PERIYAR UNIVERSITY

NAAC 'A++' Grade - State University - NIRF Rank 56–State Public University Rank 25

SALEM - 636 011, Tamil Nadu, India.

CENTRE FOR DISTANCE AND ONLINE EDUCATION

(CDOE)

M.Sc. MATHEMATICS

SEMESTER - I

ELECTIVE COURSE: GRAPH THEORY AND APPLICATIONS

(Candidates admitted from 2024 onwards)

PERIYAR UNIVERSITY

CENTRE FOