a_j x_j = a_1 x_1 + a_2 x_2 + \cdots + a_p x_p = b$$

Where $a_1, a_2, \dots, a_p$ are the first p – column of A associated with the positive variables $x_1, x_2, \dots, x_p$. Two cases arise about the vectors a_j (j = 1, 2, …, p).

Case 1: Column Vectors $a_1, a_2, \dots, a_p$ are Linearly Independent. If vectors $a_1, a_2, \dots, a_p$ are linearly independent, then p ≤ m. If p = m (i.e., rank of matrix A), then the given solution is a unique non-degenerate basic feasible solution. On the other hand if p < m, then there are m – p additional column of A, which together with $a_1, a_2, \dots, a_p$ columns, form a basis (i.e., linearly independent system) for E^m. Thus, a degenerate basic feasible solution can be formed assigning zero value to the m – p variables, i.e., $x_{p+1} = x_{p+2} = \cdots = x_m = 0$, corresponding to selected m – p columns of A.

Case 2: Column Vectors $a_1, a_2, \ldots, a_p$ are Linearly Dependent. If column vectors $a_1, a_2, \ldots, a_p$ are linearly dependent, then p > m. Thus we have to reduce the number of positive variables step by step until the columns associated with the positive variables are linearly independent. If $a_1, a_2, \ldots, a_p$ are linearly dependent, then there exist scalars $\lambda_1, \lambda_2, \ldots, \lambda_p$, with at least one λ_j positive such that $\lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_p a_p = 0$. Let x_r be the variable selected first to be reduced to zero. Thus, the vector a_r amongst p vectors, for which λr ≠ 0, can be expressed in terms of the remaining p – 1 vectors as follows:

$$\lambda_r a_r = -\sum_{j \neq r}^{p} \lambda_j a_j \; ; j = 1, 2, \ldots, p$$

(or) $$a_r = -\sum_{j \neq r}^{p} \left(\frac{\lambda_j}{\lambda_r}\right) a_j \; ; j = 1, 2, \ldots, p \qquad \rightarrow (12)$$

We substitute expression a_r from equation (12) in the expression:

$$-\sum_{j \neq r}^{p} \lambda_j a_j = b \; ; \; x_j > 0$$

and obtain $$-\sum_{j \neq r}^{p} \left(x_j - x_r \left(\frac{\lambda_j}{\lambda_r}\right)\right) a_j = b \; ; j = 1, 2, \ldots, p \qquad \rightarrow (13)$$

In this manner a feasible solution with at the most p – 1 positive variables is obtained. To ensure that these p – 1 variables be non-negative, we choose a column vector a_r in such a way that:

$$x_j - x_r \left(\frac{\lambda_j}{\lambda_r}\right) \geq 0 \; ; j = 1, 2, \ldots, p \; ; j \neq r$$

Obviously, for any j for which $\lambda_j = 0$, the above condition is satisfied. But if $\lambda_j \neq 0$, then we get:

$$\frac{x_j}{\lambda_j} - \frac{x_r}{\lambda_r} \geq 0 \; ; \; \lambda_j > 0 \qquad \rightarrow (14a)$$

and $$\frac{x_j}{\lambda_j} - \frac{x_r}{\lambda_r} \leq 0 \; ; \; \lambda_j < 0 \qquad \rightarrow (14b)$$

These two inequalities provide a method of selecting the vector ar such that p – 1 variables in equation (13) are non-negative. The maximum value of $\frac{x_r}{\lambda_r}$, for which equation (14) is satisfied and which will help in selecting a_r, is given by:

$$\frac{x_r}{\lambda_r} = Min\left\{\frac{x_j}{\lambda_j}, \lambda_j > 0\right\}$$

For this positive value of $\frac{x_r}{\lambda_r}$, each variable in equation (13) will be non-negative and a feasible solution can be obtained that has at most (p – 1) positive variables.

If the columns corresponding to these positive variables are linearly independent, then the current solution is a basic feasible solution. Otherwise, the process of eliminating the positive variables one by one is carried out till a feasible solution is obtained, such that columns of A corresponding to the positive variables are linearly independent. Then, Case I would apply and we would have a basic feasible solution.

Working Rule:

1) Compute all $\frac{x_r}{\lambda_r}$ (j = 1, 2, . . ., p) for which $\lambda_j > 0$ and choose the minimum value.

2) Reduce to zero the value of the variable corresponding to the minimum ratio $\frac{x_r}{\lambda_r}$ by using the relationship

$$a_r = \sum_{j\neq r}^{p} \left(\frac{\lambda_j}{\lambda_r}\right) a_j \ ; \ \lambda_r \neq 0$$

Here at least one λ_j must be positive; however, if all $\lambda_j \leq 0$, then multiply the equation: $\lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_p a_p = 0$ by – 1 and obtain new values of $\lambda_j \geq 0$.

3) The values of the new variables are given by $x_j' = x_j - \left(\frac{x_r}{\lambda_r}\right)\lambda_j$.

Example 3.4 Let $x_1 = 2, x_2 = 4 \ and \ x_3 = 1$ be a feasible solution to the system of equations: (i) $2x_1 - x_2 + 2x_3 = 2$ (ii) $x_1 + 4x_2 = 18$

Reduce the given feasible solution to a basic feasible solution.

Solution: We first write the system of equations in matrix notation as:

$$\begin{bmatrix} 2 & -1 & 2 \\ 1 & 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 18 \end{bmatrix}$$

where $A = \begin{bmatrix} 2 & -1 & 2 \\ 1 & 4 & 0 \end{bmatrix}$; $b = \begin{bmatrix} 2 \\ 18 \end{bmatrix}$ $and \ x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

This system of equations can also be expressed as:

$$a_1 x_1 + a_2 x_2 + a_3 x_3 = b$$

Since it is given that $x_1 = 2, x_2 = 4 \ and \ x_3 = 1$, we have $2a_1 + 4a_2 + a_3 = b$

where $a_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$; $a_2 = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$ $and \ a_3 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ are column vactors of A.

Since rank (A) = 2, only two out of three column vectors a_1, a_2 and a_3 are linearly independent. Assuming that these vectors are linearly dependent, we express one of them as a linear combination of the remaining two as:

$$a_3 = \lambda_1 a_1 + \lambda_2 a_2$$

$$\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \lambda_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \lambda_2 \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

This gives $2 = 2\lambda_1 - \lambda_2$ and $0 = \lambda_1 + 4\lambda_2$ where λ_1 and λ_2 are scalars, not all zero. On solving these two equations, we get $\lambda_1 = \frac{8}{9}$ and $\lambda_2 = -\frac{2}{9}$. Substituting values of λ_1 and λ_2 in the linear combination, we get:

$a_3 = \lambda_1 a_1 + \lambda_2 a_2 = \frac{8}{9} a_1 - \frac{2}{9} a_2$ (or) $\frac{8}{9} a_1 - \frac{2}{9} a_2 - a_3 = 0$

where $\lambda_1 = \frac{8}{9}, \lambda_2 = -\frac{2}{9}$ and $\lambda_3 = -1.$

To reduce the number of positive variables, the vector to be removed is chosen in accordance with

$$\frac{x_r}{\lambda_r} = Min\left\{\frac{x_j}{\lambda_j}, \lambda_j > 0\right\} = Min\left\{\frac{x_1}{\lambda_1}, \frac{x_2}{\lambda_2}, \frac{x_3}{\lambda_3}\right\} = Min\left\{\frac{2}{8/9}, \frac{4}{-2/9}, \frac{1}{-1}\right\} = \frac{9}{4}$$

Since $\frac{x_r}{\lambda_r} = \frac{9}{4}$ corresponds to a vector a_1, it should be removed in order to obtain a new solution with two non – negative variables. The values of the new variables are given by:

$$x_j' = x_j - \frac{x_r}{\lambda_r}(\lambda_j)$$

This gives $x_1' = x_1 - \frac{x_r}{\lambda_r}(\lambda_1) = 2 - \left(\frac{9}{4}\right)\left(\frac{8}{9}\right) \quad = 0$

$$x_2' = x_2 - \frac{x_r}{\lambda_r}(\lambda_2) = 4 - \left(\frac{9}{4}\right)\left(-\frac{2}{9}\right) = \frac{9}{2}$$

$$x_3' = x_3 - \frac{x_r}{\lambda_r}(\lambda_3) = 1 - \left(\frac{9}{4}\right)(-1) = \frac{13}{4}$$

The new solution (0, 9/2, 13/4), so obtained, is also a feasible solution. As the two column vectors a_2 and a_3 associated with non-zero variables x_2 and x_3 are linearly independent, therefore, the required basic feasible solution is: $x_1 = 0, x_2 = \frac{9}{2}, x_3 = \frac{13}{4}$. This result can be verified by substituting the values of x_1, x_2 and x_3 in the equation, $a_1 x_1 + a_2 x_2 + a_3 x_3 = b.$

Exapmle 3.5 Show that the feasible solution: $x_1 = 1, x_2 = 0, x_3 = 1$ and Z = 6 to the system of equations: $x_1 + x_2 + x_3 = 2$ and $x_1 - x_2 + x_3 = 2$ with Max Z = $2x_1 +$

$3x_2 + 4x_3$ is not a basic feasible solution.

Solution: We first write the system of equations in matrix notation as:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

where $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$; $b = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

This system of equations can also be expressed as:

$$a_1x_1 + a_2x_2 + a_3x_3 = b$$

Since it is given that $x_1 = 1, x_2 = 0$ and $x_3 = 1$, we have $a_1 + a_3 = b$

where $a_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$; $a_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ are column vactors of A.

Since rank (A) = 2, only two out of three column vectors a_1, a_2 and a_3 are linearly independent. Assuming that these vectors are linearly dependent, we express one of them as a linear combination of the remaining two as:

$$a_3 = \lambda_1 a_1 + \lambda_2 a_2$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

This gives $1 = \lambda_1 + \lambda_2$ and $1 = \lambda_1 - \lambda_2$ where λ_1 and λ_2 are scalars, not all zero.

On solving these two equations, we get $\lambda_1 = 1$ and $\lambda_2 = 0$. Substituting values of λ_1 and λ_2 in the linear combination, we get:

$$a_3 = \lambda_1 a_1 + \lambda_2 a_2 = a_1$$

(a) We take $a_1 - a_3 = 0$, where $\lambda_1 = 1, \lambda_2 = 0$ and $\lambda_3 = -1$. To reduce the number of positive variables, the vector to be removed ischosen in accordance with the theorem 3.4,

i.e.,

$$\frac{x_r}{\lambda_r} = Min \left\{ \frac{x_j}{\lambda_j}, \lambda_j > 0 \right\} = Min \left\{ \frac{x_1}{\lambda_1}, \frac{x_2}{\lambda_2}, \frac{x_3}{\lambda_3} \right\} = Min \left\{ \frac{1}{1}, \frac{1}{-1} \right\} = 1$$

Since $\frac{x_r}{\lambda_r} = 1$ corresponds to the vector a_1, it should be removed to obtain a new solution with not more than two non – negative variables. The vlues of the new variables are given by:

$$x_j' = x_j - \frac{x_r}{\lambda_r}(\lambda_j)$$

This gives $x_1' = x_1 - \frac{x_r}{\lambda_r}(\lambda_1) = 1 - 1(1) \quad = 0$

$$x_2' = x_2 - \frac{x_r}{\lambda_r}(\lambda_2) = 0 - 1(0) \quad = 0$$

$$x_3' = x_3 - \frac{x_r}{\lambda_r}(\lambda_3) = 1 - 1(-1) = 2$$

(b) By taking $a_3 - a_1 = 0$, we shall have $\lambda_1 = 1, \lambda_2 = 0 \; and \; \lambda_3 = 1$. In this case:

$$\frac{x_r}{\lambda_r} = Min \left\{\frac{x_j}{\lambda_j}, \lambda_j > 0\right\} = 1$$

Thus, the vector a_3 can be removed to obtain a new solution with not more than two non – negative variables. The vlues of the new variables are given by:

$$x_1' = x_1 - \frac{x_r}{\lambda_r}(\lambda_1) = 1 - 1(-1) = 2$$

$$x_2' = x_2 - \frac{x_r}{\lambda_r}(\lambda_2) = 0 - 1(0) \quad = 0$$

$$x_3' = x_3 - \frac{x_r}{\lambda_r}(\lambda_3) = 1 - 1(1) \quad = 0$$

The new solutions (0, 0, 2) and (2, 0, 0), so obtained, have two linearly independent column vectors, and hence both of these solutions are basic feasible. However, these two solutions are different from the given (1, 0, 1) solution. Thus, the given solution is not basic.

3.5 ALTERNATIVE OPTIMAL SOLUTIONS

The optimal value of the objective function of an LP problem is always unique but the set of basic variables yielding this optimal value need not be unique. There may be two or more basic feasible solutions that give the same value of the objective function.

Theorem 3.5: Given an optimal basic feasible solution to an LP problem and for some column a_j of A but not in B, $c_j - z_j = 0$, $y_{ij} \leq 0$ for all i = 1, 2, ..., m or $y_{ij} > 0$ for at least one i, then a_j may be inserted into the basis to yield an alternative optimal solution.

Proof: Left as an exercise for the reader.

Remark: For selecting the column vector ar of A but not in B to enter into the basis, the following formula may be used:

(i) $c_r - z_r = Max\{c_j - z_j\}$ for maximization problem

(ii) $c_r - z_r = Min\{c_j - z_j\}$ for minimization problem.

3.6 UNBOUNDED SOLUTION

Theorem 3.6: Given any basic feasible solution to an LP problem. If for this solution there is some column aj in A but not in B for which $c_j - z_j > 0$ and $y_{ij} \leq 0$ (i = 1, 2, . . ., m), then the problem has an unbounded solution, if the objective function is to be maximized.

Proof: If we introduce any column vector a_j of A for which all $y_{ij} \leq 0$ (i = 1, 2, ..., m) into the basis matrix B, then a_j must enter into the basis either at a negative level or at the zero level. Thus, a new basic feasible solution will be infeasible, unless aa_j enters at a zero level. Let x_B be the basic feasible solution to the given LP problem so that:

$$Bx_B = b \ (or) \ \sum_{i=1}^{m} x_{Bi}\beta_i = b \qquad \rightarrow (15)$$

The value of the bojective function at this solution is given by:

$$Z = c_B x_B = \sum_{i=1}^{m} c_{Bi} x_{Bi}$$

By adding and subtracting λa_j in equation (15), where λ be any scalar and a_j the vector entering the basis, we have:

$$\sum_{i=1}^{m} c_{Bi}\beta_i + \lambda a_j - \lambda a_j = b$$

$$\sum_{i=1}^{m} x_{Bi}\beta_i + \lambda a_j - \lambda \sum_{i=1}^{m} y_{ij}\beta_i = b$$

$$\sum_{i=1}^{m} (x_{Bi} - \lambda y_{ij})\beta_i + \lambda a_j = b \qquad \rightarrow (16)$$

Equation (16) represents a new solution and is given by:

$$\hat{x}_{Bi} = x_{Bi} - \lambda y_{ij} \ and \ \hat{x}_{B_{m+1}} = \lambda \, ; i = 1, 2, \dots, m$$

Since all $y_{ij} \leq 0$, we get $\hat{x}_{Bi} \geq 0$ when $\lambda > 0$. Thus equation (16) is a feasible solution in which m + 1 variables may be at a positive level. But in general, this may not be the basic solution because there are more positive variables than constraints.

The new value of the objective function Z for this solution is given by:

$$\hat{Z} = \sum_{i=1}^{m} c_{Bi}\hat{x}_{Bi} = \sum_{i=1}^{m} c_{Bi}(x_{Bi} - \lambda y_{ij}) + \lambda c_j$$

$$= \sum_{i=1}^{m} c_{Bi}x_{Bi} + \lambda\left(c_j - \sum_{i=1}^{m} c_{Bi}\lambda y_{ij}\right)$$

$$= Z + \lambda(c_j - z_j)\ ;\ \lambda = \frac{x_{Br}}{y_{rj}}$$

For a sufficiently large value of λ and $c_j - z_j > 0$, the value of Z can be increased up to infinity. Similarly, in the case of a minimization LP problem we make λ sufficiently small so that the value of Z can be decreased up to infinity, for $c_j - z_j < 0$. Such a solution is unbounded.

3.7 OPTIMALITY CONDITION

In this section we shall develop the criterion of optimality of an LP problem solution, i.e., when the iterative procedure of solving an LP problem may be stopped.

Theorem 3.7: Given a basic feasible solution to the LP problem, $x_B = B^{-1}b = (x_{B_1}, x_{B_2}, \dots, x_{B_m})$ and Z = Z* such that $c_j - z_j \leq 0$ for every column a_j in A but not in B. Then Z is the maximum value of objective function Z and x_B is an optimal basic feasible solution.

Proof: Let $x_B = B^{-1}b$ be a feasible solution to the given LP problem and $Z = c_B x_B$ be the corresponding value of objective function.

Let $x_j \geq 0$ (j = 1, 2, …, n) be any feasible solution to the same LP problem. Then the system Ax = b can be expressed in terms of column vectors of A as:

$$a_1x_1 + a_2x_2 + a_nx_n = b \qquad \rightarrow (17)$$

The value of the objective function at this solution is given by:

Z* = cx = $c_1x_1 + c_2x_2 + \cdots + c_nx_n$

Any column vector a_j of A can be expressed as linear combination of column vectors β_i of B. i.e., $a_j = \sum_{i=1}^{m} y_{ij}\beta_i$

Substituting value of a_j (j = 1, 2, …, n) in equation (18), we have:

$$x_1 \sum_{i=1}^{m} y_{i1}\beta_i + x_2 \sum_{i=1}^{m} y_{i2}\beta_i + \cdots + x_n \sum_{i=1}^{m} y_{in}\beta_i = b$$

$$\left\{\sum_{i=1}^{m} x_j y_{1j}\right\}\beta_1 + \left\{\sum_{i=1}^{m} x_j y_{2j}\right\}\beta_2 + \cdots + \left\{\sum_{i=1}^{m} x_j y_{mj}\right\}\beta_m = b$$

Let for every column a_j in A but not in B, $c_j - z_j \leq 0$. Now we prove that Z is more than value of the objective function Z* for any other feasible solution. For all columns vector of A in B, i.e., $a_j \in B$, we have:

$y_j = B^{-1}a_j = B^{-1}\beta_i = e_i$ (unit vector) provided a_j is in column i for B. Then:

$$c_j - z_j = c_j - c_B y_j = c_j - c_B e_i = c_j - c_j = 0$$

Thus $c_j - z_j = 0$ for all columns of A in B. Applying the assumption that $c_j - z_j \leq 0$ for all columns in A, then from equation (17), we have:

$\sum_{i=1}^{m}(c_j - z_j)x_j \leq 0$

$$\sum_{j=1}^{n} c_j x_j \leq \sum_{j=1}^{n} z_j x_j \qquad = \sum_{j=1}^{n} x_j \left\{\sum_{i=1}^{m} c_{Bi} y_{ij}\right\}$$

$$= \left\{\sum_{j=1}^{n} x_j y_{1j}\right\}c_{B1} + \left\{\sum_{j=1}^{n} x_j y_{2j}\right\}c_{B2} + \cdots + \left\{\sum_{j=1}^{n} x_j y_{mj}\right\}c_{Bm}$$

$$= x_{B_1}c_{B_1} + x_{B_2}c_{B_2} + \cdots + x_{B_m}c_{B_m} = Z$$

(or) Z* ≤ Z

This complets the proof of the theorem.

3.8 SOME COMPLICATIONS AND THEIR RESOLUTION

In this section we will discuss some of the complications that may arise in applying the simplex method, and their resolution.

3.8.1 Unrestricted Variables

In many situations, one or more of the variables can have either positive, negative or zero value. Such variables are called unrestricted variables. Since the use of the simplex method requires that all the decision variables must be non-negative at each iteration, therefore in order to convert an LP problem that involves unrestricted variables into an equivalent problem having only restricted variables, we have to express each of unrestricted variables as the difference of two non-negative variables.

Suppose variable x_r be unrestricted in sign. We define two new variables say x_r' and x_r'' such that: $x_r = x_r' - x_r''$; $x_r', x_r'' \geq 0$

If $x_r' \geq x_r''$, then $x_r \geq 0$ and if $x_r' \leq x_r''$, then $x_r \leq 0$. Also if $x_r' = x_r''$, then $x_r = 0$. Hence, depending on the values of x_r' and x_r'', x_r can have any sign.

The unrestricted variable must be replaced by the two new variables, both in the objective function and the constraints set of an LP problem. That is, if we have the following LP problem:

Maximize Z = $\sum_{j \neq r}^{n} c_j x_j + c_r x_r$ subject to the constraints

$\sum_{j \neq r}^{n} a_{ij} x_j + a_{ir} x_r = b_i$; $i = 1, 2, \ldots, m$ and $x_j \geq 0$, x_r unrestricted in sign ; j = 1, 2, …, n, j ≠ r, then it can be converted into equivalent standard form as follows:

Maximize Z = $\sum_{j \neq r}^{n} c_j x_j + c_r (x_r' - x_r'')$ = subject to the constraints

$\sum_{j \neq r}^{n} a_{ij} x_j + a_{ir} (x_r' - x_r'') = b_i$; $i = 1, 2, \ldots, m$ and $x_j, x_r', x_r'' \geq 0$; j = 1, 2, …, n, j ≠ r.

Since the vectors corresponding to the variables x_r' and x_r'' are linearly dependent, both of them cannot simultaneously appear in the basis. Thus, any of the following three cases may arise at the optimal solution:

(i) $x_r' = 0 \quad \rightarrow x_r = -x_r''$

(ii) $x_r'' = 0 \quad \rightarrow x_r = x_r'$

(iii) $x_r' = x'' = 0 \rightarrow x_r = 0$

This indicates that the value of x_r is determined by x_r' and x_r''.

3.8.2 Degeneracy and its Resolution

While applying the simplex method for solving an LP problem, the minimum ratio is calculated at each iteration in order to decide the basic variable to leave the basis. Sometimes this ratio is not uniquely determined or values of one or more basic variables in the solution values column become zero. This situation raises the problem of degeneracy.

Degeneracy may occur either at the first iteration, or at some subsequent iteration. The simplex method always starts with basis matrix B, the initial basic feasible solution is given by$x_B = B^{-1}b = IB = b$. Thus, the degeneracy may occur at the first iteration, only if at least one basic variable appears with zero value in x_B column.

The degeneracy at the subsequent iteration will occur only if the minimum ratio

$\left\{\frac{x_{Bi}}{y_{ik}}\ ;\ y_{ik} > 0\right\}$ is same for two or more current basic variables. Let the minimum ratio values:

$$\frac{x_{B1}}{y_{1k}} = \frac{x_{B2}}{y_{2k}} = \cdots = \frac{x_{Bp}}{y_{pk}}$$

be the same. Then an outgoing vector cannot be uniquely determined. If we select any of the basic variable as an outgoing variable, then the remaining (p – 1) variables appear with zero value at the next iteration and, therefore, an arbitrary choice of the outgoing variable may cause the next solution to be degenerate. Also, in this case the value of Z remains unimproved.

Suppose a_k is the key column vector in the simplex table in which at least one $y_{ik} \geq 0$ and $c_j - z_j > 0$ (maximization case), then in the next solution we shall obtain an improved value of objective function Z and the solution shall be non-degenerate, provided $x_{Bi} = 0$, for some $y_{ik} \leq 0$. But $x_{Bi} = 0$ for some $y_{ik} > 0$, then the next solution would be degenerate with unimproved value of Z, i.e.,

$\hat{Z} = Z + \frac{x_{Br}}{y_{rk}}(c_k - z_k) = Z$, since $\frac{x_{Br}}{y_{rk}} = 0$ for some r.

Cycling: If at any subsequent iteration the value of two or more basic variables is zero (i.e., $x_{Bi} = 0$ for some i) and $y_{ik} > 0$, then the minimum ratio will be zero corresponding to these variables. This may cause the simplex method to cycle indefinitely. That is, the solution obtained in one iteration may appear again after few more iterations and, therefore, no optimal solution will actually be arrived at.

Resolution of degeneracy:

There are two methods of resolving degeneracy:

1) Perturbation Method, and

2) Generalized Simplex Method

Perturbation Method:

Let us consider the following linear programming problem: Maximize Z = cx subject to Ax = b; and x ≥ 0 where c, $x^T \in E^n$, b is an (m × 1) matrix and A is an (m × n) matrix.

If x_{Bi} (i = 1, 2, ..., m) represents a basic feasible solution of the given LP problem, then for some basis formed from the columns a_i in A, we have:

$$b = \sum_{i=1}^{m} x_{Bi} a_i$$

This solution will be degenerate only if at least one $x_{Bi} = 0$. This degeneracy occurs because of some basis formed from the columns of A. We need not have positive value of each basis vector in order to write b as a linear combination of x_{Bi}. In other words, vector b that lies on an edge of the convex cone, determined by its vectors and the corresponding solution, would be non-degenerate once the vector b lies inside the convex cone. Hence, if b is slightly changed (perturbed) to b(ε), in such a way that it lies inside the convex cone determined by its basis, the corresponding solution would be non-degenerate.

Let B be the basis matrix at any iteration. The solution at this iteration would be given by $x_B = B^{-1}b$. Suppose that we replace b > 0 of the given LP problem by:

$$b = b(\varepsilon) + \sum_{j=1}^{n} a_j \varepsilon^j$$

where $\varepsilon > 0$ is an arbitrary small positive number and a_j are the columns of A. Then the number e is chosen in such a way that it gives a non-degenerate basic feasible solution to the following perturbed LP problem:

Maximize Z = cx subject to Ax = b(ε); and x ≥ 0

For breaking the tie, it will be necessary to have an explicit value of ε. However, it is assumed that $0 < \varepsilon < \varepsilon_{max}$, where ε_{max} denotes the maximum permissible value of ε, depending on the nature of problem. After solving the LP problem if e is equated zero, then the solution to original LP problem can be obtained.

If the basis B of the original problem is retained and b is replaced by b(ε), then the basic feasible solution to the perturbed LP problem is given by:

$$x_B(\varepsilon) = B^{-1}b(\varepsilon) = B^{-1}\left(b + \sum_{j=1}^{n} a_j \varepsilon^j\right) = B^{-1}b + \sum_{j=1}^{n} B^{-1}a_j \varepsilon^j = x_B + \sum_{j=1}^{n} y_j \varepsilon^j$$

Let the basis matrix B consist of the first m columns of A denoted by y_j (j = 1, 2, ..., m). Then, y_j obviously represents a unit vector e_j with 1 at jth position. Thus,

$$x_B(\varepsilon) = x_B + \sum_{j=1}^{m} \varepsilon^j e_j + \sum_{j=m+1}^{n} y_j \varepsilon^j$$

Thus, it is possible to have $x_{Bi}(\varepsilon) > 0$, even if $x_{Bi} = 0$ because ε^j is positive and its higher order terms in ε cannot exceed ε^j and therefore cannot be less than zero.

Given a non-degenerate basic feasible solution to the perturbed problem $x_B(\varepsilon) > 0, 0 < \varepsilon < \varepsilon_{max}$, the value of objective function $Z(\varepsilon)$ for the given solution is given by$Z(\varepsilon) = c_B x_B(\varepsilon)$. Substituting the value of $x_B(\varepsilon)$, we get:

$$Z(\varepsilon) = c_B x_B + c_B \sum_{j=1}^{m} \varepsilon^j e_j + c_B \sum_{j=m+1}^{n} y_j \varepsilon^j$$

Here, it may be noted that only b was perturbed and not A, therefore, there will be no change in $y_j = B^{-1} a_j$. Also there is no change in the cost vector c. Hence $c_j - z_j = c_j - c_B y_j$ are the same for the perturbed problem as well as the original LP problem. At any iteration, simplex table in both the cases differ only in x_B column. Thus, once the optimal basic feasible solution to the perturbed LP problem has been obtained, the same can also be obtained to the original LP problem by letting $\varepsilon = 0$.

Selection of the vector leaving the basis: If a_k is the key column and all $y_{jk} \leq 0$, then there is an unbounded solution to the perturbed LP problem and also to the original LP problem. But if at least one $y_{jk} > 0$, then the column vector to be removed from the basis is selected by calculating the ratio:

$$\frac{x_{Br}}{y_{rk}} = Min\left\{\frac{x_{Bi}(\varepsilon)}{y_{ik}}, y_{ik} > 0\right\} = Min\left\{\frac{x_{Bi}(\varepsilon)}{y_{ik}} + \frac{e^i}{y_{ik}} + \sum_{j=m+1}^{n} \varepsilon e^j \left(\frac{y_{ij}}{y_{ik}}\right)\ ;\ y_{ik} > 0\right\}$$

Let us sum up

We have learned about the reduction of any feasible to a basic feasible solution, alternative optimal solution, unbounded solution,optimality condition ,some complications and their resolution.

Check Your Progress

36. Obtain all the basic feasible solution of the following system of linear equation:

$x_1 + 2x_2 + x_3 = 4$ and $2x_1 + x_2 + 5x_3 = 5$

37.. What do you mean by an optimal basic feasible solution to an LP problem? Is the solution: $x_1 = 1, x_2 = \frac{1}{2}, x_3 = x_4 = x_5 = 0$, a basic solution of the equation.

38. Consider the system of equation: (i) $x_1 + 2x_2 + 4x_3 + x_4 = 7$;

(ii) $2x_1 - x_2 + 3x_3 - 2x_4 = 4$.Here $x_1 = 1, x_2 = 1, x_3 = 1$ and $x_4 = 0$ is a feasible solution. Reduce this feasible solution to two different basic fusible solution.

39. If $x_1 = 2, x_2 = 3, x_3 = 1$ be a feasible solution of the following LP problem, then find the feasible solution: $Max\ Z = x_1 + 2x_2 + 4x_3$

Subject to $2x_1 + x_2 + 4x_3 = 11$

$3x_1 + x_2 + 5x_3 = 14$

and $x_1,\ x_2,\ x_3\ \geq 0$

Unit Summary

Finding and evaluating all basic feasible solutions of an LP problem with more than two variables by using graphical method becomes difficult and complicated. Thus, an efficient method called the simplex method was developed by G.B. Dantzig in 1947 for solving a general class of LP model. This method is an iterative procedure of moving from one extreme point to another of the solution space. It leads to the optimal solution point and/or indicates that there exists an unbounded solution, in a finite number of steps.

we discussed as to how an LP model can be stated in its canonical and standard form, along with certain important theorems that help to understand the procedure of

(i) reducing a feasible solution to a basic feasible solution,

(ii) improving a basic feasible solution,

(iii) conditions of an alternative and unbounded solution,

(iv) optimality condition, and

(v) certain complications while applying the simplex method and their resolution

Glossary

- ε_{max} - the maximum permissible value of ε.
- Optimize - maximize or minimize

Self- Assessment Questions

1. What is meant by a basic solution of an LP problem?

2. What is meant by a basic solution to the system of m linear non homogeneous equations in n unknowns (m < n) Ax = b?

3. When is a basic solution to Ax = b said to be degenerate?

4. Explain the meaning of basic feasible solution.

5. Define: Basic feasible solution, optimum solution, optimum basic feasible solution.

6. Define (i) Feasible solution; (ii) Basic solution; (iii) Basic feasible solution; (iv) Unbounded solution.

7. (a) Define a basic solution to a given system of m simultaneous linear equations in n unknowns.

(b) How many basic feasible solutions are there to a given system of 3 simultaneous linear equations in 4 unknowns.

8. What are slack and surplus variables?

9. What is the effect of converting the inequalities in the constraints into equalities by adding slack and surplus variables in the objective function?

10. Write the standard form of an LP problem in the matrix form.

ACTIVITIES

1. Prove that the set of all feasible solutions to an LP problem is a convex set.

2. Establish that every vertex of the convex set of feasible solutions is a basic feasible solution.

3. Show that every basic feasible solution to an LP problem corresponds to an extreme point of the convex set of the feasible solutions.

4. Prove that the objective function of an LP problem assumes its optimum value at an extreme point of the convex set generated by the set of all feasible solutions.

5. Prove that if the objective function assumes its optimal value at more than one extreme points, then every convex combination of these extreme points also gives the optimal value.

Suggested Readings

1. J. K. Sharma, *Operations Research*, Theory and Applications, Third Edition (2007) Macmillan India Ltd
2. Hamdy A. Taha, *Operations Research,* (seventh edition) Prentice - Hall of India Private Limited, New Delhi, 1997.
3. F.S. Hillier & J.Lieberman *Introduction to Operation Research* (7th Edition) Tata-McGraw Hill ompany, New Delhi, 2001.
4. Beightler. C, D.Phillips, B. Wilde ,*Foundations of Optimization* (2nd Edition) Prentice Hall Pvt Ltd., New York, 1979
5. S.S. Rao - *Optimization Theory and Applications*, Wiley Eastern Ltd. New Delhi. 1990

UNIT- IV

REVISED SIMPLEX METHOD

REVISED SIMPLEX METHOD

Objectives:

After studying this unit students should learn to derive two standard forms of the revised simplex method and their computational procedure. Develop a knowledge base about relevant information required at each iteration of the revised simplex method. Appreciate the use of revised simplex method in comparison to the usual simplex method.

Use modified simplex method to solve any LP problem in which basic variables value is restricted with both lower and upper bounded value. Appreciate certain modifications required in the feasibility condition of the simplex method before solving any bounded variables LP problem.

4.1 INTRODUCTION

The revised simplex method is another efficient method, developed by G B Dantzig, for solving LP problems. It is efficient in the sense that at each iteration, we need not recompute values of all the variables, namely: $y_j, c_j - z_j$, x_B and Z while moving from one iteration to next in search of an improved solution of an LP problem. In simplex method, at each iteration it was necessary to calculate $c_j - z_j$ corresponding to non basic variable columns in order to decide whether the current solution is optimal or not. If not, then in order to select the non-basic variable to enter into the basis matrix B, we first need to know $y_j = B^{-1}a_j$, where y_j refers to the updated column a_j in the simplex table. If all $y_j \leq 0$, then the optimal solution is unbounded. Otherwise, apply the minimum ratio rule to decide which basic variable should leave the basis. Update, the basis matrix B by replacing an outgoing vector with an incoming vector.

In the revised simplex method we only need to recompute values of B^{-1}, x_B, $c_B B^{-1}$ and Z. Value of all these variables can be computed directly from their definition provided B^{-1} is known. At each iteration, B^{-1} is calculated from its previous value when only one yj is changed at each iteration for which the non basic variable is entered into the basis. Thus, the relevant information to be known at each iteration of the revised simplex method are:

(i) Coefficient of non-basic variables in the objective function, and

(ii) Coefficient of the variable to be entered into the basis in the set of constraints.

4.2 STANDARD FORMS FOR REVISED SIMPLEX METHOD

There are two standard forms of the revised simplex method.

Standard form I: In this form, it is assumed that an identity matrix is available after adding slack variables and thus there is no need of adding artificial variables.

Standard form II: In this form artificial variables are also added in order to have identity matrix. Thus, a two-phase simplex method is used to handle artificial variables.

4.2.1 Revised Simplex Method in Standard Form I

In standard form I of the revised simplex method, the objective function is also treated as another constraint. With the result, we deal with (m + 1) dimensional basis matrix B instead of m-dimensional. The reason for doing so is explained in the later part of this chapter.

Consider the LP problem in its standard form:

Max Z = $c_1x_2 + c_2x_2 + \cdots + c_nx_n + 0.x_{n+1} + 0.x_{n+2} + \cdots + 0.x_{n+m}$ → (1)

subject to the constraints

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + x_{n+1} = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + x_{n+2} = b_2 \quad \rightarrow (2)$$

$$\ldots$$

$$\ldots$$

$$\ldots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n + x_{n+m} = b_m$$

and $x_1, x_2, \dots, x_{n+m} \geq 0$ → (3)

In order to solve LP problem (3) using revised simplex method, the objective function (1) is also considered as one of the constraints equation in which value of Z can be made as large as possible and unrestricted in sign. Thus, the set of constraints can be written as:

$$Z - c_1x_1 - c_2x_2 - \cdots - c_nx_n - 0.x_{n+1} - 0.x_{n+2} - \cdots - 0.x_{n+m} \quad = 0$$

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + 0.x_{n+1} \quad = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + \quad x_{n+2} \quad = b_1 \quad \rightarrow (4)$$

……

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$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n + \quad x_{n+m} \quad = b_m$$

and $x_1, x_2, \dots, x_{n+m} \geq 0$

In matrix notations, the system of equations (4) can be expressed as:

Z – cx = 0

Ax = b and x ≥ 0

In the system of equations (4), there are (m + 1) simultaneous linear equations in (n + m +1) variables (Z, x_1, x_2, …, x_{n+m}). The aim now is to solve (4) such that Z is as large as possible and unrestricted in sign, subject to the conditions x_1, x_2, …, x_{n+m} ≥ 0. By rewriting equation (4) in a more symmetric notations as follows, we get:

$$1.x_0 + a_{01}x_1 + a_{02}x_2 + \cdots + a_{0n}x_n + a_{0,n+1}x_{n+1} - a_{0,n+2}x_{n+2} - \cdots - a_{0,n+m}x_{n+m} = 0$$

$$0.x_0 + a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + \quad x_{n+1} \quad = b_1$$

$$0.x_0 + a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + \quad x_{n+2} \quad = b_2$$

………

………

………

$0.x_0 + a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n + \qquad x_{n+m} = b_m$

where Z = x_0 and $-c_j = a_{0\,j}$ (j = 1, 2, …, n + m). In matrix notations it may also be written as:

$$\begin{bmatrix} 1 & a_{01} & a_{02} & \cdots & a_{0n} & a_{0,n+1} & \cdots & a_{0,n+m} \\ 0 & a_{11} & a_{12} & \cdots & a_{1n} & 1 & \cdots & 0 \\ 0 & a_{21} & a_{22} & \cdots & a_{2n} & 0 & \cdots & 0 \\ & \vdots & & & & & & \vdots \\ 0 & a_{m_1} & a_{m_2} & \cdots & a_{m_n} & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{n+m} \end{bmatrix} = \begin{bmatrix} 0 \\ b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

(or) $\begin{bmatrix} 1 & a_0 \\ 0 & A \end{bmatrix} \begin{bmatrix} x_0 \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ b \end{bmatrix}$ where $a_0 = (a_{01}, a_{02}, \dots, a_{0\,n+m})$.

Using the matrix notations the system of equations (4) can be written in original notation as:

$$\begin{bmatrix} 1 & -c \\ 0 & A \end{bmatrix} \begin{bmatrix} z \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ b \end{bmatrix} ; x \geq 0$$

It may be noted that for standard from I in (4) all column vectors now have m + 1 components instead of m components, and basis matrix B os of order (m + 1) rather than m. Then, corresponding to each column a_j of matrix A, a new (m + 1) component vector $[a_{0j}, a_{1j}, a_{2j}, \dots, a_{mj}]$ is defined as:

$$a_j^{(1)} = [-c_j, a_{1j}, a_{2j}, \dots, a_{mj}]$$

$$= [-c_j, a_j] = [a_{0\,j}, a_j] \qquad j = 1, 2, \dots, n$$

Similarly, corresponding to the m – component vector b in Ax = b, (m +1) – component vector $b^{(1)}$ can be written as:

$$b^{(1)} = [o, b_1, b_2, \dots, b_m] = [0, b]$$

The column corresponding to Z (i.e., x_0) is the (m + 1) – component unit vector and is denoted by $e^{(1)}$. It will always be the first column of the basis matrix B_1. The basis matrix B_1 of order (m + 1) in terms of $e^{(1)}$ and the remaining mcolumns $a_j^{(1)}$ can be expressed as:

$$B_1 = \left[e^1, \beta_1^{(1)}, \beta_2^{(1)}, \dots, \beta_m^{(1)}\right] = \left[\beta_0^{(1)}, \beta_1^{(1)}, \beta_2^{(1)}, \dots, \beta_m^{(1)}\right]$$

where $e^{(1)} = \beta_0^{(1)}, \beta_i^{(1)}$ (i = 1, 2, …, m) are m linearly independent vectors of $a_j^{(1)}$ corresponding to a_j. Obviously B_1 in the partitionrd form of the matrices, which can be written as:

$$\left[\begin{array}{c|cccc} 1 & -c_{B1} & -c_{B2} & \cdots & -c_{Bm} \\ \hline 0 & \beta_{11} & \beta_{12} & \cdots & \beta_{1m} \\ 0 & \beta_{21} & \beta_{22} & \cdots & \beta_{2m} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & \beta_{m1} & \beta_{m2} & \cdots & \beta_{mn} \end{array}\right]$$

$$B_1 = \begin{bmatrix} 1 & -c_B \\ 0 & B \end{bmatrix} == \left[\beta_0^{(1)}, \beta_1^{(1)}, \beta_2^{(1)}, \dots, \beta_m^{(1)}\right] \qquad \rightarrow (6)$$

B = $(\beta_1, \beta_2, \dots, \beta_m)$ is the basis matrix for the system Zx = b containing those columns a_j of A that are also the column $a_j^{(1)}$ of basis matrix B_1 and $c_B = \left(c_{B_1}, c_{B_2}, \dots, c_{B_m}\right)$ are the coefficient of basic variables x_{B_i} (i = 1, 2, …, m) in the equation, $Z - c_1x_1 - c_2x_2 - \cdots - c_nx_n = 0$. Equation (6) shows the conversion process of basis matrix B of Ax = b to the basis matrix B_1 of equation (5) and vice versa.

Calculation of Inverse of B_1^{-1}:

Since B is invertible and is known, therefore inverse of matrix B_1 is given by:

$$B_1^{-1} = \begin{bmatrix} 1 & c_B B^{-1} \\ 0 & B^{-1} \end{bmatrix}$$

The elements, $B^{-1}, c_B B^{-1}$ of matrix B_1^{-1} are known. Therefore, B_1^{-1} is also known. Further it may be verified that $BB^{-1} = I_{m+1}$.

Since vector $a_j^{(1)}$, not in the basis matrix B_1, can be expressed as the linear combination of the column vectors, $\beta_0^{(1)}, \beta_1^{(1)}, \beta_2^{(1)}, \dots, \beta_m^{(1)}$ in B_1, therefore,

$$a_j^{(1)} = y_{0j}\beta_0^{(1)} + y_{1j}\beta_1^{(1)} + y_{2j}\beta_2^{(1)} + \cdots + y_{mj}\beta_m^{(1)}$$

$$= \left[y_{0j}, y_{1j}, y_{2j}, \dots, y_{mj}\right]\left[\beta_0^{(1)}, \beta_1^{(1)}, \beta_2^{(1)}, \dots, \beta_m^{(1)}\right]$$

$$= y_j^{(1)} B_1$$

Thus, $y_j^{(1)} = B_1^{-1} a_j^{(1)} = \begin{bmatrix} 1 & c_B B^{-1} \\ 0 & B^{-1} \end{bmatrix} \begin{bmatrix} -c_j \\ a_j \end{bmatrix}$; $j = 1, 2, \ldots, n$

$$= \begin{bmatrix} -c_j + c_B B^{-1} a_j \\ B^{-1} a_j \end{bmatrix} = \begin{bmatrix} z_j - c_j \\ y_j \end{bmatrix}$$

Here it may be noted that the first component of $y_j^{(1)}$ is $z_j - c_j$ (it is used as optimality criterion) and the last m components constitute the vector $y_j = B^{-1} a_j = (y_{1j}, y_{2j}, \ldots, y_{mj})$.

Remark:

One advantage of treating objective function Z as one of the constraints is that $z_j - c_j$ for any column a_j, not in the basis, can be calculated by taking the product of the first row of B_1^{-1} with $a_j^{(1)}$, not in the basis B_1,

$$z_j - c_j = \{First\ row\ of\ B_1^{-1}\}\left\{column\ vector\ a_j^{(1)}\ not\ in\ the\ basis\ B_1\right\}$$

Further the (m + 1) components of $x_B^{(1)}$ can also be defined as:

$$x_B^{(1)} = B_1^{-1} b^{(1)} = \begin{bmatrix} 1 & c_B B^{-1} \\ 0 & B^{-1} \end{bmatrix} \begin{bmatrix} 0 \\ b \end{bmatrix} = \begin{bmatrix} c_B B^{-1} b \\ B^{-1} b \end{bmatrix} = \begin{bmatrix} Z \\ x_B \end{bmatrix}$$

Thus, $x_B^{(1)}$ represents the basic solution (but not necessarily feasible) of LP problem (4), where the first component represents the value of the objective function Z and the remaining m – components, x_{B_i} represents the basic solution for system of constraints Ax = b, corresponding to the basis matrix B.

4.3 COMPUTATIONAL PROCEDURE FOR STANDARD FORM I

For the initial basis matrix in revised simplex method, the columns $a_j^{(1)}$ which form the initial identity matrix I are used. Since simplex method always start with an initial basis (identity) matrix B of order m, therefore, for the revised simplex method the inverse of the initial basis matrix can be written as:

$$B_1^{-1} = \begin{bmatrix} 1 & c_B B^{-1} \\ 0 & B^{-1} \end{bmatrix} = \begin{bmatrix} 1 & c_B \\ 0 & I_m \end{bmatrix} ; B = I_m = B^{-1} \quad \rightarrow (7)$$

Further, if columns of matrix A form an initial basis matrix of order m that corresponds to the slack or surplus variables, then c_{B_i} = 0 (i = 1, 2, . . ., m). Thus Eq. (7) reduces to the form:

$$B_1^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & I_m \end{bmatrix} = \left[\begin{array}{c|cccc} 1 & 0 & 0 & \cdots & 0 \\ \hline 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{array}\right] = I_{m+1}$$

This implies that the inverse of initial basis matrix B_1 will be I_{m+1} to start the revised simplex procedure. The initial basic solution to equation (4) is given by:

$$x_B^{(1)} = \begin{bmatrix} 1 & 0 \\ 0 & I_m \end{bmatrix} \begin{bmatrix} 0 \\ b \end{bmatrix} = \begin{bmatrix} o \\ b \end{bmatrix}$$

This solution is feasible because the last m-components are non-negative, and the first component Z can have any sign.

After obtaining a basic feasible solution of equation (4) and the inverse (= I_{m+1}) of the initial basis matrix, B_1^{-1}, we need to improve the solution by using the revised simplex method. For this, we first calculate $c_j - z_j$ for each column $a_j^{(1)}$ not in the basis B_1, by taking scalar product of the first row of B_1^{-1} with each $a_j^{(1)}$ as explained earlier. The vector $a_k^{(1)}$ to enter the basis is determined by the criterion.

$c_k - z_k$ = Max $\{c_j - z_j : c_j - z_j > 0\}$, for all j

Since x_0 (= Z) is always desired in the basis, the first column $\beta_0^{(1)}(= e^{(1)})$ of the initial basis matrix inverse B_1^{-1} = I_{m+1} will never be removed from the basis at any iteration. The vector to be removed from the basis is determined by the criterion:

$\frac{x_{Br}}{y_{rk}} = Min\left\{\frac{x_{Bi}}{y_{ik}}, y_{ik} > 0\right\}$, for all i where y_{ik} (i = 1, 2, …, m) are the components of vector $y_k^{(1)}$, and $y_k^{(1)} = B_1^{-1} a_k^{(1)} = \begin{bmatrix} z_k - c_k \\ y_k \end{bmatrix}$

Since we start with an identity matrix B_1, the new inverse denoted by B_0^{-1} shall be obtained by multiplying the basis matrix inverse B_1^{-1} at the previous iteration by an elementary matrix E, where E is the inverse of an identity matrix with rth column replaced y_k.

Remarks:

1) If there is a tie in the selection of the key column, then choose the column from left to right (i.e. smallest index j).

2) A tie in selecting the outgoing vector can be broken by any of the methods discussed earlier.

4.3.1 Steps of the Procedure

The revised simplex method can be summarized in the following steps:

Step 1: Express the given problem in standard form. Express the given problem in the revised simplex form by considering the objective function as one of the constraints, and adding the slack and surplus variables, if needed, to the inequalities in order to convert them into equalities.

Step 2: Obtain initial basic feasible solution. Start with initial basis matrix B = I_m and then find B_1^{-1} and $B_1^{-1}b$ to form the initial revised simplex table as shown in Table 4.1.

Basic Variables **B**	*Solution Values* $b\ (=\mathbf{x}_B^{(1)})$	*Basis Inverse,* $\mathbf{B}_1^{-1}$ $\boldsymbol{\beta}_0^{(1)}$ $(=Z)$	$\boldsymbol{\beta}_1^{(1)}$ $(=s_1)$	$\boldsymbol{\beta}_2^{(1)}$ $(=s_2)$	...	$\boldsymbol{\beta}_m^{(1)}$ $(=s_m)$	$\mathbf{y}_k^{(1)}$
Z	0	1	0	0	...	0	$c_k - z_k$
$x_{B_1} = s_1$	b_1	0	1	0	...	0	y_{1k}
$x_{B_2} = s_2$	b_2	0	0	1	...	0	y_{2k}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$x_{B_m} = s_m$	b_m	0	0	0	0	1	y_{mk}

Table 4.1: Initial Revised Simplex Table

Step 3: Select a variable to enter into the basis (key column). For each non – basic variable, calculate $c_j - z_j$ by using the formula:

$$c_j - z_j = c_j - c_B B_1^{-1} a_j^{(1)}$$

where $B_1^{-1} a_j^{(1)}$ represents the product of the first row of B_1^{-1} and successive columns of A not in B_1^{-1}.

(i) If all $c_j - z_j \leq 0$, then the currect basic solution is optimal. Otherwise go to step 4.

(ii) If one or more $c_j - z_j$ are positive, then the variable to enter into the basis may be selected by using the formula: $c_j - z_j = Max\left\{c_j - z_j : \ c_j - z_j > 0\right\}$.

Step 4: Select a variable to leave the basis (key row).

Calculate $y_k^{(1)} = B_1^{-1} a_k^{(1)} = a_k^{(1)}$; $(k = 1)$ where $a_k^{(1)} = \left[-c_j, a_k\right]$. If all $y_{ik} \leq 0$, the optimal solution is unbounded. But if at least one $y_{ik} > 0$, then the variable to be removed from the basis is determined by calculating the ratio:

$\frac{x_{Br}}{y_{rk}} = Min\left\{\frac{x_{Bi}}{y_{ri}} \ ; \ y_{ik} > 0\right\}$.

i.e., the vector $\beta_r^{(1)}$ is selected to leave the basis and go to step 5.

If the minimum ratio is not unique, i.e. the ratio is same for more than one row, then the resulting basic feasible solution will be degenerate. To avoid cycling from taking place, the usual method of resolving the degeneracy is applied.

Step 5: Update the current solution.

Update the initial table by introducing a non – basic variable $x_k\left(= a_k^{(1)}\right)$ into the basis and removing basic variable $x_k\left(= \beta_r^{(1)}\right)$ from the basis.

Repeat Steps 3 to 5 until an optimal solution is obtained or there is an indication for an unbounded solution.

Example 4.1

Use the revised simplex method to solve the following LP problem:

Maximize Z = $2x_1 + x_2$ subject to the constraints

(i) $3x_1 + 4x_2 \leq 6$ (ii) $6x_1 + x_2 \leq 3$ and $x_1, x_2 \geq 0$

Solution:

Step 1: By introducing slack variables s_1 *and* s_2 to constraints in order to convert them to equations and by considering the objective function as one of the constraints, the given LP problem can be rewritten as:

(i) $Z - x_1 - x_2 = 0$, (ii) $3x_1 + 4x_2 + s_1 = 6$, (iii) $6x_1 + x_2 + s_2 = 3$ and $x_1, x_2, s_1, s_2 \geq 0$

This new system of constraints equations can be expressed in the matrix forms as follows:

$$\begin{array}{c} \\ \end{array}\overset{\begin{array}{ccccc}\mathbf{e}^{(1)} & \mathbf{a}_1^{(1)} & \mathbf{a}_2^{(1)} & \mathbf{a}_3^{(1)} & \mathbf{a}_4^{(1)}\end{array}}{\left[\begin{array}{c|cccc} 1 & -2 & -1 & 0 & 0 \\ \hline 0 & 3 & 4 & 1 & 0 \\ 0 & 6 & 1 & 0 & 1 \end{array}\right]} \begin{bmatrix} Z \\ x_1 \\ x_2 \\ s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \\ 3 \end{bmatrix}$$

(or) $\begin{bmatrix} 1 & -c \\ 0 & A \end{bmatrix} \begin{bmatrix} Z \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ b \end{bmatrix}$; $x \geq 0$

where $e^{(1)} = \beta_0^{(1)}, a_3^{(1)} = \beta_1^{(1)}$ *and* $a_4^{(1)} = \beta_2^{(1)}$

Step 2: The basis matrix B_1 of order (2 + 1) = 3 can be expressed as:

$$B_1 = \left[\beta_0^{(1)}, \beta_1^{(1)}, \beta_2^{(1)}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then $B_1^{-1} = \begin{bmatrix} 1 & c_B B^{-1} \\ 0 & B^{-1} \end{bmatrix} = 1$; $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \left[\beta_1^{(1)}, \beta_2^{(1)}\right]$; $c_B = [0, 0]$

The initial basic feasible solution: $s_1 = 6, s_2 = 3$ and Max Z = 0, is shown in Table 4.2.

Basic Variables **B**	*Solution Values* $b\ (= \mathbf{x}_B^{(1)})$	*Basis Inverse,* $\mathbf{B}_1^{-1}$ $\boldsymbol{\beta}_0^{(1)}$ $(= Z)$	$\boldsymbol{\beta}_1^{(1)}$ $(= s_1)$	$\boldsymbol{\beta}_2^{(1)}$ $(= s_2)$	$\mathbf{y}_k^{(1)}$	*Additional Table** $\mathbf{a}_1^{(1)}$ $(= x_1)$	$\mathbf{a}_2^{(1)}$ $(= x_2)$
Z	*0*	*1*	*0*	*0*	$c_k - z_k$	*– 2*	*– 1*
s_1	6	0	1	0		3	4
s_2	3	0	0	1		6	1

*Additional table for those column vectors $\mathbf{a}_j^{(1)}$ that are not in the basis matrix $\mathbf{B}_1$.

Table 4.2

Iteration 1:

Step 3: To select the vector corresponding to a non – basic variable to enter into the basis, we compute:

$$c_k - z_k = Max\{(c_j - z_j) > 0\,; j = 1,2\}$$

$$= Max\ \left\{-(First\ row\ of\ B_1^{-1})\left(Columns\ a_j^{(1)}\ not\ in\ basis, B_1\right)\right\}$$

$$= Max\left\{-(1,0,0)\begin{bmatrix}-2 & -1\\ 3 & 4\\ 6 & 1\end{bmatrix}\right\}$$

$$= Max\{-(-2,-1)\}$$

= 2 (corresponds to $c_1 - z_1$)

Thus, vector $a_1^{(1)}(= x_1)$ is selected to enter into the basis, for k = 1.

Step 4: To select a basic variable to leave the basis, given the entering non – basic variable x_1, we compute $y_k^{(1)}$ for k = 1, as follows:

$y_1^{(1)} = B_1^{-1}a_1^{(1)} = a_1^{(1)} = \begin{bmatrix}-2\\ 3\\ 6\end{bmatrix}$, for k = 1 and $x_B^{(1)} = B_1^{-1}b = b = \begin{bmatrix}0\\ 6\\ 3\end{bmatrix}$

After having selected the non-basic variable x_1 to enter into the basis, we shall calculate the minimum ratio to select the basic variable to leave the basis.

$$\frac{x_{Br}}{y_{rk}} = Min\left\{\frac{x_{Bi}}{y_{ri}}\ ;\ y_{ik} > 0\right\}$$

$= Min\ \left\{\frac{x_{B1}}{y_{11}}, \frac{x_{B2}}{y_{21}}\right\}$;k = 1

$$= Min\ \left\{\frac{6}{3}, \frac{3}{6}\right\}$$

$= \frac{3}{6}$ (corresponds to $\left(\frac{x_{B2}}{y_{21}}\right)$)

Thus, vector $\beta_2^{(1)}(= s_2)$ for r = 2 is selected to leave the basis.

Table 4.2 is again reproduced with the new entries in column $y_1^{(1)}$ and the minimum ratio, as shown in Table 4.3

Basic Variables **B**	*Solution Values* $b\,(=\mathbf{x}_B^{(1)})$	*Basis Inverse,* $\mathbf{B}_1^{-1}$			$\mathbf{y}_1^{(1)}$ $(c_k - z_k)$	*Min. Ratio* $\mathbf{x}_B^{(1)}/\mathbf{y}_1^{(1)}$
		$\boldsymbol{\beta}_0^{(1)}$ $(= Z)$	$\boldsymbol{\beta}_1^{(1)}$ $(= s_1)$	$\boldsymbol{\beta}_2^{(1)}$ $(= s_2)$		
Z	*0*	*1*	*0*	*0*	*– 2*	*–*
s_1	6	0	1	0	3	6/3
s_2	3	0	0	1 ↑	(6)	3/6 →

Table 4.3

Step 5: The initial basic feasible solution shown in Table 4.3 is now updated by replacing variable s_2 with the variable x_1 in the basis. For this we apply the following row operations in the same way as in the simplex method.

	$\mathbf{x}_B^{(1)}$	$\boldsymbol{\beta}_1^{(1)}$	$\boldsymbol{\beta}_2^{(1)}$	$\mathbf{y}_1^{(1)}$
$R_1 \rightarrow$	0	0	0	– 2
$R_2 \rightarrow$	6	1	0	3
$R_3 \rightarrow$	3	0	1	6

R_3 (new) $\rightarrow R_3$ (old) $\div$ 6 (*key element*); R_1 (new) $\rightarrow R_1$ (old) + 2 R_3 (new)
R_2 (new) $\rightarrow R_2$ (old) – 3 R_3 (new)

While determining the entries of the new table for an improved solution, it should be remembered that column $\beta_0^{(1)}$ will never change. Thus, entries in $x_B^{(1)}, y_1^{(1)}, \beta_1^{(1)}, \beta_2^{(1)}$ columns will be changed due to the above mentioned row operations. The improved solution is shown in Table 4.4.

Basic Variables $\mathbf{B}$	Solution Values $b(=\mathbf{x}_B^{(1)})$	Basis Inverse, $\mathbf{B}_1^{-1}$			$\mathbf{y}_2^{(1)}$ $(c_k - z_k)$	Min. Ratio $\mathbf{x}_B^{(1)}/\mathbf{y}_2^{(1)}$	Additional Table	
		$\boldsymbol{\beta}_0^{(1)}$ $(=Z)$	$\boldsymbol{\beta}_1^{(1)}$ $(=s_1)$	$\boldsymbol{\beta}_2^{(1)}$ $(=x_1)$			$\mathbf{a}_4^{(1)}$ $(=s_2)$	$\mathbf{a}_2^{(1)}$ $(=x_2)$
Z	1	1	0	1/3	– 2/3	– 2	0	– 1
s_1	9/2	0	1	– 1/2	(7/2)	$\frac{9/2}{7/2}$ →	0	4
x_1	1/2	0	0 ↑	1/6	1/6	$\frac{1/2}{1/6}$	1	1

Table 4.4

The column vectors not in the basis and new basis matrix, as shown in Table 4.4 are:

$$a_2^{(1)} = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} \text{ and } a_4^{(1)} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ and } B_1^{-1} = \begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{6} \end{bmatrix}$$

Iteration 2: Repeat Steps 3 to 5 to get the new improved solution.

Step 3: To select the vector corresponding to a non – basic variable to enter into the basis in Table 4.4, we compute:

$$c_k - z_k = Max\{(c_j - z_j) > 0\, ; j = 2, 4\}$$

$$= Max\ \{-(First\ row\ of\ B_1^{-1})\big(Columns\ a_j^{(1)}\ not\ in\ basis\big)\}$$

$$= Max\left\{-\left(1, 0, \frac{1}{3}\right)\begin{bmatrix} -1 & 0 \\ 4 & 0 \\ 1 & 1 \end{bmatrix}\right\}$$

$$= Max\left\{-\left(-1 + \frac{1}{3}, \frac{1}{3}\right)\right\}$$

$$= \frac{2}{3} \text{ (corresponds to } c_2 - z_2)$$

Thus, vector $a_2^{(1)}(= x_2)$ is selected to enter into the basis, for k = 2.

Step 4: In order to find the vector $\beta_r^{(1)}$ corresponding to basic variables to leave the basis, we first compute $y_k^{(1)}$ for k = 2, as follows:

$$y_2^{(1)} = B_1^{-1} a_2^{(1)} = a_2^{(1)} = \begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{6} \end{bmatrix} \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} \\ \frac{7}{2} \\ \frac{1}{6} \end{bmatrix}$$

The values $y_2^{(1)}$ are shown in Table 4.4.

The minimum ration for a predetermined value of k (= 2) is given by:

$$\frac{x_{Br}}{y_{rk}} = Min\left\{\frac{x_{Bi}}{y_{i2}}\ ;\ y_{i2} > 0\right\}$$

$= Min\left\{\frac{x_{B1}}{y_{12}}, \frac{x_{B2}}{y_{22}}\right\}$;k = 1

$$= Min\left\{\frac{9/2}{7/2}, \frac{1/2}{1/6}\right\}$$

$= \frac{9}{7}$ (corresponds to $\left(\frac{x_{B2}}{y_{12}}\right)$)

Thus, vector $\beta_1^{(1)} (= s_1)$ for r = 1 is selected to leave the basis, as shown in Table 4.4.

Step 5: The solution shown in Table 4.4 is now updated by replacing variable s_1 with the variable x_2 into the basis. For this we apply the following row operations in the same way as in iteration 1:

	$\mathbf{x}_B^{(1)}$	$\boldsymbol{\beta}_1^{(1)}$	$\boldsymbol{\beta}_2^{(1)}$	$\mathbf{y}_2^{(1)}$
$R_1 \rightarrow$	1	0	1/3	– 2/3
$R_2 \rightarrow$	9/2	1	– 1/2	(7/2)
$R_3 \rightarrow$	1/2	0	1/6	1/6

R_2 (new) $\rightarrow R_2$ (old) × (2/7) (*key element*); R_1 (new) $\rightarrow R_1$ (old) + (2/3) R_2 (new)
R_3 (new) $\rightarrow R_3$ (old) – (1/6) R_2 (new)

The improved solution is shown in Table 4.5.

Basic Variables B	Solution Values $\mathbf{b}\ (= \mathbf{x}_B^{(1)})$	Basis Inverse, $\mathbf{B}_1^{-1}$ $\boldsymbol{\beta}_0^{(1)}$ $(= Z)$	$\boldsymbol{\beta}_1^{(1)}$ $(= x_2)$	$\boldsymbol{\beta}_2^{(1)}$ $(= x_1)$	Additional Table $\mathbf{a}_4^{(1)}$ $(= s_2)$	$\mathbf{a}_3^{(1)}$ $(= s_1)$
Z	*13/7*	*1*	*4/21*	*5/21*	*0*	*0*
x_2	9/7	0	2/7	– 1/7	0	1
x_1	2/7	0	– 1/21	4/21	1	0

Table 4.5

The column vectors not in the basis as shown in Table 4.5 are:

$$a_3^{(1)} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } a_4^{(1)} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ and } B_1^{-1} = \begin{bmatrix} 1 & \frac{4}{21} & \frac{5}{21} \\ 0 & \frac{2}{7} & -\frac{1}{7} \\ 0 & -\frac{2}{21} & \frac{4}{21} \end{bmatrix}$$

Iteration 3: Repeat Steps 3 to 5 to get the new improved solution.

Step 3: To select the vector corresponding to a non – basic variable to enter into the basis in Table 4.5, we compute:

$$c_k - z_k = Max\{(c_j - z_j) > 0\,; j = 3, 4\}$$

$$= Max\ \left\{-(First\ row\ of\ B_1^{-1})\left(Columns\ a_j^{(1)}\ not\ in\ basis\right)\right\}$$

$$= Max\left\{-\left(1, \frac{4}{24}, \frac{5}{21}\right)\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$

$$= Max\left\{-\left(\frac{4}{21}, \frac{5}{21}\right)\right\}$$

Since all $c_j - z_j < 0$ (j = 3, 4), the current solution shown in Table 4.5 is optimal.

Thus, the optimal solution is: $x_1 = \frac{2}{7}, x_2 = \frac{9}{7}$ and Max Z = $\frac{13}{7}$.

Remark: Once the revised simplex method for solving an LP problem is fully understood there is no need of giving details about the steps of the algorithm. This is illustrated in the following two examples.

Example 4.2 Use the revised simplex method to solve the following LP problem:

Maximize Z = $3x_1 + 5x_2$ subject to the constraints

(i) $x_1 \leq 4$ (ii) $x_2 \leq 6$ (iii) $3x_1 + 2x_2 \leq 18$ and $x_1, x_2 \geq 0$

Solution: We express the given LP problem in the standard form I of the revised simplex method as follows:

(i) $Z - 3x_1 + 5x_2 = 0$ (ii) $x_1 + s_1 = 4$ (iii) $x_2 + s_2 = 6$

(iv) $3x_1 + 2x_2 + s_3 = 18$ and $x_1, x_2, s_1, s_2, s_3 \geq 0$

Now we represent the new system of constraints equations in the matrix form as follows:

$$\begin{array}{c} \\ \end{array}\begin{matrix} \mathbf{e}^{(1)} & \mathbf{a}_1^{(1)} & \mathbf{a}_2^{(1)} & \mathbf{a}_3^{(1)} & \mathbf{a}_4^{(1)} & \mathbf{a}_5^{(1)} \end{matrix}$$

$$\left[\begin{array}{c|ccccc} 1 & -3 & -5 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 3 & 2 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} Z \\ x_1 \\ x_2 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 6 \\ 18 \end{bmatrix}$$

where $e^{(1)} = \beta_0^{(1)}, a_3^{(1)} = \beta_1^{(1)}, a_4^{(1)} = \beta_2^{(1)}$ and $a_5^{(1)} = \beta_3^{(1)}$

The basis matrix B_1 of order (3 + 1) = 4 can be expressed as:

$$B_1 = \left[\beta_0^{(1)}, \beta_1^{(1)}, \beta_2^{(1)}, \beta_3^{(1)}\right] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Then, $B_1^{-1} = \begin{bmatrix} 1 & c_B B^{-1} \\ 0 & B^{-1} \end{bmatrix} = 1\,; B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \left[\beta_1^{(1)}, \beta_2^{(1)}, \beta_3^{(1)}\right]; \; c_B = [0,0]$

The initial basic feasible solution is shown in Table 4.6.

Basic Variables B	Solution Values $b\ (= x_B^{(1)})$	Basis Inverse, B_1^{-1}				$y_k^{(1)}$	Additional Table	
		$\beta_0^{(1)}$ (= Z)	$\beta_1^{(1)}$ (= s_1)	$\beta_2^{(1)}$ (= s_2)	$\beta_3^{(1)}$ (= s_3)		$a_1^{(1)}$ (= x_1)	$a_2^{(1)}$ (= x_2)
Z	0	1	0	0	0	$c_k - z_k$	– 3	– 5
s_1	4	0	1	0	0		1	0
s_2	6	0	0	1	0		0	1
s_3	18	0	0	0	1		3	2

Table 4.6

Iteration 1: To select the vector corresponding to a non – basic variable to enter into the basis, we compute:

$$c_k - z_k = Max\{(c_j - z_j) > 0\ ; j = 1, 2\}$$

$$= Max\ \{-(First\ row\ of\ B_1^{-1})(Columns\ a_j^{(1)}\ not\ in\ basis)\}$$

$$= Max\left\{-(1, 0, 0,0)\begin{bmatrix} -3 & -5 \\ 1 & 0 \\ 0 & 1 \\ 3 & 2 \end{bmatrix}\right\}$$

$$= Max\{-(-3, -5)\}$$

= 5 (corresponds to $c_2 - z_2$)

Thus, vector $a_2^{(1)}(= x_2)$ is selected to enter into the basis, for k = 2.

To select the basic variable to leave the basis, we compute:

$$y_k^{(1)} = B_1^{-1}a_k^{(1)} = a_k^{(1)} = \begin{bmatrix} -5 \\ 0 \\ 1 \\ 2 \end{bmatrix}\text{; k = 2 and } x_B^{(1)} = B_1^{-1}b = b = \begin{bmatrix} 0 \\ 4 \\ 6 \\ 18 \end{bmatrix}$$

After having selected the non-basic variable, x_2, to enter the basis, we shall calculate the minimum ratio in order to select the basic variable to leave the basis:

$$\frac{x_{Br}}{y_{rk}} = Min\left\{\frac{x_{Bi}}{y_{i2}}\ ;\ y_{i2} > 0\right\}$$

$$= Min\ \left\{\frac{x_{B1}}{y_{12}}, \frac{x_{B2}}{y_{22}}, \frac{x_{B3}}{y_{32}}\right\}\text{;k = 2}$$

$$= Min\left\{\frac{4}{0}, \frac{6}{1}, \frac{18}{2}\right\}$$

$$= 6 \text{ (corresponds to } \left(\frac{x_{B2}}{y_{22}}\right))$$

Thus, vector $\beta_2^{(1)}(= s_2)$ for r = 2 is selected to leave the basis.

Table 4.6 is again reproduced with the new entries in column $y_k^{(1)}$ and minimum ratio, as shown in Table 4.7.

Basic Variables **B**	*Solution Values* $\mathbf{b}\ (= \mathbf{x}_B^{(1)})$	*Basis Inverse,* $\mathbf{B}_1^{-1}$				$\mathbf{y}_2^{(1)}$ $(c_k - z_k)$	*Min. Ratio* $\mathbf{x}_B^{(1)}/\mathbf{y}_2^{(1)}$
		$\boldsymbol{\beta}_0^{(1)}$ $(= Z)$	$\boldsymbol{\beta}_1^{(1)}$ $(= s_1)$	$\boldsymbol{\beta}_2^{(1)}$ $(= s_2)$	$\boldsymbol{\beta}_3^{(1)}$ $(= s_3)$		
Z	*0*	*1*	*0*	*0*	*0*	*– 5*	–
s_1	4	0	1	0	0	0	–
s_2	6	0	0	1	0	(1)	6/1 = 6 →
s_3	18	0	0	0 ↑	1	2	18/2 = 9

Table 4.7

The initial basic feasible solution shown in Table 4.7 is now updated by introducing variable x_2 into the basis and removing s_2 from the basis.

	$\mathbf{x}_B^{(1)}$	$\boldsymbol{\beta}_1^{(1)}$	$\boldsymbol{\beta}_2^{(1)}$	$\boldsymbol{\beta}_3^{(1)}$	$\mathbf{y}_2^{(1)}$
$R_1 \rightarrow$	0	0	0	0	– 5
$R_2 \rightarrow$	4	1	0	0	0
$R_3 \rightarrow$	6	0	1	0	(1)
$R_4 \rightarrow$	18	0	0	1	2

For this we apply the following row operations:

$R_3(new) \rightarrow R_3(old) \div 1(key\ element)$; $R_4(new) \rightarrow R_4(old) - R_3(new)$;

$R_1(new) \rightarrow R_1(old) + 5R_3(new)$

The improved solution is shown in Table 4.8.

Basic Variables B	Solution Values $b\ (= x_B^{(1)})$	Basis Inverse, B_1^{-1} $\beta_0^{(1)}$ $(= Z)$	$\beta_1^{(1)}$ $(= s_1)$	$\beta_2^{(1)}$ $(= x_2)$	$\beta_3^{(1)}$ $(= s_3)$	$y_1^{(1)}$	Min. Ratio $x_B^{(1)}/y_2^{(1)}$	Additional Table $a_1^{(1)}$ $(= x_1)$	$a_4^{(1)}$ $(= s_2)$
Z	*30*	*1*	*0*	*5*	*0*	*– 3*	–	*– 3*	*0*
s_1	4	0	1	0	0	0	–	1	0
x_2	6	0	0	1	0	1	6/1 = 6	0	1
s_3	6	0	0	– 2	1	(3)	6/3 = 2 →	3	0

Table 4.8

The column vectors not in the basis and new basis matrix are given below. These are also shown in Table 4.8.

$$a_1^{(1)} = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 3 \end{bmatrix} ;\ a_4^{(1)} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} ;\ B_1^{-1} = \begin{bmatrix} 1 & 0 & 5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix}$$

Iteration 2: Again to select the vector corresponding to non – basis vectors $a_1^{(1)}$ and $a_4^{(1)}$ to enter into the basis, we compute:

$$c_k - z_k = Max\{(c_j - z_j) > 0\ ; j = 1, 4\}$$

$$= Max\ \left\{-(First\ row\ of\ B_1^{-1})\left(Columns\ a_j^{(1)}\ not\ in\ basis\right)\right\}$$

$$= Max \left\{-(1, 0, 5, 0) \begin{bmatrix} -3 & 0 \\ 1 & 0 \\ 0 & 1 \\ 3 & 0 \end{bmatrix}\right\}$$

$$= Max\{-(-3, 5)\}$$

= 3 (corresponds to $c_1 - z_1$)

Thus, vector $a_1^{(1)} (= x_1)$ is selected to enter into the basis, for k = 1.

To select the basic variable to leave the basis, we compute:

$$y_k^{(1)} = B_1^{-1} a_k^{(1)} = \begin{bmatrix} 1 & 0 & 5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 3 \end{bmatrix} \text{; k = 1}$$

The values of $y_1^{(1)}$ are shown in Table 4.8.

The minimum ratio for predetermined value of k (= 1) is given by:

$$\frac{x_{Br}}{y_{rk}} = Min\left\{\frac{x_{Bi}}{y_{i1}}\ ;\ y_{i1} > 0\right\}$$

$$= Min\left\{\frac{x_{B1}}{y_{11}}, \frac{x_{B2}}{y_{21}}, \frac{x_{B3}}{y_{31}}\right\}$$

$$= Min\left\{\frac{4}{1}, \frac{6}{0}, \frac{6}{3}\right\}$$

$$= \frac{6}{3} \text{ (corresponds to } \left(\frac{x_{B3}}{y_{31}}\right))$$

Thus, vector $\beta_3^{(1)} (= s_3)$ for r = 3 is selected to leave the basis, as shown in Table 4.8.

The solution shown in Table 4.8 is now updated by introducing variable x_1 into the basis and removing variables s_3 from the basis.

	$\mathbf{x}_B^{(1)}$	$\boldsymbol{\beta}_1^{(1)}$	$\boldsymbol{\beta}_2^{(1)}$	$\boldsymbol{\beta}_3^{(1)}$	$\mathbf{y}_3^{(1)}$
$R_1 \rightarrow$	30	0	5	0	– 3
$R_2 \rightarrow$	4	1	0	0	0
$R_3 \rightarrow$	6	0	1	0	1
$R_4 \rightarrow$	6	0	– 2	1	(3)

For this we apply the following row operations:

$R_4(new) \rightarrow R_4(old) \div 3(key\ element)$; $R_3(new) \rightarrow R_3(old) - R_4(new)$;

$$R_1(new) \rightarrow R_1(old) + 3R_4(new)$$

The improved solution is shown in Table 4.9.

Basic Variables B	Solution Values $b(= x_B^{(1)})$	Basis Inverse, B_1^{-1}				Additional Table	
		$\beta_0^{(1)}$ $(= Z)$	$\beta_1^{(1)}$ $(= s_1)$	$\beta_2^{(1)}$ $(= x_2)$	$\beta_3^{(1)}$ $(= x_1)$	$a_4^{(1)}$ $(= s_2)$	$a_5^{(1)}$ $(= s_3)$
Z	*36*	*1*	*0*	*3*	*1*	*0*	*0*
s_1	4	0	1	2/3	– 1/3	0	0
x_2	6	0	0	1	0	1	0
x_1	2	0	0	– 2/3	1/3	0	1

Table 4.9

The columns vectors not in the basis and the basis matrix, as shown in Table 4.9 are:

$$a_4^{(1)} = \begin{bmatrix} 3 \\ 2/3 \\ 1 \\ -2/3 \end{bmatrix} ;\ a_5^{(1)} = \begin{bmatrix} 1 \\ -1/3 \\ 0 \\ 1/3 \end{bmatrix} ;\ B_1^{-1} = \begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 2/3 & -1/3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2/3 & 1/3 \end{bmatrix}$$

Iteration 3: The procedure illustrated in iterations 1 and 2 is repeated to update the solution as shown in Tabl2 4.9

First to select the vector to enter into the basis, we compute:

$$c_k - z_k = Max\{(c_j - z_j) > 0\ ; j = 3, 4\}$$

$$= Max\ \left\{-(First\ row\ of\ B_1^{-1})\left(Columns\ a_j^{(1)}\ not\ in\ basis\right)\right\}$$

$$= Max\left\{-(1, 0, 3, 1)\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$

$$= Max\{-(3, 1)\}$$

Since all $c_k - z_k < 0$, the current solution shown in Table 4.9 is optimal.

Thus, the optimal solution is: $x_1 = 2$, $x_2 = 6$ and Max Z = 36.

Example 4.3 Use the revised simplex method to solve the following LP problem:

Maximize Z = $x_1 + x_2 + 3x_3$ subject to the constraints

(i) $3x_1 + 2x_2 + x_3 \leq 3$ (ii) $2x_1 + x_2 + 2x_3 \leq 2$ and $x_1, x_2, x_3 \geq 0$

Solution: Introduce slack variables s_1 and s_2 to the constraints in order to convert them into equations and consider objective function as one of the constraints. The LP problem can be written as:

(i) $Z - x_1 - x_2 - x_3 = 0$ (ii) $2x_1 + x_2 + 2x_3 + s_1 = 3$

(iii) $2x_1 + x_2 + 2x_3 + s_2 = 2$ and $x_1, x_2, x_3, s_1, s_2 \geq 0$

The initial basis matrix B_1 is given by:

$$B_1 = \left[\beta_0^{(1)}, \beta_1^{(1)}, \beta_2^{(1)}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The initial basic feasible solution: $s_1 = 3, s_2 = 2$ and Z = 0 is shown in Table 4.10.

Basic Variables B	Solution Values $b(=x_B^{(1)})$	Basis Inverse, B_1^{-1}				Additional Table		
		$\beta_0^{(1)}$ $(=Z)$	$\beta_1^{(1)}$ $(=s_1)$	$\beta_2^{(1)}$ $(=s_2)$	$y_k^{(1)}$	$a_1^{(1)}$ $(=x_1)$	$a_2^{(1)}$ $(=x_2)$	$a_3^{(1)}$ $(=x_3)$
Z	*0*	*1*	*0*	*0*	$c_k - z_k$	*– 1*	*– 1*	*– 3*
s_1	3	0	1	0	0	3	2	1
s_2	2	0	0	1	1	2	1	2

Table 4.10

Iteration 1: To select a non – basic variable out of x_1, x_2 *and* x_3 to enter into the basis, we compute:

$$c_k - z_k = Max\{(c_j - z_j) > 0\,;\, j = 1, 2, 3\}$$

$$= Max\ \left\{-(First\ row\ of\ B_1^{-1})\left(Columns\ a_j^{(1)}\ not\ in\ basis\right)\right\}$$

$$= Max\left\{-(1,0,0)\begin{bmatrix}-1 & -1 & -3\\ 3 & 2 & 1\\ 2 & 1 & 1\end{bmatrix}\right\}$$

$$= Max\{-(1,1,3)\}$$

= 3 (corresponds to $c_3 - z_3$)

Thus, vector $a_3^{(1)} (= x_3)$ is selected to enter into the basis, for k = 3.

Now, to select the basic variable to leave the basis, we compute:

$y_3^{(1)} = B_1^{-1} a_3^{(1)} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}-3\\ 1\\ 2\end{bmatrix} = \begin{bmatrix}-3\\ 1\\ 2\end{bmatrix}$; k = 3 and $x_B^{(1)} = B_1^{-1} b = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0\\ 3\\ 2\end{bmatrix} = \begin{bmatrix}0\\ 3\\ 2\end{bmatrix}$

The values of $y_3^{(1)}$ and $x_B^{(1)}$ are shown in Table 4.11.

Basic Variables B	*Solution Values* $b\ (= \mathbf{x}_B^{(1)})$	*Basis Inverse,* $\mathbf{B}_1^{-1}$ $\boldsymbol{\beta}_0^{(1)}$ $(= Z)$	$\boldsymbol{\beta}_1^{(1)}$ $(= s_1)$	$\boldsymbol{\beta}_2^{(1)}$ $(= s_2)$	$\mathbf{y}_k^{(1)}$ $(k = 3)$	*Min. Ratio* $\mathbf{x}_B^{(1)}/\mathbf{y}_3^{(1)}$
Z	*0*	*1*	*0*	*0*	*– 3*	–
s_1	3	0	1	0	1	3/1 = 3
s_2	2	0	0	1 ↑	(2)	2/2 = 1 →

Table 4.11

Vector to be removed from the basis is determined by applying the minimum ratio rule shown in Table 4.11.

$$\frac{x_{Br}}{y_{rk}} = Min\left\{\frac{x_{Bi}}{y_{ik}}\ ;\ y_{ik} > 0\right\}$$

$$= Min\left\{\frac{x_{B1}}{y_{13}}, \frac{x_{B2}}{y_{23}}\right\}$$

$$= Min\left\{\frac{3}{1}, \frac{2}{2}\right\}$$

= 1 (corresponds to $\left(\frac{x_{B2}}{y_{23}}\right)$)

i.e., r = 2 and therefore basic variable s_2 is to be removed from the basis and the updated solution is shown in Table 4.12.

Basic Variables $\mathbf{B}$	Solution Values $\mathbf{b}(=\mathbf{x}_B^{(1)})$	Basis Inverse, $\mathbf{B}_1^{-1}$ $\boldsymbol{\beta}_0^{(1)}$ $(=Z)$	$\boldsymbol{\beta}_1^{(1)}$ $(=s_1)$	$\boldsymbol{\beta}_2^{(1)}$ $(=x_3)$	$\mathbf{y}_k^{(1)}$ $(k=3)$	Additional Table $\mathbf{a}_1^{(1)}$ $(=x_1)$	$\mathbf{a}_2^{(1)}$ $(=x_2)$	$\mathbf{a}_5^{(1)}$ $(=s_2)$
Z	*3*	*1*	*0*	*3/2*		– *1*	– *1*	*0*
s_1	2	0	1	– 1/2		3	2	0
x_3	1	0	0	1/2		2	1	1

Table 4.12

The column vectors not in the basis are:

$$a_1^{(1)} = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix};\ a_2^{(1)} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix};\ a_5^{(1)} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Iteration 2: Again repeat Steps 3 to 5 to get te new improved solution, if possible.

$$c_k - z_k = Max\left\{-(First\ row\ of\ B_1^{-1})\left(Columns\ a_j^{(1)}\ not\ in\ basis\right)\right\}$$

$$= Max\left\{-\left(1, 0, \tfrac{3}{2}\right)\begin{bmatrix} -1 & -1 & 0 \\ 3 & 2 & 0 \\ 2 & 1 & 1 \end{bmatrix}\right\}$$

$$= Max\left\{-\left(-2, -\tfrac{1}{2}, -\tfrac{3}{2}\right)\right\}$$

Since all $c_k - z_k < 0$, the current basic feasible solution is optimal as shown in Table 4.12. Thus, the required optimal solution is: $x_1 = 0, x_2 = 0, x_3 = 1$ and Max Z = 3.

4.4 COMPARISON OF SIMPLEX METHOD AND REVISED SIMPLEX METHOD

Consider an LP problem with constraints Ax = b, where A is a matrix of order m × n. If initially artificial variables are not added for obtaining the initial basis matrix, then for solving LP problem by the simplex method we need to transform n + 1 columns (n corresponding to columns of A and the last corresponding to x_B column) at each iteration. Also, at each iteration one variable is introduced into the basis and one is removed from it. Thus, this increase computational time because procedure involves evaluation of n – m + 1 columns. Furthermore, for each of these columns, we need to transform m + 1 elements (m corresponding to y_j and the last corresponding to $(c_j - z_j)$. For moving from one iteration to another we also need to calculate the minimum ratio $\frac{x_{Bi}}{y_{ik}}$. Hence, in all, we have to perform multiplication (m + 1) (n – m + 1) times and addition m (n – m + 1) times.

In the revised simplex method, there are m + 1 rows and m + 2 columns. So for moving from one iteration to another we have to make (m + 1)2 multiplication operations in order to get an improved solution in addition to m (n – m) operations for calculating $(c_j - z_j)$'s.

1. In the revised simplex method we need to make (m +1) × (m + 2) entries in each table, while in the simplex method there are (m + 1) (m + 1) entries in each table.

2. If the number of variables, n, is significantly larger than the number of constraints m, then the computational efforts of the revised simplex method is smaller than that of the simplex method.

3. Revised simplex method reduces the cumulative round-off error while calculating $(c_j - z_j)$'s and updated column y_k due to the use of original data.

4. The inverse of the current basis matrix is obtained automatically.

One disadvantage of the revised simplex method is that while updating the table to move from one solution to another, an additional table of original non-basic variable, not in the basis, is required. This may cause some computational errors.

Let us sum up

We have learned to derive two standard forms of the revised simplex method and their computational procedure. Developed a knowledge base about relevant information required at each iteration of the revised simplex method.

Check Your Progress

Use the revised simplex method to solve the following LP problems:

40. $Max\ Z = x_1 + 2x_2$

Subject to (i) $x_1 + x_2 \leq 3$

(ii) $x_1 + 2x_2 \leq 5$

(iii) $3x_1 + x_2 \leq 6$

and $x_1,\ x_2 \geq 0$

41. $Max\ Z = 2x_1 + x_2$

Subject to (i) $3x_1 + 4x_2 \leq 6$

(ii) $6x_1 + x_2 \leq 3$

and $x_1,\ x_2 \geq 0$

42. $Max\ Z = x_1 + x_2$

Subject to (i) $3x_1 + 3x_2 \leq 6$

(ii) $x_1 + 4x_2 \leq 4$

and $x_1,\ x_2 \geq 0$

43. $Max\ Z = 6x_1 - 2x_2 + 3x_3$

Subject to (i) $2x_1 - x_2 + 2x_3 \leq 2$

(ii) $x_1 + 4x_3 \leq 4$

and $x_1,\ x_2, x_3 \geq 0$

44. $Max\ Z = 3x_1 + 2x_2 + 5x_3$

Subject to (i) $x_1 + 2x_2 + x_3 \le 430$

(ii) $3x_1 + 2x_3 \le 460$

(iii) $x_1 + 4x_2 \le 420$

and $x_1,\ x_2, x_3\ \ \ge 0$

4.5 BOUNDED VARIABLES LP PROBLEM

In addition to the constraints in any LP problem, the value of some or all variables is restricted with lower and upper limits. In such cases the standard form of an LP problem appears as:

Optimize (Max or Min) Z = cx subject to the constraints Ax = b and l ≤ x ≤ u where l = $(l_1, l_2, \ldots, l_n)$ and u = $(u_1, u_2, \ldots, u_n)$ denote the lower and upper constraints bounds for variable x respectively. Other symbols have their usual meaning.

The inequality constraints l ≤ x ≤ u in the LP model can be converted into equality constraints by introducing slack and/or surplus variables s′ and s″ as follows:

x ≥ l or x – s″ = l, s″ ≥ 0 and x ≤ u or x + s′ = u, s′ ≥ 0

Thus, the given LP model contains m + n constraints equations with 3n variables. However, this size can be reduced to simply Ax = b.

The lower bound constraints l ≤ x can also be written as: x = l + s″, s″ ≥ 0, and therefore, with this substitution variable x can be eliminated from all the constraints.

The upper bound constraints x ≤ u can also be written as: x = u – s′, s′ ≥ 0. Such substitution, however, does not ensure non-negative value of x. It is in this context that a special technique known as bounded variable simplex method was developed in order to overcome this difficulty.

In bounded variable simplex method, the optimality condition for a solution is the same as the simplex method, discussed earlier. But the inclusion of constraints x + s′ = u in the simplex table requires modification in the feasibility condition of the simplex method due to the following reasons:

(i) A basic variable should become a non-basic variable at its upper bound (in usual simplex method all non-basic variables are at zero level).

(ii) When a non-basic variable becomes a basic variable, its value should not exceed its upper bound and should also not disturb the non-negativity and upper bound conditions of all existing basic variables.

4.6 THE SIMPLEX ALGORITHM

Step 1: (i) If the objective function of a given LP problem is of minimization, then convert it into that of maximization by using the following relationship:

Minimize Z = – Maximize Z* ; Z* = – Z

(ii) Check whether all b_I (i = 1, 2, . . ., m) are positive. If any one is negative, then multiply the corresponding constraint by –1 in order to make it positive.

(iii) Express the mathematical model of the given LP problem in standard form by adding slack/or surplus variables.

Step 2: Obtain an initial basic feasible solution. If any of the basic variables is at a positive lower bound, then substitute it out at its lower bound.

Step 3: Calculate $c_j - z_j$ as usual for all non-basic feasible. Examine values of $c_j - z_j$.

(i) If all $c_j - z_j \leq 0$, then the current basic feasible solution is the optimal solution.

(ii) If at least one $c_j - z_j > 0$ and this column has at least one entry positive (i.e., $y_{ij} > 0$) for some row i, then this indicates that an improvement in the value of objective function, Z is possible.

Step 4: If Case (ii) of Step 3 holds true, then select a non-basic variable to enter into the new solution according to the following criterion:

$$c_k - z_k = Min\left\{c_j - z_j \,:\, c_j - z_j > 0\right\}$$

Step 5: After identifying the column vector (non-basic variable) that will enter the basis matrix B, the vector to be removed from B is calculated. For this calculate the quantities:

$\theta_1 = Min\left\{\frac{x_{Bi}}{y_{ir}}, y_{ir} > 0\right\}$; $\theta_2 = Min\left\{\frac{u_r - x_{Bi}}{-y_{ir}}, y_{ir} < 0\right\}$ and $\theta = Min\,\{\theta_1, \theta_2, u_r\}$

where u_r is the upper bound for the variable x_r in the current basic feasible solution. Obviously, if all $y_{ir} > 0, \theta_2 = \infty$.

(i) If $\theta = \theta_1$, then the basic variable x_k (column vector a_k) is removed from the basis and is replaced by non-basic variable, say x_r (column vector a_r), as usual, by applying row operations.

(ii) If $\theta = \theta_1$, then the basic variable x_k (column vector a_k) is removed and replaced with a non-basic variable x_r (column vector a_r). But at this stage value of basic variable $x_r = x_{Br}$ is not at upper bound. This must be substituted out by using the relationship:

$$(x_{Bk})_r = (x_{Bk})'_r - y_{kr}u_r \,; 0 \le (x_{Bk})'_r \le u_r$$

where $(x_{Bk})'_r$ denotes the value of variables x_r.

The value of non-basic variable x_r is given at its upper bound value while the remaining non – basic variables are put at zero value by using the relationship:

$$x_r = u_r - x'_r \,; 0 \le x'_r \le u_r$$

(iii) If$\theta = u_r$, the variable x_r is given its upper bound value while the remaining non-basic variables are put at zero value by the relationship:

$$x_r = u_r - x'_r \,; 0 \le x'_r \le u_r$$

Step 6: Go to Step 4 and repeat the procedure until all θ entries in the $c_j - z_j$ row are either negative or zero.

Example 4.4 Solve the following LP problem:

Maximize Z = $3x_1 + 2x_2$ subject to the constraints

(i) $x_1 - 3x_2 \le 3$ (ii) $x_1 - 2x_2 \le 4$ (iii) $2x_1 + x_2 \le 20$ (iv) $x_1 + 3x_2 \le 30$

(v) $-x_1 + x_2 \le 6$ and $0 \le x_1 \le 8 \,; 0 \le x_2 \le 6$

Solution: We first add non – negative slack variable s_i (i = 1, 2, 3, 4, 5) to convert inequality constraints to equations. The standard form of LP problem then becomes:

Maximize $Z = 3x_1 + 2x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4 + 0s_5$ subject to the constraints

(i) $x_1 - 3x_2 + s_1 = 3$ (ii) $x_1 - 2x_2 + s_2 = 4$ (iii) $2x_1 + x_2 + s_3 = 20$

(iv) $x_1 + 3x_2 + s_4 = 30$ (v) $-x_1 + x_2 + s_5 = 6$ and $x_1, x_2, s_1, s_2, s_3, s_4, s_5 \ge 0$

The initial basic feasible solution to this problem is:

$$x_{B1} = s_1 = 3, x_{B2} = s_2 = 4, x_{B3} = s_3 = 20, x_{B4} = s_4 = 30, x_{B5} = s_5 = 6.$$

Since there are no upper bounds specified for these basic variables, arbitarily assume that all of them have upper bound at ∞. i.e., $s_1 = s_2 = s_3 = s_4 = s_5 = \infty$. This solution can also be read from the initial simplex Table 4.13.

		$u_i \rightarrow$	8	6	∞	∞	∞	∞	∞	
		$c_j \rightarrow$	3	2	0	0	0	0	0	
c_B	Basic Variables B	Solution Values $b(=x_B)$	x_1	x_2	s_1	s_2	s_3	s_4	s_5	$u_i - x_{Bi}$
0	s_1	3	(1)	−3	1	0	0	0	0	∞ − 3 = ∞ →
0	s_2	4	1	−2	0	1	0	0	0	∞ − 4 = ∞
0	s_3	20	2	1	0	0	1	0	0	∞ − 20 = ∞
0	s_4	30	1	3	0	0	0	1	0	∞ − 30 = ∞
0	s_5	6	−1	1	0	0	0	0	1	∞ − 6 = ∞
$Z = 0$		z_j	0	0	0	0	0	0	0	
		$c_j - z_j$	3 ↑	2	0	0	0	0	0	

Table 4.13 Initial Solution

Since $c_1 = z_1 = 3$ is largest positive, variable x_1 is eligible to enter into the basis. As none of the basic variables s_1 to s_5 are at their upper bound, thus, for deciding about the variable to leave the basis, we compute:

$\theta_1 = Min \left\{\frac{x_{Bi}}{y_{i1}}, y_{i1} > 0\right\} = Min \left\{\frac{3}{1}, \frac{4}{1}, \frac{20}{2}, \frac{30}{1}\right\} = 3$ (corresponds to x_1)

$\theta_2 = Min \left\{\frac{u_i - x_{Bi}}{-y_{i1}}, y_{i1} < 0\right\} = \frac{\infty - 6}{-(-1)} = \infty$ (corresponds to s_5) and $u_1 = 8$.

Therefore $\theta = Min\,\{\theta_1, \theta_2, u_1\} = Min\,\{3, \infty, 8\} = 3$ (corresponding to θ_1)

Thus, s_1 is eligible to leave the basis and therefore $y_{11} = 1$ becomes the key element.

Introduce x_1 into the basis and remove s_1 from the basis by applying row operations in the same manner as discussed earlier. The improved solution is shown in Table 4.14.

		$u_i \rightarrow$	8	6	∞	∞	∞	∞	∞	
		$c_j \rightarrow$	3	2	0	0	0	0	0	
c_B	Basic Variables B	Solution Values $b\ (= x_B)$	x_1	x_2	s_1	s_2	s_3	s_4	s_5	$u_i - x_{Bi}$
3	x_1	3	1	−3	1	0	0	0	0	$8 - 3 = 5$
0	s_2	1	0	(1)	−1	1	0	0	0	$\infty - 1 = \infty$ →
0	s_3	14	0	7	−2	0	1	0	0	$\infty - 14 = \infty$
0	s_4	27	0	6	−1	0	0	1	0	$\infty - 27 = \infty$
0	s_5	9	0	−2	1	0	0	0	1	$\infty - 9 = \infty$
$Z = 9$		z_j	0	−9	3	0	0	0	0	
		$c_j - z_j$	0	11 ↑	−3	0	0	0	0	

Table 4.14

Since $c_2 - z_2 = 11$ is largest positive, variable x_2 is eligible to enter into the basis. For deciding which variable should leave the basis, we compute:

$\theta_1 = Min \left\{\frac{x_{Bi}}{y_{i2}}, y_{i2} > 0\right\} \quad = Min \left\{\frac{1}{1}, \frac{14}{7}, \frac{27}{6}\right\} \quad = 1$ (corresponds to x_2)

$\theta_2 = Min \left\{\frac{u_i - x_{Bi}}{-y_{i2}}, y_{i2} < 0\right\} = Min \left\{\frac{8-3}{-(-3)}, \frac{\infty}{-(-2)}\right\} = \frac{5}{3}$ (corresponds to x_1) and $u_2 = 6$.

Therefore, $\theta = Min\,\{\theta_1, \theta_2, u_2\} = Min \left\{1, \frac{5}{3}, 6\right\} = 1$ (corresponds to s_2)

Thus, s_2 will leave the basis and $y_{22} = 1$ becomes the key element.

Introduce x_2 into the basis and remove s_2 from the basis as usual. The improved solution is shown in Table 4.15. Since $c_3 - z_3$ is largest positive, therefore variable s_1 is eligible to enter into the basis. We compute:

$\theta_1 = Min \left\{\frac{x_{Bi}}{y_{i3}}, y_{i3} > 0\right\} \quad = Min \left\{\frac{7}{5}, \frac{21}{5}\right\} \quad = \frac{7}{5}$ (corresponds to s_3)

$\theta_2 = Min \left\{\frac{u_i - x_{Bi}}{-y_{i1}}, y_{i1} < 0\right\} = Min \left\{\frac{8-6}{-(-2)}, \frac{6-1}{-(-1)}, \frac{\infty}{-(-1)}\right\} = 1$ (corresponds to x_1)

and $u_3 = \infty$.

Therefore, $\theta = Min\,\{\theta_1, \theta_2, u_3\} = Min \left\{\frac{7}{5}, 1, \infty\right\} = 1$ (corresponds to x_1)

Thus, x_1 will leave the basis and $y_{13} = -2$ becomes the key element.

		$u_i \rightarrow$	8	6	∞	∞	∞	∞	∞	
		$c_j \rightarrow$	3	2	0	0	0	0	0	
c_B	Basic Variables B	Solution Values $b (= x_B)$	x_1	x_2	s_1	s_2	s_3	s_4	s_5	$u_i - x_{Bi}$
3	x_1	6	1	0	(−2)	3	0	0	0	8 − 6 = 2 →
2	x_2	1	0	1	−1	1	0	0	0	6 − 1 = 5
0	s_3	7	0	0	5	−7	1	0	0	∞ − 7 = ∞
0	s_4	21	0	0	5	−6	0	1	0	∞ − 21 = ∞
0	s_5	11	0	0	−1	2	0	0	1	∞ − 11 = ∞
Z = 20		z_j	3	2	−8	11	0	0	0	
		$c_j - z_j$	0	0	8 ↑	−11	0	0	0	

Table 4.15

Introduce s_1 into the basis and remove x_1 from the basis, as usual. The improved solution is shown in Table 4.16.

		$u_i \rightarrow$	8	6	∞	∞	∞	∞	∞	
		$c_j \rightarrow$	3	2	0	0	0	0	0	
c_B	Basic Variables B	Solution Values $b (= x_B)$	x_1	x_2	s_1	s_2	s_3	s_4	s_5	$u_i - x_{Bi}$
0	s_1	−3	−1/2	0	1	−3/2	0	0	0	∞ − (−3) = ∞
2	x_2	−2	−1/2	1	0	−1/2	0	0	0	6 − (−2) = 8
0	s_3	22	5/2	0	0	1/2	1	0	0	∞ − 22 = ∞
0	s_4	36	5/2	0	0	3/2	0	1	0	∞ − 36 = ∞
0	s_5	8	−1/2	0	0	1/2	0	0	1	∞ − 8 = ∞
Z = −4		z_j	−1	2	0	−1	0	0	0	
		$c_j - z_j$	4	0	0	1	0	0	0	

Table 4.16

Since $c_1 - z_1 = 4$ is largest positive, therefore variable x_1 is eligible to enter into the basis. Also the upper bound for variable x_1 is 8; therefore we update the value of basic variables by using relationship and data of Table 4.16, as follows:

$$x_{B_1} = s_1 = x'_{B_1} - y_{11}u_1 = -3 - \left(-\frac{1}{2}\right)8 = 1$$

$$x_{B_2} = s_2 = x'_{B_2} - y_{21}u_1 = -2 - \left(-\frac{1}{2}\right)8 = 2$$

$$x_{B_3} = s_3 = x'_{B_3} - y_{31}u_1 = 22 - \left(\frac{5}{2}\right)8 \quad = 1$$

$$x_{B_4} = s_4 = x'_{B_4} - y_{41}u_1 = 36 - \left(\frac{5}{2}\right)8 \quad = 1$$

$$x_{B_5} = s_5 = x'_{B_5} - y_{51}u_1 = 8 - \left(-\frac{1}{2}\right)8 \quad = 1$$

Also one of the non – basic variables x_1 at its upper bound can be brought at zero level by using the substitution:

$$x_1 = u_1 - x'_1 = 8 - x'_1 \,; 0 \le x'_1 \le 8$$

The data of Table 4.16 can now be updated by substituting new values of basic variables as well as non – basic variables, as shown in Table 4.17. Since $c_4 - z_4$ is the only positive value, s_2 will enter into the basis. For deciding which variable should leave the basis, we compute:

$\theta_1 = Min\left\{\frac{x_{Bi}}{y_{i4}}, y_{i4} > 0\right\} = Min\left\{\frac{2}{1/2}, \frac{16}{3/2}, \frac{12}{1/2}\right\} = Min\left\{4, \frac{32}{3}, 24\right\} = 4$ (corresponds to s_3)

		$u_i \rightarrow$	8	6	∞	∞	∞	∞	∞	
		$c_j \rightarrow$	– 3	2	0	0	0	0	0	
c_B	Basic Variables B	Solution Values $b\ (= x_B)$	x'_1	x_2	s_1	s_2	s_3	s_4	s_5	$u_i - x_{Bi}$
0	s_1	1	1/2	0	1	– 3/2	0	0	0	∞ – 1 = ∞
2	x_2	2	1/2	1	0	– 1/2	0	0	0	6 – 2 = 4
0	s_3	2	– 5/2	0	0	(1/2)	1	0	0	∞ – 2 = ∞ →
0	s_4	16	– 5/2	0	0	3/2	0	1	0	∞ – 16 = ∞
0	s_5	12	1/2	0	0	1/2	0	0	1	∞ – 12 = ∞
$Z = 4 + 24 = 28$	z_j		1	2	0	– 1	0	0	0	
	$c_j - z_j$		– 4	0	0	1 ↑	0	0	0	

Table 4.17

$\theta_2 = Min\left\{\frac{u_i - x_{Bi}}{-y_{i4}}, y_{i4} < 0\right\} \quad = Min\left\{\frac{\infty}{-(-3/2)}, \frac{6-2}{-(-1/2)}\right\} = 8$ (corresponds to x_2)

and $u_4 = \infty$.

Therefore, $\theta = Min\,\{\theta_1, \theta_2, u_4\} = Min\,\{4, 8, \infty\} = 4$ (corresponds to s_3)

Thus, variable s_3 will leave the basis and $y_{34} = \frac{1}{2}$ becomes the key element.

Introduce s_2 into the basis and remove s_3 from the basis as usual. The improved solution is shown is Table 4.18.

		$u_j \rightarrow$	8	6	∞	∞	∞	∞	∞	
		$c_j \rightarrow$	– 3	2	0	0	0	0	0	
c_B	Basic Variables B	Solution Values $b\ (= x_B)$	x_1'	x_2	s_1	s_2	s_3	s_4	s_5	$u_i - x_{Bi}$
0	s_1	7	– 7	0	1	0	3	0	0	∞ – 1 = ∞
2	x_2	4	(– 2)	1	0	0	1	0	0	6 – 2 = 4 →
0	s_2	4	– 5	0	0	1	2	0	0	∞ – 2 = ∞
0	s_4	12	5	0	0	0	– 3	1	0	∞ – 16 = ∞
0	s_5	10	3	0	0	0	– 1	0	1	∞ – 12 = ∞
Z = 8 + 24 = 32		z_j	– 4	2	0	0	2	0	0	
		$c_j - z_j$	1 ↑	0	0	0	– 2	0	0	

Table 4.18

Since $c_1 - z_1$ is the only positive value, variable x_1' will enter the basis. To decide which variable will leave the bais, we compute:

$\theta_1 = Min \left\{\frac{x_{Bi}}{y_{i1}}, y_{i1} > 0\right\} \quad = Min \left\{\frac{12}{5}, \frac{10}{3}\right\} \quad = \frac{12}{5}$ (corresponds to s_4)

$\theta_2 = Min \left\{\frac{u_i - x_{Bi}}{-y_{i1}}, y_{i1} < 0\right\} = Min \left\{\frac{\infty}{-(-7)}, \frac{6-4}{-(-2)}, \frac{\infty}{-(-5)}\right\} \quad = 1$ (corresponds to x_2)

and $u_1 = 8$.

Therefore, $\theta = Min\,\{\theta_1, \theta_2, u_1\} = Min \left\{\frac{12}{5}, 1, 8\right\} = 1$ (corresponds to x_2)

Thus, x_2 will leave the basis and $y_{21} = -2$ becomes the key element.

Introduce x_1' into the basis and remove x_2 from the basis. The new solution is shown in Table 4.19.

		$u_j \rightarrow$	8	6	∞	∞	∞	∞	∞	
		$c_j \rightarrow$	– 3	2	0	0	0	0	0	
c_B	Basic Variables B	Solution Values $b\ (= x_B)$	x_1'	x_2	s_1	s_2	s_3	s_4	s_5	$u_i - x_{Bi}$
0	s_1	– 7	0	– 7/2	1	0	– 1/2	0	0	∞ + 7 = ∞
– 3	x'_1	– 2	1	– 1/2	0	0	– 1/2	0	0	– 3 + 2 = – 1
0	s_2	– 6	0	– 5/2	0	1	– 1/2	0	0	∞ + 6 = ∞
0	s_4	22	0	5/2	0	0	– 1/2	1	0	∞ – 22 = ∞
0	s_5	16	0	3/2	0	0	1/2	0	1	∞ – 16 = ∞
Z = 24 + 6		z_j	– 3	3/2	0	0	3/2	0	0	
= 30		$c_j - z_j$	0	1/2	0	0	– 3/2	0	0	

Table 4.19

Since the upper bound for variable x_2 is 6, we update the value of basic variables by using relationship and data of Table 4.19, as follows:

$$x_{B_1} = s_1 = x'_{B_1} - y_{12}u_2 = -7 - \left(-\frac{7}{2}\right)6 = 14$$

$$x_{B_2} = s_2 = x'_{B_2} - y_{22}u_2 = -2 - \left(-\frac{1}{2}\right)6 = 1$$

$$x_{B_3} = s_3 = x'_{B_3} - y_{32}u_2 = -6 - \left(-\frac{5}{2}\right)6 = 9$$

$$x_{B_4} = s_4 = x'_{B_4} - y_{42}u_2 = 22 - \left(\frac{5}{2}\right)6 \quad = 7$$

$$x_{B_5} = s_5 = x'_{B_5} - y_{51}u_1 = 16 - \left(\frac{3}{2}\right)6 \quad = 7$$

The non – basic variable x_2 at its upper bound can be brought at zero level by using the substitution: $x_2 = u_2 - x'_2 = 6 - x'_2\,; 0 \leq x'_2 \leq 6$

The data of Yable 4.19 can now be updated by substituting new values of basic variables and non – basic variables as shown in Table 4.20.

		$u_j \rightarrow$	8	6	∞	∞	∞	∞	∞
		$c_j \rightarrow$	– 3	– 2	0	0	0	0	0
c_B	Basic Variables B	Solution Values $b\ (= x_B)$	x'_1	x'_2	s_1	s_2	s_3	s_4	s_5
0	s_1	14	0	7/2	1	0	– 1/2	0	0
– 3	x'_1	1	1	1/2	0	0	– 1/2	0	0
0	s_2	9	0	5/2	0	1	– 1/2	0	0
0	s_4	7	0	– 5/2	0	0	– 1/2	1	0
0	s_5	7	0	– 3/2	0	0	1/2	0	1
$Z = 24 - 3 = 21$		z_j	– 3	– 3/2	0	0	3/2	0	0
		$c_j - z_j$	0	– 1/2	0	0	– 3/2	0	0

Table 4.20

In table 4.20, all $c_j - z_j \leq 0$, an optimal solution is arrived at with values of variables as: $x'_1 = 1$ (or) $x_1 = u_1 - x'_1 = 8 - 1\,;\ x_2 = u_2 - x'_2 = 6 - 0 = 6$ and Max Z = 33.

Example 4.5 Solve the following LP problem:

Maximize Z = $3x_1 + 2x_2 + 2x_3$ subject to the constraints

(i) $x_1 + 2x_2 + 2x_3 \leq 14$ (ii) $2x_1 + 4x_2 + 3x_3 \leq 23$ (iii) $0 \leq x_1 \leq 4$

(iv) $2 \leq x_2 \leq 5$ (v) $0 \leq x_3 \leq 3$

Solution: The variable x_2 has a positive lower bound, therefore taking $x'_2 = x_2 - 2$ (or) $x_2 = x'_2 + 2$. Then the fourth constraint of a given LP problem can be written as $0 \leq x'_2 \leq 3$ and new LP problem will become:

Maximize $Z = 3x_1 + 5(x_2' + 2) + 2x_3 = 3x_1 + 5x_2' + 2x_3 + 10$ subject to the constraints

(i) $x_1 + 2(x_2' + 2) + 2x_3 \leq 14$ (or) $x_1 + 2x_2' + 2x_3 \leq 10$

(ii) $2x_1 + 4(x_2' + 2) + 3x_3 \leq 23$ (or) $2x_1 + 4x_2' + 3x_3 \leq 15$

(iii) $0 \leq x_1 \leq 4$ (iv) $0 \leq x_2' \leq 3$ (v) $0 \leq x_3 \leq 3$

We now introduce non – negative slack variables s_1 and s_2 to convert inequality constraints to equations. The standard form of LP problem then becomes:

Maximize $Z = 3x_1 + 5x_2' + 2x_3 + 10 + 0s_1 + 0s_2$ subject to the constraints

(i) $x_1 + 2x_2' + 2x_3 + s_1 = 10$ (ii) $2x_1 + 4x_2' + 3x_3 + x_2 = 15$

and $x_1, x_2', x_3, s_1, s_2 \geq 0$

The initial basic feasible solution to this problem is: $x_{B_1} = s_1 = 10, x_{B_2} = s_2 = 15$. Also, for the basic variables s_1 and s_2 no upper bounds are specified, it is, therefore, assumed that both of these have an upper bound at ∞. The initial basic feasible solution can be read from the initial simplex Table 4.21.

		$u_j \rightarrow$	4	3	3	∞	∞	
		$c_j \rightarrow$	3	5	2	0	0	
c_B	Basic Variables B	Solution Values b $(= x_B)$	x_1	x_2'	x_3	s_1	s_2	$u_i - x_{Bi}$
0	s_1	10	1	2	2	1	0	$\infty - 10 = \infty$
0	s_2	15	2	(4)	3	0	1	$\infty - 15 = \infty$ →
$Z = 10$		z_j	0	0	0	0	0	
		$c_j - z_j$	3	5 ↑	2	0	0	

Table 4.21 Initial Solution

Since $c_2 - z_2 = 5$ is largest positive, variable x_2' will enter the basis. As none of the basic variables s_1 and s_2 are their upper bound, thus for deciding which variable will leave the basis, we compute:

$\theta_1 = Min \left\{\frac{x_{Bi}}{y_{i2}}, y_{i2} > 0\right\} = Min \left\{\frac{10}{2}, \frac{15}{4}\right\} = \frac{15}{4}$ (corresponds to s_2)

$\theta_2 = \infty$, because all entries in column 2 are positive, i.e., $y_{12} > 0$ for all i and $u_2 = 3$.

Therefore, $\theta = Min\,\{\theta_1, \theta_2, u_2\} = MIn \left\{\frac{15}{4}, \infty, 3\right\} = 3$ (corresponds to u_2)

Thus, x_2' is substituted at its upper bound and remains non – basic.

The non – basic variable x_2' at its upper bound can now be put at zero value by using the substitution: $x_2' = u_2 - x_2'' = 3 - x_2''$; $0 \le x_2'' \le 3$

The value of other basic variables are updated by using Table 4.22.

		$u_i \rightarrow$	4	3	3	∞	∞	
		$c_j \rightarrow$	3	– 5	2	0	0	
c_B	Basic Variables B	Solution Values b (= x_B)	x_1	x_2''	x_3	s_1	s_2	$u_i - x_{Bi}$
0	s_1	4	1	– 2	2	1	0	∞ – 4 = ∞
0	s_2	3	(2)	– 4	3	0	1	∞ – 3 = ∞ →
Z = 15 + 10		z_j	0	0	0	0	0	
= 25		$c_j - z_j$	3 ↑	– 5	2	0	0	

$$x_{B_1} = s_1 = x'_{B_1} - y_{12} u_2 = 10 - 2 \times 3 = 4$$
$$x_{B_2} = s_2 = x'_{B_2} - y_{22} u_2 = 15 - 4 \times 3 = 3$$

Table 4.22

The data in Table 4.21 can be now updated by putting new value of basic variables and non – basic variables x_2' as shown in Table 4.22.

In Table 4.22, $c_1 - z_1 = 3$ is largest positive, therefore, variable x_1 will enter into the basis. For deciding which variable should leave the basis, we compute:

$\theta_1 = Min \left\{\frac{x_{Bi}}{y_{i1}}, y_{i1} > 0\right\} = Min \left\{\frac{4}{1}, \frac{3}{1}\right\} = 3$ (corresponds to s_2)

$\theta_2 = \infty$, because all entries in column 1 are positive, i.e., $y_{i1} > 0$ for all i and $u_1 = 4$.

Therefore, $\theta = Min\,\{\theta_1, \theta_2, u_2\} = MIn\,\{3, \infty, 4\} = 3$ (corresponds to θ_1)

Thus, variable s_2 will leave the basis and $y_{21} = 2$ becomes the key element.

Introduce x_1 into the basis and remove s_2 from the basis, as usual. The improved solution is shown in Table 4.23.

		$u_i \rightarrow$	4	3	3	∞	∞	
		$c_j \rightarrow$	3	– 5	2	0	0	
c_B	Basic Variables B	Solution Values b (= x_B)	x_1	x_2''	x_3	s_1	s_2	$u_i - x_{Bi}$
0	s_1	5/2	0	0	1/2	1	– 3/2	∞ – 5/2 = ∞
3	x_1	3/2	1	(–2)	3/2	0	1/2	3 – 3/2 = 3/2 →
Z = 25 + 9/2		z_j	3	– 6	9/2	0	3/2	
= 59/2		$c_j - z_j$	0	1 ↑	– 5/2	0	– 3/2	

Table 4.23

In Table 4.23, $c_2 - z_2 = 1$ is the only positive value, therefore, variable x_2'' will enter into the basis. For deciding which variable should leave the basis, we compute:

$\theta_1 = \infty$, because all entries in column 2 are either negative or zero, i.e., $y_{i2} > 0$ for all i (corresponds to s_2)

$\theta_2 = Min\left\{\frac{u_i - x_{Bi}}{-y_{i2}}, y_{i2} < 0\right\} = Min\left\{\infty, \frac{5/2}{-(-2)}\right\} = \frac{5}{4}$, and $u_2 = 3$.

Therefore, $\theta = Min\,\{\theta_1, \theta_2, u_2\} = Min\left\{\infty, \frac{5}{4}, 3\right\} = \frac{5}{4}$ (corresponds to θ_2)

Thus, variable x_1 will leave the basis. To put $x_1$at its upper bound, substitute $x_1 = 4 - x_1'$ in Table 4.23. The improved solution is shown in Table 4.24

		$u_i \rightarrow$	4	3	3	∞	∞
		$c_j \rightarrow$	3	– 5	2	0	0
c_B	Basic Variables B	Solution Values $b\ (= x_B)$	x_1	x_2''	x_3	s_1	s_2
0	s_1	5/2	0	0	1/2	1	– 1/2
– 5	x_2''	– 3/4	– 1/2	1	– 3/4	0	– 1/4
Z = 25 + 15/4		z_j	5/2	– 5	15/4	0	5/4
= 115/4		$c_j - z_j$	1/2	1	– 7/4	0	– 5/4

Table 4.24

The value of non – basic variable x_1 at its upper bound 4 can be put zero level by substituting

$x_1 = 4 - x_1'\,; 0 \leq x_1' \leq 4.$

The value of other basic variables are updated by using the relationship:

$$x_{B_1} = s_1 = x_{B_1}' - y_{11}u_1 = \frac{5}{2} - 0 \times 4 \qquad = \frac{5}{2}$$

$$x_{B_2} = x_2'' = x_{B_2}' - y_{21}u_1 = -\frac{3}{4} - (-\frac{1}{2}) \times 4 = \frac{5}{2}$$

The data in Table 4.24 can now be updated by putting new values of basic variables and non – basic variable x_1' as shown in Table 4.25.

		$u_i \rightarrow$	4	3	3	∞	∞
		$c_j \rightarrow$	– 3	– 5	2	0	0
c_B	Basic Variables B	Solution Values $b\ (= x_B)$	x_1'	x_2''	x_3	s_1	s_2
0	s_1	5/2	0	0	1/2	1	– 1/2
– 5	x_2''	5/4	1/2	1	– 3/4	0	– 1/4
Z = 123/4		z_j	– 5/2	– 5	15/4	0	5/4
		$c_j - z_j$	– 1/2	0	– 7/4	0	– 5/4

Table 4.25

In table 4.25, all $c_j - z_j \leq 0$, an optimal solution is arrived at with values of variables:

$$x_1' = 0\ (or)\ 4 - x_1 = 0\ (or)\ x_1 = 4$$

$$x_2' = \frac{5}{4}\ (or)\ 3 - x_2' = \frac{5}{4}\ (or) 3 - (x_2 - 2) = \frac{5}{4}\ (or)\ x_2 = \frac{15}{4}$$

and Max Z = $\frac{123}{4}$.

Let us sum up

We have learned about bounded variable LPP to find the solution by using simplex algorithm.

Check Your Progress

Solve the following LP problems

45. $Max\ Z = x_2 + 3x_3$

Subject to $x_1 + x_2 + x_3 \leq 10$

$$x_1 - 2x_3 \leq 0$$

$$2x_2 - x_3 \leq 10$$

$$0 \leq x_1 \leq 8;\ 0 \leq\ x_2 \leq 4;\ x_3\ \ \geq 0$$

46. $$Max\ Z = 4x_1 + 4x_2 + 3x_3$$

Subject to
$$-x_1 + 2x_2 + 3x_3 \leq 15$$
$$-x_2 + x_3 \leq 4$$
$$2x_1 + x_2 - x_3 \leq 6$$
$$x_1 - x_2 + 2x_3 \leq 10$$
$$0 \leq x_1 \leq 8;\ x_2 \geq 0;\ x_3\ \leq 4$$

Unit Summary

The revised simplex method is another efficient method. It is efficient in the sense that at each iteration, we need not recompute values of all the variables, in the simplex table, while moving from one iteration to next in search of an improved solution of an LP problem. In the usual simplex method, at each iteration it was necessary to calculate $c_j - z_j$ corresponding to non-basic variable columns in order to decide whether the current solution is optimal or not. If not, then in order to select the non-basic variable to enter into the basis matrix B, we first need to know $y_j = B^{-1}\,a_j$, where y_j refers to the updated column a_j in the simplex table being examined. If $y_j \leq 0$, then the optimal solution is unbounded. Otherwise, apply the minimum ratio rule to decide which basic variable should leave the basis.

In bounded variable simplex method, the optimality condition for a solution is the same as the simplex method. But the inclusion of upper bound x ≤ u or x = u – s′ or constraints x + s′ = u in the simplex table requires modification in the feasibility condition of the simplex method due to the following reasons:

A basic variable should become a non-basic variable at its upper bound (in usual simplex method all non-basic variables are at zero level).

When a non-basic variable becomes a basic variable, its value should not exceed its upper bound and should also not disturb the non-negativity and upper bound conditions of all existing basic variables.

Glossary

- Max Z – maximize Z
- Min Z – minimize Z

Self- Assessment Questions

1. Formulate a linear programming problem in the form of the revised simplex method.
2. Develop a computations algorithm for solving a linear programming problem by the revised

simplex method.

3. Compare the revised simplex method with simplex method and bring out the salient points of the

difference.

4. Give a brief outline for the standard form I of the revised simplex method.
5. What is the difference between simplex method and revised simplex method? When and where should the two be applied?

ACTIVITIES

Solve the following LP problems:

1. $Max\ Z = 4x_1 + 10x_2 + 9x_3 + 11x_4$

Subject to $2x_1 + 2x_2 + 2x_3 + 2x_4 \le 5$

$48x_1 + 80x_2 + 160x_3 + 240 \le 257$

$0 \le x_j \le 1; j = 1,2,3,4.\ .$

2. $Max\ Z = 3x_1 + x_2 + x_3 + 7x_4$

Subject to $2x_1 + 3x_2 - x_3 + 4x_4 \le 40$

$-2x_1 + 2x_2 + 5x_3 - x_4 \le 35$

$x_1 + x_2 - 2x_3 + 3x_4 \le 100$

$x_1 \ge 2, x_2 \ge 1, x_3 \ge 3, x_4 \ge 4\ .$

Suggested Readings

1. J. K. Sharma, *Operations Research*, Theory and Applications, Third Edition (2007) Macmillan India Ltd
2. Hamdy A. Taha, *Operations Research,* (seventh edition) Prentice - Hall of India Private Limited, New Delhi, 1997.
3. F.S. Hillier & J.Lieberman *Introduction to Operation Research* (7^{th} Edition) Tata-McGraw Hill ompany, New Delhi, 2001.
4. Beightler. C, D.Phillips, B. Wilde ,*Foundations of Optimization* (2^{nd} Edition) Prentice Hall Pvt Ltd., New York, 1979
5. S.S. Rao - *Optimization Theory and Applications*, Wiley Eastern Ltd. New Delhi. 1990

UNIT – V

PARAMETRIC LINEAR PROGRAMMING

PARAMETRIC LINEAR PROGRAMMING

Objectives

After studying this unit you should be able to appreciate the need of parametric analysis to find various basic feasible solutions of any LP problem, Which become optimal one after the other due to continuous variations in the parameters of LP problem. Perform parametric analysis to study variation in the objective function coefficients and resources availability. Take care of change in the optimal solution due to variation in LP model parameters over a range of variation.

Appreciate the need of a goal programming approach for solving multi-objective decision problems. Distinguish between LP and GP approaches for solving a business decision problem. Formulate GP model of the given multi-objective decision problem. Understand the method of assigning different ranks and weights to unequal multiple goals. Use simplex method for solving a GP model.

5.1 INTRODUCTION

Once an LP model based on real-life data has been solved, the decision-maker desires to know how the solution will change if parameters, such as cost (or profit) c_j, availability of resources b_i and the technological coefficients a_{ij} are changed. We have already discussed the need to perform a sensitivity analysis in order to consider the impact of discrete changes in its parameters on optimal solution of LP model. In this chapter, we will discuss another parameter variation analysis also called parametric analysis to find various basic feasible solutions of an LP model that become optimal one after the other, due to continuous variations in the parameters. Since LP model parameters change as a linear function of a single parameter, this technique is known as linear parametric programming.

The purpose of this analysis is to keep to a minimum the additional efforts required to take care of changes in the optimal solution due to variation in LP model parameters over a range of variation. In this chapter we will perform parametric analysis only for the following two parameters (evaluation of other parameters, over a range, is also possible but tend to be more complex.)

(i) Variation in objective function coefficients, c_j

(ii) Variation in resources availability (Right-hand side values), b_i

Let λ be the unknown (positive or negative) scalar parameter with which coefficients in the LP model vary. We start the analysis at optimal solution obtained at λ = 0. Then, using the optimality and feasibility conditions of the simplex method we determine the range of λ for which the optimal solution at λ = 0 remains unchanged. Let λ lies between 0 and λ_1. This means 0 ≤ λ ≤ λ_1 is the range of λ beyond which the current solution will become infeasible and/or non-optimal. Thus at λ = λ_1 a new solution is determined which remains optimal and feasible in other interval, say λ_1 ≤ λ ≤ λ_2. Again a new solution at λ = λ_2 is obtained. The process of determining the range of λ is repeated till a stage is reached beyond which the solution either does not change or exist.

5.2 VARIATION IN THE OBJECTIVE FUNCTION COEFFICIENTS

Let us define the parametric linear programming model as follow:

$$\text{Maximize } Z = \sum_{j=1}^{n}\left(c_j + \lambda c_j'\right)x_j$$

subject to the constraints

$$\sum_{j=1}^{n} a_{ij}x_j = b_i \,;\, i = 1, 2, \dots, m \text{ and } x_j \geq 0 \,;\, j = 1, 2, \dots, n$$

where $\lambda c_j'$ represents predetermined variation in the parameter c and $\lambda \geq 0$ is a scalar parameter. Now the aim is to determine such consecutive values of λ at which the current optimal basic feasible solution tends to change with a change in the coefficients c_j. Such consecutive values of λ are called critical (range) values of λ and are measured from λ = 0. Thus, the given LP problem is initially solved by using simplex method at λ = 0. Since changes in cost coefficient c_j only affect the optimality of the current solution, therefore as λ changes only $c_j - z_j$ values are affected. Hence, for the perturbed LP problem let us calculate $c_j - z_j$ values corresponding to non-basic variable columns in the optimal simplex table as follows:

$$c_j(\lambda) - z_j(\lambda) = c_j(\lambda) - c_B(\lambda)B^{-1}a_j = \left(c_j + \lambda c_j'\right) - (c_B + \lambda c_B')y_j \; ; \; y_j = B^{-1}a_j$$

$$= \left(c_j - c_B y_j\right) + \lambda\left(c_j' - c_B' y_j\right) = \left(c_j - z_j\right) + \lambda\left(c_j' - z_j'\right) ; \; z_j = c_B y_j$$

For a solution to be optimal for all values of λ we must have $c_j(\lambda) - z_j(\lambda) \le 0$ (maximization case) and $c_j(\lambda) - z_j(\lambda) \ge 0$ (minimization case). These inequalities, for a given solution, are also used for determining the range $\lambda_1 \le \lambda \le \lambda_2$, within which the current solution remains optimal as follows:

$$\lambda = Min \left\{\frac{-\left(c_j - z_j\right)}{\left(c_j' - z_j'\right)}\right\}$$

where $\left(c_j' - z_j'\right) > 0$ for maximization and $\left(c_j' - z_j'\right) < 0$ for minimization.

Example 5.1 Consider the parametric linear programming problem:

Maximize Z = $(3 - 6\lambda)x_1 + (2 - 2\lambda)x_2 + (5 + 5\lambda)x_3$

subject to the constraints

(i) $x_1 + 2x_2 + x_3 \le 430$ (ii) $3x_1 + 2x_3 \le 460$ (iii) $3x_1 + 4x_2 \le 420$

and $x_1, x_2, x_3 \ge 0$. Perform the parametric analysis and identify all the critical values of the

parameter λ.

Solution:

The given parametric LP problem can be written in its standard form as:

Maximize Z = $(3 - 6\lambda)x_1 + (2 - 2\lambda)x_2 + (5 + 5\lambda)x_3 + 0s_1 + 0s_2 + 0s_3$

subject to the constraints

(i) $x_1 + 2x_2 + x_3 + s_1 = 430$ (ii) $3x_1 + 2x_3 + s_2 = 460$

(iii) $3x_1 + 4x_2 + s_3 = 420$ and $x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$.

According to the problem, we have:

$$c(\lambda) = c_j + \lambda c'_j = (3, 2, 5, 0, 0, 0) + \lambda(-6, -2, 5, 0, 0, 0)$$

Solving the given LP problem with $\lambda = 0$. The optimal solution at $\lambda = 0$ is shown at table 5.1.

		$c_j \rightarrow$	3	2	5	0	0	0
c_B	Basic Variables B	Solution Values $b\ (= x_B)$	x_1	x_2	x_3	s_1	s_2	s_3
2	x_2	100	– 1/4	1	0	1/2	– 1/4	0
5	x_3	230	3/2	0	1	0	1/2	0
0	s_3	20	2	0	0	– 2	1	1
$Z = 1{,}350$		z_j	7	2	5	1	2	0
		$c_j - z_j$	– 4	0	0	– 1	– 2	0

Table 5.1: Optimal Solution at $\lambda = 0$

The optimal solution is $x_1 = 0, x_2 = 100, x_3 = 230\ and\ Max\ Z = 1350$.

In order to find the first critical (or range) value of λ in which the solution shown in Table 5.1 remains optimal. We first find $c'_j - z'_j$ values corresponding to non – basic variables $x_1, s_1\ and\ s_2$ columns as follows:

$$c'_j - z'_j = c'_j - c'_B y_j = (-6, 0, 0) - (-2, 5, 0)\begin{bmatrix} -1/2 & 1/2 & -1/4 \\ 3/2 & 0 & 1/2 \\ 2 & -2 & 1 \end{bmatrix}; j = 1, 4, 5$$

$$= (-6, 0, 0) - \left[\frac{1}{2} + \frac{15}{2}, -1, \frac{1}{2} + \frac{5}{2}\right] = (-6, 0, 0) - (8, -1, 3) = (-14, 1, -3)$$

For a maximization LP problem, the current solution will remain optimal provided all $c_j(\lambda) - z_j(\lambda) \leq 0$. Since $c'_j - z'_j > 0$, the first critical value of λ is given by:

$$\lambda_1 = Min\left\{\frac{-(c_j - z_j)}{(c'_j - z'_j) > 0}\right\} = -\frac{(c_4 - z_4)}{c'_4 - z'_4} = -\frac{(-1)}{1} = 1$$

This means that for $\lambda_1 \in [0, 1]$, the solution given in Table 5.1 remains optimal. The objective function value in thisinterval is given by:

$$Z(\lambda) = Z + Z'(\lambda) = c_B x_B + \lambda c'_B x_B = 1350 + 950\lambda$$

Now, for values of λ other than zero in the interval [0, 1], we compute $c_j(\lambda) - z_j(\lambda)$ values for non – basic variables x_1, s_1, s_2 as shown in Table 5.2.

$$c_1(\lambda) - z_1(\lambda) = (c_1 - z_1) + \lambda(c'_1 - z'_1) = -4 - 14\lambda \leq 0 \; (or) \; \lambda \geq \frac{2}{7}$$

$$c_4(\lambda) - z_4(\lambda) = (c_4 - z_4) + \lambda(c'_4 - z'_4) = -1 + \lambda \quad \leq 0 \; (or) \; \lambda \geq 1$$

$$c_5(\lambda) - z_5(\lambda) = (c_5 - z_5) + \lambda(c'_5 - z'_5) = -2 - 3\lambda \quad \leq 0 \; (or) \; \lambda \geq -\frac{2}{3}$$

The optimal solution of any value of λ in the interval [0,1] is given in Table 5.2.

			$c'_j \rightarrow$	-6	-2	5	0	0	0
			$c_j \rightarrow$	3	2	5	0	0	0
c'_B	c_B	Basic Variables B	Solution Values $b\,(= x_B)$	x_1	x_2	x_3	s_1	s_2	s_3
2	2	x_2	100	$-1/4$	1	0	1/2	$-1/4$	0
5	5	x_3	230	3/2	0	1	0	1/2	0
0	0	s_3	20	2	0	0	-2	1	1
$Z(\lambda) = 1{,}350 + 950\lambda$			$c_j - z_j$	-4	0	0	-1	-2	0
			$c'_j - z'_j$	-14	0	0	1	-3	0
			$c_j(\lambda) - z_j(\lambda)$	$-4 - 14\lambda$	0	0	$-1 + \lambda$	$-2 - 3\lambda$	0

Table 5.2

At $\lambda = 1, c_4(\lambda) - z_4(\lambda) = 0$ in the $'s_1'$ column. But for $\lambda > 1, c_4(\lambda) - z_4(\lambda) > 0$ for non – basic variable s_1 and hence the solution in Table 5.2 no longer remains optimal.

We now enter variable s_1 in the solution to find new optimal solution. The new optimal solution shown in Table 5.3 is $x_1 = 0, x_2 = 0, x_3 = 230$ and

Max Z = 2300.

			$c'_j \rightarrow$	−6	−2	5	0	0	0
			$c_j \rightarrow$	3	2	5	0	0	0
c'_B	c_B	Basic Variables B	Solution Values $b (= x_B)$	x_1	x_2	x_3	s_1	s_2	s_3
0	0	s_1	200	−1/2	2	0	1	−1/2	0
5	5	x_3	230	3/2	0	1	0	1/2	0
0	0	s_3	420	1	4	0	0	0	1
$Z(\lambda) = 2{,}300$			$c_j - z_j$	−9/2	2	0	0	−5/2	0
			$c'_j - z'_j$	−27/2	−2	0	0	−5/2	0
			$c_j(\lambda) - z_j(\lambda)$	−18	0	0	0	−5	0

Table 5.3: Optimal Solution at λ = 1

The solution shown in Table 5.3 will be optimal if all $c_j(\lambda) - z_j(\lambda) \leq 0$, j = 1, 2, 5. The check the optimality we compute these values for the non-basic variables x_1, x_2 and s_2 as follows:

$$c_1(\lambda) - z_1(\lambda) = (c_1 - z_1) + \lambda(c'_1 - z'_1) = -\frac{9}{2} - \frac{27}{2}\lambda \leq 0\ (or)\ \lambda \geq -\frac{1}{3}$$

$$c_2(\lambda) - z_2(\lambda) = (c_2 - z_2) + \lambda(c'_2 - z'_2) = 2 - 2\lambda \quad \leq 0\ (or)\ \lambda \geq 1,$$

where is true

$$c_5(\lambda) - z_5(\lambda) = (c_5 - z_5) + \lambda(c'_5 - z'_5) = -\frac{5}{2} - \frac{5}{2}\lambda\ \leq 0\ (or)\ \lambda \geq -1$$

Therefore, for λ = 1, the $c_j(\lambda) - z_j(\lambda) \leq 0$ for all non – basic variable columns and hence the solution in Table 5.3 is optimal: $x_1 = x_2 = 0, x_3 = 230$ and Max Z = 2,300.

For λ ≤ –2/3 , $c_j(\lambda) - z_j(\lambda)$ value for non-basic variable s_2 becomes positive and again solution shown in Table 5.3 no longer remains optimal. Entering variable s_2 in the basis to find new optimal solution. The variable s_2 will replace basic variable s_3 in the basis. The new optimal solution is shown in Table 5.4.

			$c'_j \rightarrow$	-6	-2	5	0	0	0
			$c_j \rightarrow$	6	2	5	0	0	0
c'_B	c_B	Basic Variables B	Solution Values $b\ (=x_B)$	x_1	x_2	x_3	s_1	s_2	s_3
–2	2	x_2	105	1/4	1	0	0	0	1/4
5	5	x_3	220	1/2	0	1	1	0	– 1/2
0	0	s_2	20	2	0	0	–2	1	1
Z = 1310 + 890 λ			$c_j - z_j$	0	0	0	–5	0	2
			$c'_j - z'_j$	– 8	0	0	–5	0	3

Table 5.4: Optimal Solution

Solution shown in Table 5.4 will be optimal only when c_jλ) – c_j(λ) ≤ 0 ; j = 1, 4, 6

$$c_1(\lambda) - z_1(\lambda) = (c_1 - z_1) + \lambda(c'_1 - z'_1) = 0 - 8\lambda \quad \leq 0\ (or)\ \lambda \geq 0$$

$$c_4(\lambda) - z_4(\lambda) = (c_4 - z_4) + \lambda(c'_4 - z'_4) = -5 - 5\lambda \ \leq 0\ (or)\ \lambda \geq -1$$

$$c_6(\lambda) - z_6(\lambda) = (c_6 - z_6) + \lambda(c'_6 - z'_6) = 2 + 3\lambda \quad \leq 0\ (or)\ \lambda \geq \frac{2}{3}$$

Thus for –1 ≤ λ ≤ –2/3, the optimal solution is: $x_1 = 0, x_2 = 105, x_3 = 220$ and Max Z = 1310 + 890 λ. Hence Table 5.2 to 5.4 give family of optimal solutions for – 2/3 ≤ λ ≤ 1, λ ≥ 1 and –1≤ λ ≤ –2/3. The critical value of λ are –2/3 and 1.

Example 5.2 Consider the linear programming problem:

Minimize Z = $-x_1 - 3x_2$

subject to the constraints

(i) $x_1 + x_2 \leq 6$ (ii) $-x_1 + 2x_2 \leq 6$ and $x_1, x_2 \geq 0$.

The optimal solution to this solution is given in Table 5.5.

		$c_j \rightarrow$	-1	-3	0	0
c_B	Basic Variables B	Solution Values $b\ (=x_B)$	x_1	x_2	s_1	s_2
-1	x_1	2	1	0	2/3	$-1/3$
-3	x_2	4	0	1	1/3	1/3
$Z = -14$		z_j	-1	-3	$-5/3$	$-2/3$
		$c_j - z_j$	0	0	5/3	2/3

Table 5.5: Optimal Solution at $\lambda = 0$

Solve this problem of the variation in the cost vector is $c' = (2, 1, 0, 0)$. Identify all the critical values of parameter λ.

Solution: The given parametric LP model in its standard form is stated as:

Minimize Z = $(-1 + 2\lambda)x_1 + (-3 + \lambda)x_2 + 0s_1 + 0s_2$ subject to the constraints

(i) $x_1 + x_2 + s_1 = 6$ (ii) $-x_1 + 2x_2 + s_2 = 6$ and $x_1, x_2, s_1, s_2 \geq 0$

When $\lambda = 0$, the given parametric LP problem reduces to the ordinary LP problem whose optimal solution: $x_1 = 2, x_2 = 4$ and Max Z = –14 is given in Table 5.5.

In order to find the first critical value of λ other than zero for which the solution shown in Table 5.6 is optimal, we first find $c_j' - z_j'$ values corresponding to the non – basic variables s_1 and s_2 as follows:

$$c_j' - z_j' = c_j' - c_B' y_j = (0, 0) - (2, 1)\begin{bmatrix} 2/3 & -1/3 \\ 1/3 & 1/3 \end{bmatrix} ; j = 3, 4$$

$$= (0, 0) - \left[\frac{5}{3}, -\frac{1}{3}\right] = \left[-\frac{5}{3}, \frac{1}{3}\right]$$

Since LP problem is of minimization, at the optimal solution we must have

$c_j(\lambda) - z_j(\lambda) \geq 0$ for all j. Also as $c_3' - z_3' < 0$, the first critical value of λ is given by

$$\lambda_1 = Min\left\{\frac{-(c_j - z_j)}{(c_j' - z_j') < 0}\right\} = -\frac{(c_3 - z_3)}{c_3' - z_3'} = \frac{-(5/3)}{-5/3} = 1$$

This means that for $\lambda_1 \in [0, 1]$, the solution in Table 5.6 remains optimal. The objective function value in this interval is given by

$$Z(\lambda) = Z + Z'(\lambda) = c_B x_B + \lambda c'_B x_B = -14 + \lambda[2, 1]\begin{bmatrix}2\\4\end{bmatrix} = -14 + 8\lambda$$

Thus for values of λ other than zero in the interval [0, 1], we compute the solution as shown in Table 5.6.

			$c'_j \rightarrow$	*2*	*1*	*0*	*0*
			$c_j \rightarrow$	*– 1*	*– 3*	*0*	*0*
c'_B	c_B	*Basic Variables* **B**	*Solution Values* $b\ (= x_B)$	x_1	x_2	s_1	s_2
2	– 1	x_1	2	1	0	2/3	– 1/3
1	– 3	x_2	4	0	1	1/3	1/3
$Z(\lambda) = -14 + 8\lambda$			$c_j - z_j$	0	0	5/3	2/3
			$c'_j - z'_j$	0	0	– 5/3	1/3
			$c_j(\lambda) - z_j(\lambda)$	0	0	$\frac{5}{3} - \frac{5}{3}\lambda$	$\frac{2}{3} + \frac{1}{3}\lambda$

Table 5.6

The optimal solution given in Table 5.6 will remain optimal if all

$$c_j(\lambda) - z_j(\lambda) \geq 0\ ; j = 3, 4$$

$$c_3(\lambda) - z_3(\lambda) = (c_3 - z_3) + \lambda(c'_3 - z'_3) = \left(\frac{5}{3}\right) - \left(\frac{5}{3}\right)\lambda$$

$$c_4(\lambda) - z_4(\lambda) = (c_4 - z_4) + \lambda(c'_4 - z'_4) = \left(\frac{2}{3}\right) + \left(\frac{1}{3}\right)\lambda$$

At $\lambda = 1$, $c_3(\lambda) - z_3(\lambda) = 0$ and $c_4(\lambda) - z_4(\lambda) > 0$. So entering variable s_1 in the basis. The new solution is shown in Table 5.7.

			$c'_j \rightarrow$	2	1	0	0
			$c_j \rightarrow$	−1	−3	0	0
c'_B	c_B	Basic Variables B	Solution Values $b\ (= x_B)$	x_1	x_2	s_1	s_2
0	0	s_1	3	3/2	0	1	−1/2
1	−3	x_2	3	−1/2	1	0	1/2
$Z(\lambda) = -9 + 3\lambda$			$c_j - z_j$	−5/2	0	0	3/2
			$c'_j - z'_j$	5/2	0	0	−1/2
			$c_j(\lambda) - z_j(\lambda)$	$-\frac{5}{2} + \frac{5}{3}\lambda$	0	0	$\frac{3}{2} - \frac{1}{2}\lambda$

Table 5.7: Optimal Solution at λ = 1

Since LP problem is of minimization, therfore for the solution to be optimal we must have $c_j(\lambda) - z_j(\lambda) \geq 0$ for all j. But $c_j(\lambda) - z_j(\lambda) \geq 0$ for non – basic variable columns 1 and 4 at λ = 1. Thus the current solution: $x_1 = 0, x_2 = 3$ and Min Z = – 6 is the optimal solution. However, we need to find the new critical value of λ in the interval [1, λ_2] over which the solution shown in Table 5.7 remains optimal. For this computing:

$$\lambda_2 = Min \left\{ \frac{-(c_j - z_j)}{(c'_j - z'_j) < 0} \right\} = -\frac{(c_4 - z_4)}{c'_4 - z'_4} = \frac{-(3/2)}{-(1/2)} = 3$$

This shows that if λ ∈ [1, 3], the $c_j(\lambda) - z_j(\lambda) \geq 0$ for j = 1, 4 and the solution shown in Table 5.7 is optimal. In the interval [1, 3], the value of objective function is given by

$$Z(\lambda) = Z + Z'(\lambda) = c_B x_B + \lambda c'_B x_B = [0, -3] \begin{bmatrix} 3 \\ 3 \end{bmatrix} + \lambda [0, 1] \begin{bmatrix} 3 \\ 3 \end{bmatrix} = -9 + 3\lambda$$

At $\lambda = 3, c_4(\lambda) - z_4(\lambda) = 0$, so entering variable s_2 into the basis and remove x_2 from the basis to get a new solution as shown in Table 5.8.

			$c'_j \rightarrow$	2	1	0	0
			$c_j \rightarrow$	-1	-3	0	0
c'_B	c_B	Basic Variables B	Solution Values $b\,(=x_B)$	x_1	x_2	s_1	s_2
0	0	s_1	6	1	1	1	0
0	0	s_2	6	-1	2	0	1
$Z = 0$			$c_j - z_j$	-1	-3	0	0
			$c'_j - z'_j$	2	1	0	0
			$c_j(\lambda) - z_j(\lambda)$	5	0	0	0

Table 5.8: Optimal Solution at $\lambda = 3$

All $c_j(\lambda) - z_j(\lambda) \geq 0$ in Table 5.8. However, we find the interval [3, λ_3] in which this solution remains optimal as follows:

$$\lambda_3 = Min\left\{\frac{-(c_1 - z_1)}{(c'_1 - z'_1) < 0}\right\}$$

But all $\left(c'_j - z'_j\right) \geq 0$, this solution will remain optimal for all values of λ in the interval [3, ∞].

5.3 VARIATION IN THE AVAILABILITY OF RESOURCES (RHS) VALUES

Let us define the parametric linear programming model as follows:

Maximize Z = $\sum_{j=1}^{n} c_j x_j$

subject to the constraints

$\sum_{j=1}^{n} a_{ij} x_j = b_i \pm \lambda b'_i\,;\, i = 1, 2, \ldots, m$ and $x_j \geq 0\,;\, j = 1, 2, \ldots, n$

where $b_i \pm \lambda b'_i$ is the predetermined variation in resource values (right-hand side values), where $\lambda \geq 0$ is a scalar parameter. Now our aim is to find the range (or critical values) of λ so that the current optimal solution remains unchanged with a change in the right-hand side constants b_i, for all i.

Let B_0 and $x_{B_0} = B_0^{-1} b$ b be the optimal basis and optimal basic feasible solution, respectively, of the original LP problem when it is solved at λ = 0.

If b is replaced by b + λb' , then the optimality condition $c_j - z_j$ will not be affected. However, such changes will affect the value of the basic variables. The new values are given by:

$$x_B(\lambda) = B^{-1}(b + \lambda b') = B^{-1}b + \lambda B^{-1}b' = x_B + \lambda x_B'$$

Now as long as x_B(λ) ≥ 0, the current basis remains optimal. Thus, this criterion can be used to determine the range of λ, within which the solution remains optimal, as follows:

$$\lambda = Min \left\{ \frac{x_{Bi}}{-x_{Bi}' < 0} \right\} = \frac{x_{Br}}{-x_{Br}'}$$

Let λ = λ_1. Then for λ ∈ [0, λ_1], the current solution remains optimal and at this solution the value of the objective function is given by $Z(\lambda) = Z + \lambda Z' = c_B x_B + \lambda c_B x_B'$. At λ_1 the current basis x_B (right hand side) is replaced by $x_B(\lambda) = B^{-1}(b + \lambda b')$ and x_{Br} is removed from the basis by the usual simplex method. The process of finding the new range [λ_1, λ_2] of values of λ is repeated, over which the new basis is optimal. The process is terminated when $x_{Bi}' = B^{-1}b' \geq 0$ for all i. This also implies that the current basis is optimal for all values of λ greater than or equal to the last value of λ.

Example 5.3 Consider the linear programming problem:

Maximize Z = $3x_1 + 2x_2 + 5x_3$

subject to the constraints

(i) $x_1 + 2x_2 + x_3 \leq 430 + \lambda$ (ii) $3x_1 + 2x_3 \leq 460 - 4\lambda$

(iii) $x_1 + 4x_2 \leq 420 - 4\lambda$ and $x_1, x_2, x_3 \geq 0$

Determine the critical value (range) of λ for which the solution remains optimal basic feasible.

Solution:

The given parametric LP problem can be written in its standard form as:

Maximize Z = $3x_1 + 2x_2 + 5x_3 + 0s_1 + 0s_2 + 0s_3$

subject to the constraints

(i) $x_1 + 2x_2 + x_3 + s_1 = 430 + \lambda$ (ii) $3x_1 + 2x_3 + s_2 = 460 - 4\lambda$

(iii) $x_1 + 4x_2 + s_3 = 420 - 4\lambda$ and $x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$.

The optimal solution when λ = 0 is shown in Table 5.9.

		$c_j \rightarrow$		3	2	5	0	0	0
c_B	Basic Variables B	Solution Values b'	b	x_1	x_2	x_3	s_1	s_2	s_3
2	x_2	3/2	100	– 1/4	1	0	1/2	– 1/4	0
5	x_3	– 2	230	3/2	0	1	0	1/2	0
0	s_3	– 10	20	2	0	0	– 2	1	1
Z = 1,350		z_j		7	2	5	1	2	0
		$c_j - z_j$		– 4	0	0	– 1	– 2	0

Table 5.9: Optimal Solution at λ = 0

In order to find the range in which the solution shown in Table 5.9 is optimal, we first calculate

$$x'_B = B^{-1}b'$$

$$\begin{bmatrix} x_2 \\ x_3 \\ s_3 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/4 & 0 \\ 0 & 1/2 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -4 \\ -4 \end{bmatrix} = \begin{bmatrix} 3/2 \\ -2 \\ -10 \end{bmatrix}$$

For a fixed λ, values of basic variables in Table 5.9 becomes:

$x_2 = 100 + \left(\frac{3}{2}\right)\lambda, x_3 = 230 - 2\lambda$ and $s_3 = 20 - 10\lambda.$

The optimal solution shown in Table 5.9 will remain optimal as long as:

$$x_2 = 100 + \left(\frac{3}{2}\right)\lambda \geq 0 \text{ (or) } \lambda \leq -\frac{200}{3}$$

$$x_3 = 230 - 2\lambda \quad \geq 0 \text{ (or) } \lambda \leq 115$$

$$s_3 = 20 - 10\lambda \quad \geq 0 \text{ (or) } \lambda \leq 2$$

Consequently the solution in Table 5.9 will remain optimal between – 200/3 and 2, i.e., – 200/3 ≤ λ ≤ 2. In particular for any λ ∈ [0, 2], the objective function value and the right hand side values are given by

$$Z(\lambda) = c_B x_B + \lambda c_B x_B' = (2, 5, 0)\begin{bmatrix}100\\230\\20\end{bmatrix} + \lambda(2, 5, 0)\begin{bmatrix}3/2\\-2\\-10\end{bmatrix} = 1350 - 7\lambda$$

$$x_B(\lambda) = x_B + \lambda x_B' = \begin{bmatrix}100\\230\\20\end{bmatrix} + \lambda\begin{bmatrix}3/2\\-2\\-10\end{bmatrix} = \begin{bmatrix}100 + 3\lambda/2\\230 - 2\lambda\\20 - 10\lambda\end{bmatrix}$$

Evidently for λ > 2, the new solution will be primal infeasible because $x_{B3}(= s_3)$ will become negative. The solution at λ = 2 is shown in Table 5.10.

		$c_j \rightarrow$	3	2	5	0	0	0
c_B	Basic Variables B	Solution Values $b (= x_B)$	x_1	x_2	x_3	s_1	s_2	s_3
2	x_1	103	– 1/4	1	0	1/2	– 1/4	0
5	x_2	226	3/2	0	1	0	1/2	0
0	s_3	0	2	0	0	(– 2)	1	1
Z = 1,336		z_j	7	2	5	1	2	0
		$c_j - z_j$	– 4	0	0	– 1	– 2	0

Table 5.10: Optimal Solution at λ = 2

For λ > 2, the basic variable s_3 becomes negative. Consequently solution becomes infeasible. Therefore dual simplex method is applied to find the new optimal solution.

Remove s_3 (because $x_{B3} \leq 0$) from the basis. Determine the ratio

$\{(c_j - z_j)/y_{rj} \ ; \ y_{rj} < 0\} = 1/2$ (corresponds to s_1), and enter s_1 into the basis. The new solution is shown in Table 5.11.

		$c_j \rightarrow$		3	2	5	0	0	0
c_B	Basic Variables B	Solution Values b′	b	x_1	x_2	x_3	s_1	s_2	s_3
2	x_2	–1	105	1/4	1	0	0	0	1/4
5	x_3	–2	203	3/2	0	1	0	1/2	0
0	s_1	5	–10	–1	0	0	1	– 1/2	– 1/2
Z = 1,336		z_j		8	2	5	0	5/2	1/2
		$c_j - z_j$		– 5	0	0	0	– 5/2	– 1/2

Table 5.11: Optimal Solution at λ > 2

In order to find the next critical value of λ in the interval [2, λ_2] in which the solution shown in Table 5.10 remains optimal, we first find

$$x_B' = B^{-1}b' = \begin{bmatrix} 0 & 0 & 1/4 \\ 0 & 1/2 & 0 \\ 1 & -1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 1 \\ -4 \\ -4 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ 5 \end{bmatrix}$$

The solution shown in Table 5.11 will remain optimal as long as basic variable

$x_2, x_3, and\ s_1$ remain non – negative, i.e.,

$x_2 = 105 - \lambda \geq 0$ (or) $\lambda \leq 105$; $x_3 = 230 - 2\lambda \geq 0$ (or) $\lambda \leq 115$;

$s_1 = -10 + 5\lambda \geq 0$ (or)$\lambda \geq 2$

Thus the solution is optimal for all values of λ in the range $2 \leq \lambda \leq 105$.

For $\lambda \in [2, 105]$, the optimal objective function value and the right hand side values are given by

$$Z(\lambda) = c_B x_B + \lambda c_B x_B' = (2, 5, 0) \begin{bmatrix} 105 \\ 230 \\ 10 \end{bmatrix} + \lambda(2, 5, 0) \begin{bmatrix} -1 \\ -2 \\ 5 \end{bmatrix} = 1360 - 12\lambda$$

$$x_B(\lambda) = x_B + \lambda x_B' = \begin{bmatrix} 105 \\ 230 \\ 10 \end{bmatrix} + \lambda \begin{bmatrix} -1 \\ -2 \\ 5 \end{bmatrix} = \begin{bmatrix} 105 - \lambda \\ 230 - 2\lambda \\ 10 + 5\lambda \end{bmatrix}$$

Evidently, if λ > 105, the new solution will be primal infeasible because basic variable x_2 becomes negative. Hence, no optimal solution exists for all

λ ≥ 105.

For λ ≤ – 200/3, the basic variable in Table 5.11 becomes negative.

Applying dual simplex method to find solution for λ ≤ – 200/3. Entering non-basic variable s_2 into the basis to replace basic variable x_2. The new optimal solution is shown in Table 5.12.

				$c_j \rightarrow$	3	2	5	0	0	0
c_B	Basic Variables B	Solution Values b'	b		x_1	x_2	x_3	s_1	s_2	s_3
0	s_2	−6	400		1	−4	0	−2	1	0
5	x_3	1	430		1	2	1	1	0	0
0	s_3	−4	420		1	4	0	0	0	1
Z = 2,150		z_j			5	10	5	5	0	0
		$c_j - z_j$			−2	−8	0	−5	0	0

Table 5.12: Optimal Solution at λ = 105

The critical values of λ for which solution shown in Table 5.12 remains optimal are calculated as follows:

$$\begin{bmatrix} s_2 \\ x_3 \\ s_3 \end{bmatrix} = B^{-1}b' = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -4 \\ -4 \end{bmatrix} = \begin{bmatrix} -6 \\ 1 \\ -4 \end{bmatrix}$$

Hence the basic solution in Table 5.12 will remain optimal provided

$x_3 = 430 + \lambda \geq 0\ (or)\ \lambda \geq -430$; $s_2 = 400 - 6\lambda \geq 0\ (or)\ \lambda \leq \frac{200}{3}$;

$s_3 = 420 - 4\lambda \geq 0\ (or)\ \lambda \leq 105$

This implies that the solution is optimal in the range $-430 \leq \lambda \leq -\frac{200}{3}$.

For λ < – 430, the basic variable x_3 in Table 5.12 becomes negative. As there is no negative entry in x_3 row of Table 5.13, the prime solution is infeasible. Hence, there exists no optimal solution to the problem for all λ < – 430.

Hence, Table 5.9, Table 5.11 and Table 5.12 gives range of λ values,

– 200/3 ≤ λ ≤ 2, 2 ≤ λ ≤ 105 and – 430 ≤ λ ≤ 200/3, respectively for which solution is optimal.

Example 5.4 Consider the linear programming problem:

Maximize Z = $4x_1 + 6x_2 + 2x_3$

subject to the constraints

(i) $x_1 + x_2 + x_3 \leq 3$ (ii) $x_1 + 4x_2 + 7x_3 \leq 9$ and $x_1, x_2, x_3 \geq 0$.

The optimal solution to this LP problem is shown below:

		$c_j \rightarrow$	4	6	2	0	0
c_B	Basic Variables B	Solution Values b (= x_B)	x_1	x_2	x_3	s_1	s_2
4	x_1	1	1	0	–1	4/3	–1/3
6	x_2	2	0	1	2	(–1/3)	1/3
$Z = 16$		z_j	4	6	8	10/3	2/3
		$c_j - z_j$	0	0	–6	–10/3	–2/3

Table 5.13: Optimal Solution

Solve the problem if the variation in right-hand side vector is: $(3, -3)^T$. Perform complete parametric analysis and identify all critical values of parameter λ.

Solution:

The given parametric LP problem can be written in its standard form as:

Maximize Z = $4x_1 + 6x_2 + 2x_3 + 0s_1 + 0s_2$

subject to the constraints

(i) $x_1 + x_2 + x_3 + s_1 = 3 + 3\lambda$ (ii) $x_1 + 4x_2 + 7x_3 + s_2 = 9 - 3\lambda$

and $x_1, x_2, x_3, s_1, s_2 \geq 0$.

The optimal solution when λ = 0 is given in Table 5.13. For values of λ other than zero, the values of right-hand side constants change because of the variation in vector b′. The new values are computed as follows:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = B^{-1}b' = \begin{bmatrix} \frac{4}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 3 \\ -3 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

As λ changes, values of basic variables x_1 and x_2 also change and solution Table 5.13 remains optimal provided value of basic variables remains non-negative, i.e., solution remains optimal provided

x_1 = 1 + 5λ ≥ 0 or λ ≥ – 1/5,

x_2 = 2 – 2λ ≥ 0 or λ ≤ 1.

Thus, solution remains optimal in the range –1/5 ≤ λ ≤ 1 and is given by:

x1 = 1 + 5λ, x_2 = 2 – 2λ, x_3 = x_4 = x_5 = 0, and Max Z = 16 + 8λ.

For λ > l, the basic variable x_2 becomes negative. Consequently solution becomes infeasible for the primal, but remains feasible for the dual because all $c_j - z_j$ ≤ 0. Apply dual simplex method to find the new optimal solution for λ > 1.

Evidently x_2 is the variable that leaves the basis. Determine the ratio {$(c_j - z_j)/y_{rj}$; y_{rj} < 0} = 10 (corresponding to s_1) and enter s_1 into the basis.

The new solution is shown in Table 5.14.

		$c_j \rightarrow$		4	6	2	0	0
c_B	*Basic Variables* **B**	*Solution Values* **b′**	**b**	x_1	x_2	x_3	s_1	s_2
4	x_1	– 3	9	1	4	7	0	1
0	s_1	6	– 6	0	– 3	– 6	1	– 1
Z = 36 – 12λ		z_j		4	16	28	0	4
		$c_j - z_j$		0	– 10	– 26	0	– 4

Table 5.14: Optimal Solution

The basic solution shown in Table 5.14 is:

$x_1 = 9 - 3\lambda\,;\ x_2 = 0\,;\ x_3 = 0\,;\ s_1 = -6 + 6\lambda\,;\ s_2 = 0$ and Max Z = 36 – 12λ.

This solution will remain optimal provided:

$x_1 = 9 - 3\lambda \geq 0\ (or)\ \lambda \leq 3\ and\ s_1 = -\ 6 + 6\lambda \geq 0\ (or)\ \lambda \geq 2.$

i.e., solution is optimal for all 1 ≤ λ ≤ 3.

For λ > 3, the basic variable x_1 becomes negative. As there is no negative coefficient in the x_1 row, the primal solution is infeasible. Hence there exists no optimal solution to the problem for all λ > 3.

For λ ≤ –1/5, the basic variable x_1 in Table 5.14 becomes negative. Consequently, solution becomes infeasible for the primal, but remains feasible for the dual, because all $c_j - z_j$ ≤ 0. Applying dual simplex method to find the new optimal solution for λ ≤ –1/5. Evidently x_1 is the variable that leaves the basis.

Determine the ratio {($c_j - z_j$) / y_{rj} ; y_{rj} < 0} = {6, 2}. Enter s_2 into the basis. The new solution is shown in Table 5.15.

		$c_j \rightarrow$		4	6	2	0	0
c_B	Basic Variables B	Solution Values b'	b	x_1	x_2	x_3	s_1	s_2
0	s_2	– 15	– 3	– 3	0	3	– 4	1
6	x_2	3	3	1	1	1	1	0
$Z = 18 + 18\lambda$		z_j		6	6	6	6	0
		$c_j - z_j$		– 2	0	– 4	– 6	0

Table 5.15: Optimal Solution

The basic solution shown in Table 5.15 is:

$x_1 = 0$; $x_2 = 3 + 3\lambda$; $x_3 = 0$; $x_4 = 0$; $x_5 = -3 - 15\lambda$ and Max Z = 18 + 18λ.

This solution will remain optimal provided x_2 = 3 + 3λ ≥ 0 or λ ≥ – 1 and x_5 = – 3 – 15λ ≥ 0 or λ ≤ –1/5. For λ < –1, the basic variable x_2 in Table 5.15 becomes negative. As there is no negative coefficient in the x_2 row, the primal solution is infeasible.

Hence there exists no optimal solution to the problem for all λ < – 1. Thus Tables 5.13, 5.14 and 5.15 give families of optimal solutions for – 1/5 ≤ λ ≤ 1, 1 ≤ λ ≤ 3 and – 1 ≤ λ ≤ 1/5 respectively.

Let Us Sum Up

We have studied about parametric linear programming and their parameters

I. Variation in objective function coefficients, c_j

II. Variation inresources availability, b_i

To solve related problems.

Check Your Progress

47. $Max\ Z = \lambda x - y$

Subject to the constraints, (i) $3x - y \geq 4,\ (ii) 2x + y \leq 3$

and $-\infty < \alpha \leq \lambda \leq \beta \leq \infty$

where ∞ is an arbitrary, and small scalar but finite and β is an arbitrary and large scalar number but finite. Perform a complete parametric programming analysis.

48. (a) $\text{Max}\, Z = (\lambda - 1)x_1 + x_2$

Subject to (i) $x_1 + 2x_2 \leq 10$, (ii) $2x_1 + x_2 \leq 11,$

(iii) $x_1 - 2x_2 \leq 3$

and $x_1, x_2 \geq 0$

(b) $Min\ Z = -\lambda x_1 - \lambda x_2 - x_3 + x_4$

Subject to (i) $3x_1 - 3x_2 - x_3 + x_4 \geq 5$

(ii) $2x_1 - 2x_2 - x_3 + x_4 \geq 3$

and $x_1, x_2, x_3, x_4 \geq 0$

Perform a complete parametric programming analysis, Identify the range of critical values of the parameter λ and all optimal basic feasible solutions.

49. $Max\ Z = 3x_1 + 2x_2 + 5x_3$

Subject to (i) $x_1 + 2x_2 + x_3 \leq 430 + 100\lambda$

(ii) $3x_1 + 2x_3 \leq 460 - 200\lambda$

(iii) $x_1 + 4x_2 \leq 420 + 400\lambda$

and $x_1, x_2, x_3 \geq 0$

Preform parametric analysis to determine the range of λ for which the solution remains optimal basic feasible.

50. $\text{Max}\, Z = (6 - \lambda)\, x_1 + (12 - \lambda)x_2 + (4 - \lambda)x_3$

Subject to (i) $3x_1 + 4x_2 + x_3 \leq 2$, (ii) $x_1 + 3x_2 + 2x_3 \leq 1$

and $x_1, x_2, x_3 \geq 0$

Perform a complete parametric programming analysis and identify all the critical values of the programming λ.

51. $\text{Max } Z = (4 - 10\lambda)\,x_1 + (8 - 4\lambda)x_2$

Subject to (i) $x_1 + x_2 \leq 4$, (ii) $2x_1 + x_2 \leq 3 - \lambda$

and $x_1, x_2 \geq 0$

Study the variation in the optimum solution with parameter λ, where $-\infty < \lambda < \infty$.

5.4 GOAL PROGRAMMING

Objectives

After studying this section, students should able to appreciate the need of a goal programming approach for solving multi-objective decision problems. Distinguish between LP and GP approaches for solving a business decision problem. Formulate GP model of the given multi-objective decision problem. Understand the method of assigning different ranks and weights to unequal multiple goals. Use simplex method for solving a GP model.

LPP is formulated and solved to optimize a single objective function involving profit or cost under the set of constraints. A single objective function is easy to solve but is not often represent the real life situation due to divergent and conflicting objectives of any business or service organization. Hence it is necessary to attain a satisfactory level of achievementamong multiple and conflicting objectives or goals. The technique of deriving a best possible “satisfactory‟ level of goal attainment is called goal programming (GP).

A problem is model into a GP model in a similar manner as that of the LP model. However, the GP model accommodates multiple and often conflicting goals in a particular priority order. A particular priority structure is established by ranking and weighting various goals and sub-goals in accordance with their importance. The priority structure helps to deal with all goals, in such a way that more important goals are achieved first at the expense of less important goals.

5.5 DIFFERENCE BETWEEN LP AND GP APPROACH

LP has two major limitations from its application point of view

(i) single objective function and (ii) same unit of measurement of various resources.

i) The LP model has a single objective function to be optimized such as profit maximization, cost minimization etc. However, in actual practice the decision maker may like to get simultaneous solution to a complex system of competing objectives. The solution of any LPP is based on the cardinal value such as profit or cost, where as GP allows an ordinal solution. As it may not be possible to obtain information about the value or cost of a goal, their upper and lower limits are determined.

ii) Whenever there are multiple incommensurable (different units of measurement) goals, LP incorporates only one of these goals in the objective function and treats the remaining goals as constraints.

In goal programming, goals are given an ordinal ranking in terms of their contributions to the organization.

5.6 CONCEPT OF GOAL PROGRAMMING

The concept of GP was introduced by Charnes and Cooper in 1961. They suggested a method for solving infeasible LPP arising from various conflicting resource constraints (goals). Some of the examples of multiple conflicting goals are

(1) maximizing profits and increase wages paid to employees.

(2) upgrade product quality and reduce product cost and

(3) reduce credit losses and increase sales.

The solution of the GP problem involves achieving some higher goals first, before the lower order goals are considered. It is not possible to achieve every goal to the extend desired by the decision maker. GP attempts to achieve a satisfactory level of goals rather than optimum solution for a single goal.

In GP, instead of trying to minimize or maximize the objective function directly as in LPP, the deviations from established goals within the given set of constraints are minimized. The deviational variables are represented in two dimensions, both positive and negative deviations from each goal and sub goal. The objective function then becomes the minimization of a sum of these deviations based on the relative importance within the priority structure assigned to each deviation.

5.7 GOAL PROGRAMMING MODEL FORMULATION

5.7.1 Single Goal with Multiple Sub goals

A goal is the result desired by a decision maker. The goal may be underachieved, fully achieved or overachieved within the given decision environment. The degree of goal achievement depends upon the „relative managerial effort applied to an activity.

If the target level for the ith goal is fully achieved then the ith constraint is written as:

$$\sum_{j=1}^{n} a_{ij}x_j = b_i$$

To allow for underachievement or overachievement, the above stated ith goal can be rewritten as: $\sum_{j=1}^{n} a_{ij}x_j + d_i^- - d_i^+ = b_i, i = 1\ to\ m$

Where, d_i^-= negative deviation from ith goal (under achievement)

d_i^+ = positive deviation from ith goal (over achievement)

Since both under and over achievement of a goal can-not be achieved simultaneously, one or both of these deviational variables ($d_i^-\ or\ d_i^+$) be zero in the solution.

i.e., $d_i^- \times d_i^+ = 0$.

i.e., if one assumes a positive value in the solution, the other must be zero and vice-versa.

Remark

The deviational variable in GP model $d_i^-\ and\ d_i^+$ are equivalent to slack and surplus variables in LPP, respectively.

The deviational variable d_i^+ (called surplus variable in LP) is **removed from objective function** of GP when over achievement is acceptable. Similarly, if under achievement is acceptable, d_i^-(called slack variable in LPP) is removed from objective function of GP. But if exact attainment of the goal is desired, then both $d_i^- \ and \ d_i^+$ are included in the objective function.

Example 5.7.1

A manufacturing firm produces two types of products A and B. The unit profit of producer A is Rs.100/- and that of product B is Rs.50/-. The goal of the firm is to earn a total profit of exactly Rs.700/- in the next week. Formulate as a GPP.

Solution:

Let x_1 and x_2 be number of units of product A and B to be produced respectively. Therefore, single goal of profit maximization is stated as: Maximize z = $100x_1$ + $50x_2$. As the goal of the firm is to earn a total profit of exactly Rs.700 per week, the above single goal can be restated to allow for under and overachievements as

$$100x_1 + 50x_2 \ + d_1^- - \ d_1^+ = 700$$

Therefore the goal programming model can be formulated as:

$$Minimize \ z = \ d_1^- + \ d_1^+$$

$$Subjecta \ to \ 100x_1 + 50x_2 \ + d_1^- - \ d_1^+ = 700; \ x_1, x_2, d_1^-, d_1^+ \geq 0$$

Where d_1^- = under achievement of the profit goal of Rs.700/-.

d_1^+= over achievement of the profit goal of Rs.700/-.

Remark:

If the profit goal is not completely achieved, the slack in the profit goal will be expressed by negative deviation d_1^- from the goal. But if the solution shows a profit in excess of Rs.700/-, the surplus in the profit will be expressed by positive deviation, d_1^+, from the goal.

If the profit goal of Rs.700/- is exactly achieved, both $d_1^- and \ d_1^+$ will be zero.

5.7.2 Equally Ranked Multiple Goals

Example 5.7.2

A manufacturing firm produces two types of products A and B. The unit profit of a producer A is Rs.100/- and that of product B is Rs.50/-. The goal of the firm is to earn a total profit of exactly Rs.700/- in the next week. Let us suppose that the manager in addition to the profit goal of Rs.700/-, also wants to achieve sales volume for products A and B close to 5 and 4 respectively. Formulate this problem as a GPP.

Solution:

The constraints of the problem can by stated as

$100x_1 + 50x_2 = 700$ (profit target goal)
$x_1 \leq 5$ and $x_2 \leq 4$ (sales target goal)
The corresponding GP model is

$$Minimize\ z = d_1^- + d_2^- + d_3^- + d_1^+$$
$$Subject\ to\ 100x_1 + 50x_2 + d_1^- - d_1^+ = 700;\ x_1 + d_2^- = 5; x_2 + d_3^- = 4$$

$$x_1, x_2, d_1^-, d_1^+,\ d_2^-,\ d_3^- \geq 0$$

Where d_1^- *and* d_1^+ are under achievement and over achievement of the profit goal of Rs.700/-, d_2^- *and* d_3^- represent underachievement of sales volume for product A and B respectively.

Remark:

Since sales target goals are given as the maximum possible sales volume, d_2^+ *and* d_3^+ are not included in the sales target constraints.

Example 5.7.3

An office equipment manufacturer produces two kinds of products, chairs and lamps. Production of either a chair or a lamp requires 1 hour of production capacity in the plant.

The plant has a maximum production capacity of 50 hours per week. Because of the limited sales capacity, the maximum numbers of chairs and lamps that can be sold are 6 and 8 per week respectively. The gross margin from the sale of a chair is Rs.90 and Rs.60 for a lamp. The plant manager decides to determine the number of units of each product that should be produced per week in consideration of the following set of goals:

i) Available production capacity should be fully utilized but not exceeded.

ii) Sales of two products should be as much as possible. iii) Overtime should not exceed 20 per cent of available production time.

Formulate this problem as a GP model so that the plant manager may achieve his goals as closely as possible.

Solution:

Let x_1, x_2 be the number of units of chair and lamp to be produced respectively.

The **first goal** is to attain the production capacity with the target established at

50 hours/week, hence the corresponding constraint is

$$x_1 + x_2 + d_1^- - d_1^+ = 50$$

Where d_1^- = under utilization of production capacity

d_1^+ = over utilization of production capacity.

The **second goal** pertains to maximization of sales, hence the sales constraints are

$$x_1 + d_2^- = 6 \; and \; x_2 + d_3^- = 8$$

As the sales goals are the maximum sales volumes, $d_2^+ and\; d_3^+$ will not appear in these constraints.

The **third goal** pertains to minimization of overtime as much as possible, the corresponding constraint is

$$d_1^+ + d_4^- - d_4^+ = \frac{20}{100}(50) = 10$$

Where d_1^+ = overtime beyond 50 hours

d_4^- = overtime less than 20 per cent of goal constraints

d_4^+ = overtime more than 20 per cent of goal constraints.

The required GP model is

$$Minimize\ z = d_1^+ + d_2^- + d_3^- + d_4^+$$

Subject to

$$x_1 + x_2 + d_1^- - d_1^+ = 50;\ x_1 + d_2^- = 6;\ x_2 + d_3^- = 8;\ d_1^+ + d_4^- - d_4^+ = 10$$

$$x_1, \quad x_2, \quad d_1^-, \quad d_1^+, \quad d_2^-, \quad d_3^-, \quad d_4^-, \quad d_4^+ \geq 0$$

5.7.3 Ranking and Weighting of Unequal Multiple Goals

Multiple and conflicting goals are usually not of equal rank. Hence to achieve these goals according to their importance a "pre-emptive" priority factor P_1, P_2,… is assigned to goal deviations in the formulation of the objective function to be minimized. The P's do not assume numerical values; they are simply a convenient way of indicating that one goal is more important than another.

The priority factors have the relationship of $P_1 >> P_2 >> P_3 \ldots P_k >> P_{k+1} >> \ldots$. Where >> means "more important than". That is, $P_j > n\, P_{j+1}$ (j= 1, 2, 3…, k), where n is very large number, implies that multiplication by 'n' cannot make a lower order goal as important as the higher order goal. Hence, a lower priority goal will never be achieved at the expense of higher priority goal.

It is possible that two or more goals may be assigned equal priority factor. Also within a given priority there may be sub goals of unequal importance which must be given due weightage.

That is different weights are assigned to the individual deviational variable with identical priority factor in the GP objective function. It is important to note that deviational variable of the same priority level must have the same unit of measurement.

5.7.4 General GP Model

The general goal linear programming model with m goals may be stated as

$$Minimize\ z = \sum_{i=1}^{m}\sum_{r=1}^{k} P_r(\,w_i^- d_i^- +\ w_i^+ d_i^+)$$

subject to linear constraints

$$\sum_{j=1}^{n} a_{ij}x_j +\ d_i^- -\ d_i^+ = b_i, i = 1\ to\ m$$

$$x_j,\ d_i^-,\ d_i^+ \geq 0$$

where x_j represents decision variable which is under the control of the DM whereas ranking coefficient P_r, weights w_i, coefficient matrix a_{ij} and constant b_i are not under direct control of DM. $d_i^-\ and\ d_i^+$ are deviational variables representing the amount of under and over achievement of i^{th} goal respectively.

5.8 GRAPHICAL SOLUTION METHOD FOR GOAL PROGRAMMING

A graphical method can also be used to solve GPP with two decision variables like LPP.

Step 1: Graph all the constraints and identify the feasible region, after setting the deviational variables to zero.

Step 2: Identify the top–priority solution. This is accomplished by determining the point or points within the feasible region that satisfies the highest priority goal.

Step 3: Move to the goal having the next highest priority and determine the best solution, such that, the best solution does not degrade the solution already achieved for goals of higher priority.

Step 4: Repeat step-3 until all priority levels have been investigated.

Example 5.8.1

A manufacturing firm produces two types of products A and B. According to past experience, production of either product A or B requires an average of one hour in the plant.

The plant has a normal production capacity of 400 hours a month. The marketing department of the firm reports that because of limited market, the maximum number of products A and B that can be sold in a month are 240 and 300 respectively. The net profit from the sale of products A and B are Rs.800 and Rs.400 respectively. The manager of the firm has set the following goals arranged in the order of importance (pre-emptive priority factor).

P1: He wants to avoid under utilization of normal production capacity.

P2: He wants to sell maximum possible units of Products A and B. Since the net profit from the sale of product A is twice the amount from product B, the manager has twice as much desire to achieve sales for product A as for product B.

P3: He wants to minimize the overtime operation of the plant as much as possible. Formulate and solve the given problem by graphical method of goal programming.

Solution: Let x_1, x_2 be the number of units of product A and B to be produced respectively.

Production capacity constraint is $x_1 + x_2 + d_1^- - d_1^+ = 400$

Where d_1^- is underutilization (idle time) of production capacity.

d_1^+ is overtime operation of the normal capacity.

As sales goals are maximum possible sales volume, positive deviation will not appear in the sales volume, hence sales constraint can be expressed as

$$x_1 + d_2^- = 240 \; and \; x_2 + d_3^- = 300$$

Where $d_2^- and \; d_3^-$ are under-achievements of the sales goals for product A and B

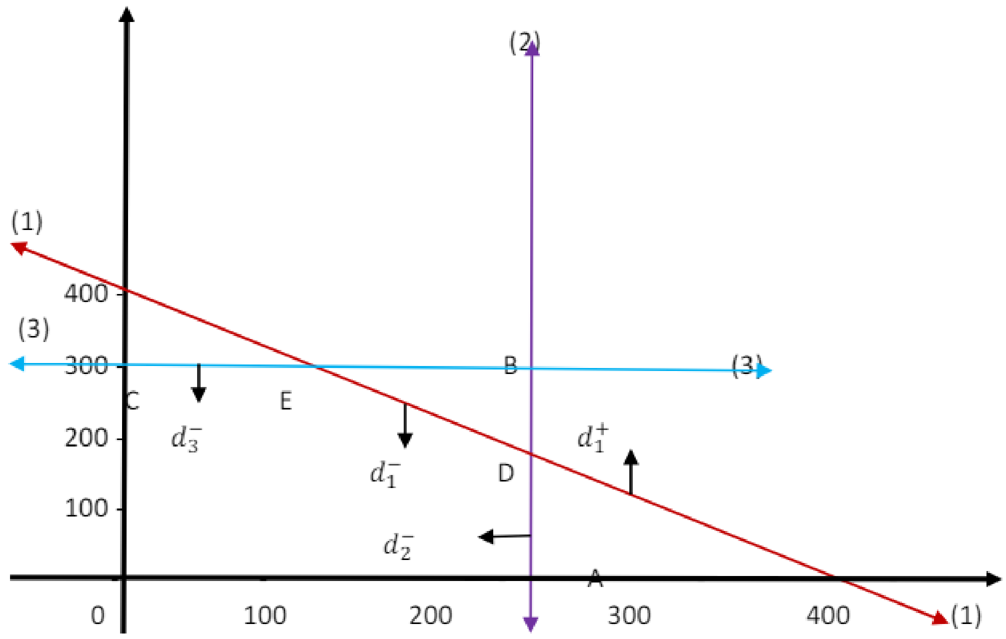
respectively. Corresponding GPP is

$$Minimize \; z = p_1 \, d_1^- + p_2(2d_2^- + d_3^-) + p_3 d_1^+$$

Subject to $x_1 + x_2 + d_1^- - d_1^+ = 400;\ x_1 + d_2^- = 240;\ x_2 + d_3^- = 300$

$x_1,\ x_2,\ d_1^-,\ d_1^+,\ d_2^-,\ d_3^- \geq 0$

Graphical solution method



As under and over utilization of the plant capacities are allowed, both the deviational variables d_1^- *and* d_1^+ are indicated by arrows in the graph. Similarly d_2^- *and* d_3^- are also indicated by arrows in the graph.

As the 1st goal is to minimize d_1^-, it is completely achieved on and above the line ED.

Hence $d_1^- = 0$. In the 2nd goal, as the differential weight of product A is twice, we try to achieve the sales goal of product A first and which can be completely achieved on and right to the line DB. Hence $d_2^- = 0$. Also, the 2nd priority given to product B is completely achieved on and above the line EB. Hence $d_3^- = 0$.

First two goals are completely achieved at the point B in the feasible region (2)B(3). Hence $d_1^- = 0, d_2^- = 0, d_3^- = 0$. The 3rd goal is to minimize over time operation, which cannot be achieved at the expense of first two goals. To find B solve equations (2) and (3), we get $x_1 = 240,\ x_2 = 300$. Using all these values in (1), we get, $d_1^+ = 140$.

The solution to this problem is

$$x_1 = 240,\ x_2 = 300, d_1^- = 0 = d_2^- = d_3^-, d_1^+ = 140$$

Remark:

If the 2nd priority is given to the over time operation and 3rd priority to sales goal constraint then the solution will be on the line DE. As the product A has more weight than product B, the solution will be at the point D which satisfies the higher weight product A at the expense of a lower weight product B. Hence the solution is
$x_1 = 240,\ x_2 = 160, d_1^- = 0 = d_2^- = d_1^+,\ d_3^- = 140$.

Example 5.8.2

A camera company manufactures two types of 35 mm cameras. The production process for manufacturing the cameras is such that two departmental operations are required. To produce their standard camera requires two hours of production time in department 1 and 3 hours in department 2. To produce their deluxe model requires 4 hours of production time in department 1 and 3 hours in department 2. Currently, 80 hours of labour are available each week in each of the departments. This labour time is somewhat restrictive factor since the company has a general policy of avoiding over time if possible.

The manufacturers profit on each standard camera is Rs 30, while the profit on the deluxe Model is Rs 40. Management has set the following goals:

P1: Avoid overtime operation in each department.

P2: Prior sales records indicate that on the average, a minimum of 10 standard and 10 deluxe cameras can be sold weekly. Management would like to meet these sales goals. Since the production time may limit producing this number of each camera, and since the deluxe camera has a higher profit margin, the sales goals should be weighed by the profit contribution for the respective cameras. i.e., Rs. 30 for the standard camera and Rs. 40 for the deluxe camera.
(We can also use weight of 3 and 4 since they have the same ratio of the profit contribution).

P3: Maximize the profit.

Solution: Let x_1 and x_2 be the number of standard and deluxe cameras to be produced

respectively. The GP model

$$Minimize\ z = p_1\,(d_1^+ + d_2^+) + p_2(3d_3^- + 4\,d_4^-) + p_3 d_5^-$$

$$Subject\ to$$

$$2x_1 + 4x_2 + d_1^- - d_1^+ = 80$$

$$3x_1 + 3x_2 + d_2^- - d_2^+ = 80$$

$$x_1 + d_3^- - d_3^+ = 10$$

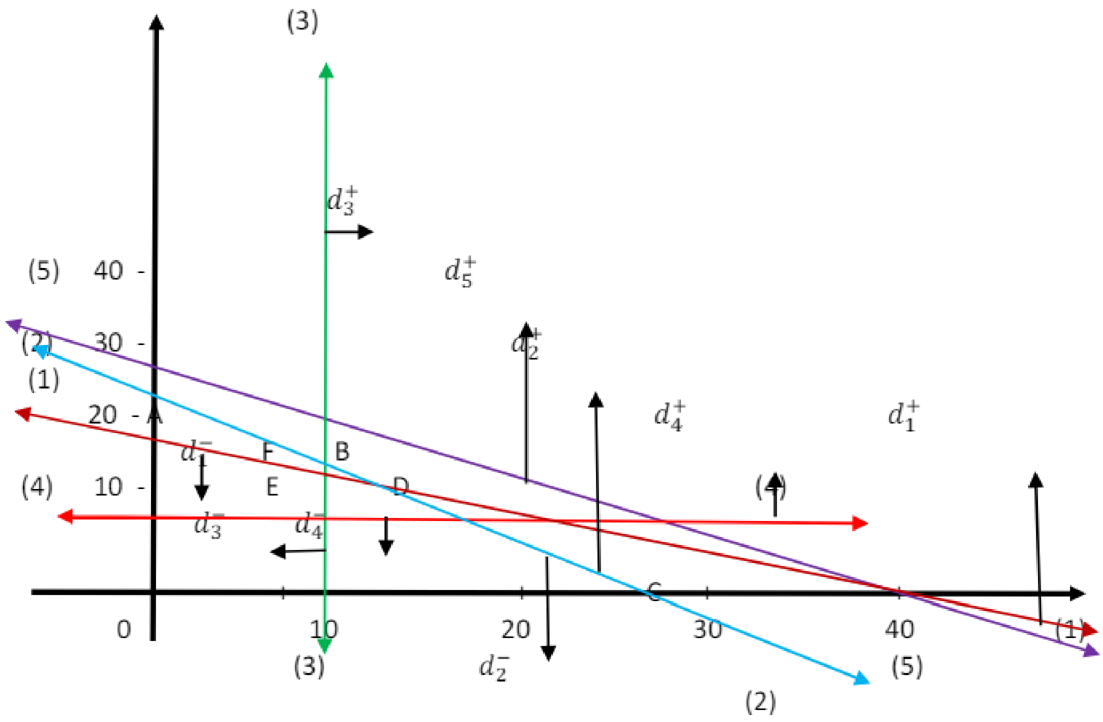
$$x_2 + d_4^- - d_4^+ = 10$$

$$30x_1 + 40x_2 + d_5^- - d_5^+ = 1200$$

$$x_1,\ x_2\ ,\ d_i^-,\ d_i^+\ \geq 0$$

The value 1200 for the R.H.S of the above equation is an arbitrary high profit goal.

Graphical solution method



The d_1^+ and d_2^+ deviational variables have a priority coefficient of P_1 in the objective function.

Hence goal constraints 1 and 2 must be considered first. If we set $d_1^+ = 0$ and $d_2^+ = 0$ then the area of feasibility is on and below the line AB and BC. Any point in this area will satisfy the condition $d_1^+=0$ and $d_2^+=0$.

Now d_3^- and d_4^- deviational variables have priority co-efficient of P_2 in the objective function.

As $P_2d_4^-$ factor has a differential weight of 4 while $P_2d_3^-$ factor has a weight of 3, we try to minimize d_4^- to zero before considering d_3^-. Hence, we can minimize both d_4^- and d_3^- to zero and still remain within the feasible region. However, minimizing these variables to zero will reduce the feasible region to EDBF. Any point in this feasible area will satisfy the conditions

$d_1^+ = 0, d_2^+ = 0, d_3^- = 0$ and $d_4^- = 0$.

The last priority level is P_3, the deviational variable associated with this priority is d_5^-. If we minimize d_5^- to zero, we will achieve all the goals. However the goal equation

(5) lies outside the current feasible region. As (5) is not feasible, P_3 level goal can only be achieved at the expense of goals with higher priorities.

In order to have an acceptable solution that does not destroy the achievement of goals with a higher priority, d_5^- must be positive. As we like to minimize d_5^- as much as possible, draw lines parallel to (5) until it contact the feasible region which is the point B on the feasible region. B is (13.33, 13.33) (solving equations 1 and 2).

The values of the deviational variables are

$$d_1^-=0,\ d_1^+=0,\ d_2^-=0,\ d_2^+=0,\ d_3^-=0,\ d_4^-=0,\ d_5^+=0.$$

Substituting these values and x_2 = 13.33 in to the respective goal constraints, we can determine: d_3^+ = 3.33, d_5^- = 266.67, d_4^+ = 3.33.

By producing 13.33 units/week of each camera, the company can achieve its first two objectives and can maximize profits. Exactly 80 hours will be used in each departmental operation and profits will be Rs.933.33. The company will under achieve its objective of Rs.1200 profit by Rs. 266.67.

5.9 Modified Simplex Method of GPP (Minimization):

Step-1: Determine the initial basic feasible solution and set up initial simplex table.

Compute z_j and $c_j - z_j$ values separately for each of the ranked goals P1, P2, … and enter at the bottom of the simplex table. First priority goal (P1) is shown at the bottom and least priority goal at the top.

Step-2: Examine $c_j - z_j$ values in the P1 row first. If all $c_j - z_j \geq 0$ at the highest priority levels, then optimum solution has been obtained. If at least one $c_j - z_j < 0$ at a priority level and there is no positive entry at the higher unachieved priority levels, in the same column, then the current solution is not optimum.

Step-3: If the target values of each goal in the solution column (x_B) is zero, the current solution is optimum.

Step-4: Examine the negative values of $c_j - z_j$ of the highest priority (P1) and choose the most negative of these. The column corresponding to this value becomes the pivot column. Otherwise move to next priority (P2) and select the most negative value of $c_j - z_j$ for determining the pivot column.

Step-5: Determine the pivot row and pivot element in the same way as in the simplex method.

Step-6: Any negative value in the $c_j - z_j$ row which has positive $c_j - z_j$ under any lower priority rows is ignored. It is because deviations from highest priority goal would be increased with the entry of this variable in the basis.

Example 5.9.1 Use modified simplex method to solve the following GPP.

$$Minimize\ z = p_1 d_1^- + p_2(2d_2^- + d_3^-) + p_3 d_1^+$$

Subject to the constraints

$$x_1 + x_2 + d_1^- - d_1^+ = 400;\ x_1 + d_2^- = 240;\ x_2 + d_3^- = 300$$

$x_1,\ x_2,\ d_1^-,\ d_1^+,\ d_2^-,\ d_3^- \geq 0$

Solution:

	c_j	0	0	p_1	$2p_2$	p_2	p_3		
c_B	$Basic\ variable$	x_1	x_2	d_1^-	d_2^-	d_3^-	d_1^+	x_B	$Ratio$
p_1	d_1^-	1	1	1	0	0	-1	400	400/1
$2p_2$	d_2^-	**1**	0	0	1	0	0	240	<u>**240/1**</u>
p_2	d_3^-	0	1	0	0	1	0	300	---
	p_3	0	0	0	0	0	1	Z=0	---
$c_j - z_j$:	p_2	-2	-1	0	0	0	0	780	---
	p_1	**-1**	-1	0	0	0	1	400	---

In the table:

$$Z = c_B x_B = 400\,p_1 + 480\,p_2 + 300\,p_2 = 400\,p_1 + 780\,p_2$$

The values $p_1 = 400,\ p_2 = 780,\ p_3 = 0$ in the x_B column represent the unachieved portion of each goal.

$$\boldsymbol{c_j - z_j = c_j - c_B x_j}$$

$$c_1 - z_1 = 0 - [\,p_1 \times 1 + 2p_2 \times 1 + p_2 \times 0] = -p_1 - 2p_2$$

$$c_2 - z_2 = 0 - [\,p_1 \times 1 + 2p_2 \times 0 + p_2 \times 1] = -p_1 - p_2$$

$$c_6 - z_6 = p_3 - [\,p_1 \times (-1) + 2p_2 \times 0 + p_2 \times 0] = p_1 + p_3$$

In the p_1 row, the most negative value is -1 corresponding to x_1 & x_2, (Also as the most negative -2 is along p_2 corresponding to x_1), we can select x_1 as entering variable.

	c_j	0	0	p_1	$2p_2$	p_2	p_3		
c_B	*Basic variable*	x_1	x_2	d_1^-	d_2^-	d_3^-	d_1^+	x_B	*Ratio*
p_1	d_1^-	0	**1**	1	− 1	0	− 1	160	<u>**160/1**</u>
0	x_1	1	0	0	1	0	0	240	---
p_2	d_3^-	0	1	0	0	1	0	300	300/1
	p_3	0	0	0	0	0	1	$z = 0$	---
$c_j - z_j$:	p_2	0	− 1	0	2	0	0	300	---
	p_1	0	**− 1**	0	1	0	1	160	---

The value of the objective function $z = c_B x_B = 160\,p_1 + 300\,p_2$ indicates that the unachieved portion of the first and second goals has decreased.

In the p_1 row, the most negative value is -1 corresponding to x_2 , select x_2 as entering variable.

	c_j	0	0	p_1	$2p_2$	p_2	p_3		
c_B	$Basic\ variable$	x_1	x_2	d_1^-	d_2^-	d_3^-	d_1^+	x_B	$Ratio$
0	x_2	0	1	1	-1	0	-1	160	---
0	x_1	1	0	0	1	0	0	240	---
p_2	d_3^-	0	0	-1	1	1	**1**	140	<u>140/1</u>
$c_j - z_j$:	p_3	0	0	0	0	0	1	$z = 0$	---
	p_2	0	0	1	1	0	$-$**1**	140	---
	p_1	0	0	1	0	0	0	0	---

All $c_j - z_j$ values in p_1 row are either positive or zero, also the value of z in p_1 row is completely minimized to zero. The most negative value in p_2 row is -1 corresponding to d_1^+ and corresponding **entry in the higher priority p_1 is not positive**, hence we can revise the table.

	c_j	0	0	p_1	$2p_2$	p_2	p_3		
c_B	$Basic\ variable$	x_1	x_2	d_1^-	d_2^-	d_3^-	d_1^+	x_B	$Ratio$
0	x_2	0	1	0	0	1	0	300	---
0	x_1	1	0	0	1	0	0	240	---
p_3	d_1^+	0	0	-1	1	1	1	140	---
$c_j - z_j$:	p_3	0	0	1	-1	-1	0	$z = 140$	---
	p_2	0	0	0	2	1	0	0	---
	p_1	0	0	1	0	0	0	0	---

All $c_j - z_j$ values in p_2 row are either positive or zero and the value of z in p_2 row is completely minimized to zero. In p_3 row there are two negative values, however, it is not possible to choose $d_2^- \; or \; d_3^-$ as the pivot column because there is already a positive value at a higher priority level p_2. Hence the solution in the above table will not improve further. The optimum solution is

$$x_1 = 240, \qquad x_2 = 300, \qquad d_1^- = \; d_2^- = \; d_3^- = 0, \qquad d_1^+ = 140.$$

Let Us Sum Up

We have learned about Goal programming model formulation and its types. Also find the solution by using Graphical method and Modified simplex method .

Check Your Progress

52. A company produces motorcycle seats. The company has two production lines. The production rate for line 1 is 50 seats per hour and for line 2, it is 60 seats per hour. The company has entered into a contract to supply 1200 seats daily to another company. Currently, the normal operation period for each line is 8 hours. The production manager of the company is trying to determine the best daily operation hours for the two lines in order to achieve the following goals:

P1: Produce and deliver 1200 seats daily.

P2: Limit the daily overtime operations hours of line-2 to 3 hours.

P3: Minimize under-utilization of regular daily operation hours of each line. Assign differential weights based on the relative productivity rate.

P4: Minimize the daily overtime operation hours of each line as much as possible. Assign differential weights based on the relative cost of overtime. It is assumed that the cost of operation is identical for the two production lines.

Formulate a goal programming model and then solve it by using graphical method.

53. Use Modified simplex method to solve

$$Minimize\ z = p_1 d_1^- + p_2 d_2^- + 2p_2 d_3^- + p_3 d_1^+$$

$$Subject\ to$$

$$10x_1 + 10x_2 + d_1^- - d_1^+ = 400$$

$$x_1 + d_2^- = 40$$

$$x_2 + d_3^- = 30$$

$$x_1,\ x_2,\ d_i^-,\ d_i^+ \geq 0$$

Unit Summary

Parametric linear programming techniques are used to determine the effect of pre-determined continuous variation in the input data, on the optimal solution of an LPP. The parametric analysis aims at finding various basic solutions that become optimal one after the other due to continuous variations in the LP model parameters. These techniques reduce computational time required to obtain the changes in the optimal solution due to variation in LP model parameters over a range of variation.

Goal Programming is an approach used for solving a multi-objective optimization problem. A problem is modeled into a GP model in a manner similar to that of an LP model. However, the GP model accommodates multiple, and often conflicting, incommensurable goals, in a particular priority order. A particular priority structure is established by ranking and weighing various goals and their subgoals, in accordance with their importance. The priority structure helps to deal with all goals that cannot be completely and/or simultaneously achieved in such a manner that more important goals are achieved first, at the expense of the less important ones. An important feature of a GP is that the goals (a specific numerical target values that the decision-maker would ideally like to achieve) are satisfied in ordinal sequence. To the extent desired by the decision-maker, attempts are made to achieve each goal sequentially rather than simultaneously, up to a satisfactory level rather than an optimal level. In GP, instead of trying to minimize or maximize the objective function directly, as in the case of an LP, the deviations from established goals within the given set of constraints are minimized. The deviational variables are represented in two dimensions– both positive and negative deviations from each goal and subgoal. These deviational variables represent the extent to which the target goals are not achieved. The objective function then becomes the minimization of a sum of these deviations, based on the relative importance within the pre-emptive priority structure assigned to each deviation.

Glossary

- c_j- Cost (or profit),
- b_i- Availability of resources
- a_{ij}- Technological coefficients
- B- Basic variables

Self- Assessment Questions

1. Write a short note on parametric linear programming.
2. Explain the basic difference between sensitivity analysis and parametric programming.
3. In a linear programming problem Min Z = **cx** subject to **Ax= b** and **x** ≥ 0 discuss the effect of
 (a) discrete changes in the requirement vector **b**.
 (b) discrete changes in the cost vector **c**.
4. Explain what is meant by a parametric linear programming problem, pointing out its chief characteristics
5. What is goal programming? Clearly state its assumptions.

Activities

1. Explain the difference between cardinal value and ordinal value.
2. Under what circumstances can cardinal weights be used in the objective function of a goal programming model? What happens if the cardinal weights

are attached to all priorities in the objective function of a goal programming model?

3. State some problem areas in management where goal program ming might be applicable.

4. 'Goal programming appears to be the most appropriate, flexible and powerful technique for complex decision problems involving multiple conflicting objectives.' Discuss.

5. What is goal programming? Why are all goal programming problems minimization problems? Why does altering the goal priorities result in a different solution to a problem? Explain.

References

1. J. K. Sharma, *Operations Research*, Theory and Applications, Third Edition (2007)Macmillan India Ltd
2. Hamdy A. Taha, *Operations Research,* (seventh edition) Prentice - Hall of IndiaPrivate Limited, New Delhi, 1997.

Suggested Readings

3. F.S. Hillier & J.Lieberman *Introduction to Operation Research* (7th Edition) Tata-McGraw Hill ompany, New Delhi, 2001.
4. Beightler. C, D.Phillips, B. Wilde ,*Foundations of Optimization* (2nd Edition) PrenticeHall Pvt Ltd., New York, 1979
5. S.S. Rao - *Optimization Theory and Applications*, Wiley Eastern Ltd. New Delhi. 1990

ANSWERS FOR CHECK YOUR PROGRESS

UNIT- I

1. $x_1 = 0, x_2 = \frac{15}{7}$ and Max Z = $\frac{45}{7}$ (cut –1: x_4)

   $x_1 = \frac{5}{4}, x_2 = \frac{5}{4}, x_4 = \frac{15}{4}$ and Max Z = $\frac{25}{4}$ (cut –2: x_4)

   $x_1 = 0, x_2 = 2$ and Max Z = 6

2. $x_1 = \frac{5}{4}, x_2 = \frac{5}{8}$ and Max Z = 15

   $x_1 = \frac{10}{3}, x_2 = 0, x_4 = \frac{25}{3}$ and Max Z = $\frac{20}{3}$

   $x_1 = 2, x_2 = 0, x_3 = 2$ and Max Z = – 16

3. $x_1 = \frac{9}{2}, x_3 = \frac{7}{2}$ and Max Z = 63

   Fractional cut (x_2) : $x_1 = \frac{32}{7}, x_2 = 3, x_3 = \frac{11}{7}$ and Max Z = 59

   Fractional cut (x_1) : $x_1 = 4, x_2 = 3$ and Max Z = 55

4. $x_1 = 0, x_2 = \frac{7}{4}$ and Max Z = $\frac{39}{4}$

   $x_1 = \frac{3}{2}, x_2 = 1, x_4 = \frac{9}{2}$

   $x_1 = 1, x_2 = 1$ and Max Z = 15.

5. $x_1 = \frac{7}{2}, x_2 = \frac{9}{5}$ and Max Z = $\frac{53}{10}$

   $x_1 = 5, x_2 = 0$ and Max Z = 5

6. $x_1 = 0, x_2 = 3$ and Max Z = 12

7. $x_1 = \frac{20}{9}, x_2 = \frac{5}{3}, x_3 = \frac{26}{9}$ and Max Z = 17

   x_1 = 2 , x_2 = 2 , x_3 = 2 and Max Z = 14

8. No integer solution.

9. Optimal solution is $x_1^* = 7/3, x_2^* = 7/3,\ z_{max} = 42$.

10. Optimal solution is $x_1^* = 0, x_2^* = 100, x_3^* = 230,\ z_{max} = 1350$.

UNIT- II

11. (a) $x_0 = (4, -1, 1)$, local minimum

(b) $x_0 = (0, 0, 0)$, local minimum

(c) $x_0 = (8, 4, 3)$, local minimum

12. Point of inflection at $x = {-b}/{3a}$

13. $R = yx = 15xe^{-x/3}; x = 3 \; or \; \infty$ (absurd); Max. P = Rs. 50 at x =100.

14. $P = R - C = 26x - x^2; x = 13, Max\ P$ =Rs. 149.

15. Let x be the vacant apartments;

Profit = Revenue – cost = (4,500 + 150x) (60 – x) - 6x

16. $x_1 = 4.95, x_2 = 2.045$, and Min $Z = 21.63$

17.. $x_1 = 0.81, x_2 = 0.35, x_3 = 0.28$ and Min $Z = 0.84$

18. $3x_1 = 1/3, x_2 = 5/6$, and Max $Z = 4.166$

19. $x_1 = 12.06, x_2 = 10.35$, and Max $Z = 80.73$

20. Formulate $L(x, y, z, \lambda) = f(x, y, z) - \lambda g(x, y, z)$; where $f(x, y, z) = xyz$, is the volume of a parallelopipped. Differentiate partially L with respect to x, y, z and λ and equate them equal to zero. Solve four equation to get first $\lambda = \left(\frac{3}{2}\right). z\ y\ z$ and then $x = \frac{a}{\sqrt{3}}, y = \frac{b}{\sqrt{3}}$, and $z = \frac{c}{\sqrt{3}}$.

21. $x_1 = 44, x_2 = 2, \lambda = 100$ and Max $Z = 4{,}900$

22. $x_1 = \frac{1}{2}, x_2 = \frac{3}{2}, \lambda_1 = 3, \lambda_2 = 0$ and Max $Z = \frac{17}{2}$

23. $x_1 = \frac{4}{13}, x_2 = \frac{33}{13}$ and Max $Z = 21.3$

24. $x_1 = \frac{48}{5}, x_2 = \frac{1}{5}$ and Max $Z = 587.72$

25. $x_1 = 1, x_2 = 1$ and Max $Z = 0$

26. (a) $x_1 = 1, x_2 = 20$ and Max $Z = 1$

(b) $x_1 = 3, x_2 = 2$ and Max $Z = 3$

27. (a) $x_1 = 2, x_2 = 2$ and Min $Z = 8$

(b) $x_1 = 0, x_2 = 1$ and Min $Z = 2$

28. (a) $x_1 = 60, x_2 = 20$ and Max $Z = 400$

(b) $x_1 = 1, x_2 = 1$ and Min $Z = 1$

29. Let x and y = quantity of product A and B to be produced, respectively.

Max Z = (200 – $2x^2$) + (500 – $2y^2$)

Subject to (i) $0.5x + 0.25y \leq 35$, (ii) $2x + 3y \leq 80$;

and $x, y \geq 0$.

30. $x_1 = 0, x_2 = 1, \lambda_1 = \frac{1}{3}, \lambda_2 = \frac{5}{3}$ and Max Z=3
31. $x_1 = \frac{4}{13}, x_2 = \frac{33}{13}$ and Max Z= $\frac{267}{13}$
32. $x_1 = \frac{1}{4}, x_2 = \frac{15}{8}$ and Max Z= $\frac{97}{16}$

UNIT- III

33. (a) (i) Basic: $x_1 = 0, \quad x_2 = 1/2$ Non-basic: $x_3 = x_4 = 0$

(ii) Basic: $x_1 = 2, \quad x_3 = 7/2$ Non-basic: $x_2 = x_4 = 0$ (infeasible also)

(iii) Basic: $x_1 = 8/3, \ x_4 = 7/3$ Non-basic: $x_2 = x_3 = 0$ (infeasible also)

(iv) Basic: $x_2 = 1/2, \ x_3 = 0$ Non-basic: $x_1 = x_4 = 0$

(v) Basic: $x_2 = 1/2, \ x_3 = 0$ Non-basic: $x_1 = x_4 = 0$

(vi) Basic: $x_3 = 0, \quad x_4 = 1$ Non-basic: $x_1 = x_2 = 0$ (infeasible also)

(b) (i) $x_1 = 0, x_2 = 6, x_3 = -2$

(ii) $x_1 = 3, x_2 = 0, x_3 = 1$

(iii) $x_1 = 2, x_2 = 2, x_3 = 0$

34. (i) Basic: $x_1 = 1, \quad x_2 = 0$ Non-basic: $x_3 = 0$ (degenerate solution)

(ii) Basic: $x_1 = 5/3, \quad x_3 = 1/3$ Non-basic: $x_1 = 0$

(iii) Basic: $x_1 = 1, \ x_3 = 0$ Non-basic: $x_2 = 0$ (degenerate solution)

35. The solution $(x_1 = 1, x_2 = \frac{1}{2}, x_3 = 0, x_4 = 0, x_5 = 0)$ is not a basic solution.
36. For each of the three possible submatrices of order 2, calculate $x_B = B^{-1}b$.

(a) $x_1 = 2, x_2 = 1, x_3 = 0$ (b) $x_1 = 0, x_2 = \frac{5}{3}, x_3 = \frac{2}{3}$

37. For each of the 10 possible submatrices of order 2, calculate $x_B = B^{-1}b$. None of the 10 basic feasible solution corresponds to the given solution. Hence the given solution is not basic.
38. (i) For $\lambda_1 = 2, \lambda_2 = 1, \lambda_3 = -1$ and $\lambda_4 = 0$, the solution is (0, 1/2, 3/2, 0)

(ii) For $\lambda_1 = -2, \lambda_2 = -1, \lambda_3 = 1$ and $\lambda_4 = 0$, the solution is (3, 2, 0, 0)

39. For $\lambda_1 = -1, \lambda_2 = -2, \lambda_3 = 1$, the basic feasible solution is:

$x_1 = 3, x_2 = 5$ and $x_3 = 0$.

UNIT – IV

40. $x_1 = 0, x_2 = 2/5$ and $Max\ Z = 5$

41. $x_1 = 2/7, x_2 = 9/7$ and $Max\ Z = 13/7$

42. $x_1 = 8/5, x_2 = 3/5$ and $Max\ Z = 11/5$

43. $x_1 = 4, x_2 = 6, x_3 = 0$ and $Max\ Z = 12$

44. $x_1 = 0, x_2 = 100, x_3 = 230$ and $Max\ Z = 1350$.

45. $x_1 = 20/3, x_2 = 0, x_3 = 10/3$ and $Max\ Z = 10$

46. $x_1 = \frac{17}{5}, x_2 = \frac{16}{5}, x'_3 = 0\ or\ 4 - x_3 = 0\ or\ x_3 = 4$ and $Max\ Z = 192/5$

UNIT – V

47. $x = \frac{8}{5}, y = -1/5$ and $Min\ Z = \frac{1}{5} + \left(\frac{8}{5}\right)\lambda$.

Problem has one characteristics solution: $-2 \leq \lambda \leq 3$ and a multiple solution for $\lambda = 3$.

48. (a) $x_1 = 0, x_2 = 5; 0 \leq \lambda \leq 3/2$

$x_1 = 4, x_2 = 3; 3/2 \leq \lambda \leq 3$

$x_1 = 5, x_2 = 1; \lambda \geq 3$

(b) $x_1 = x_2 = x_3 = 0; x_4 = 5$ and $Max\, Z = 5;\ -2 \leq \lambda \leq 3$.

49. $x_1 = 0, x_2 = 2, x_3 = 5, \lambda \leq 2.3$

50. $x_1 = 2/5, x_2 = 1/5, 0 \leq \lambda \leq 3$

51. $x_1 = 4, x_2 = 0, -\infty < \lambda < -5$

$x_1 = 0, x_2 = 5, -5 < \lambda < -1$

$x_1 = 0, x_2 = 3, -1 < \lambda < 2$

No feasible solution when $\lambda > 3$.

52. The GP model is

$$Minimize\ z = p_1 d_1^- + p_2 d_4^+ + p_3(5d_2^- + 6d_3^-) + p_4(6d_2^+ + 5d_3^+)$$

Subject to

$$50x_1 + 60x_2 + d_1^- - d_1^+ = 1200$$

$$x_1 + d_2^- - d_2^+ = 8$$

$$x_2 + d_3^- - d_3^+ = 8$$

$$x_2 + d_4^- - d_4^+ = 11$$

$$x_1,\ x_2,\ d_i^-,\ d_i^+ \geq 0$$

The solution is

$$x_1 = 10.8,\ x_2 = 11, d_1^- = 0 = d_2^- = d_3^- = d_4^+,\ d_2^+ = 2.8,\ d_3^+ = 3$$

Note: P1, P2 and P3 are completely achieved but P4 is not achieved.

53. $x_1 = 40, \quad x_2 = 30, \quad d_1^+ = 300,\ d_1^- = d_2^- = d_3^- = 0$.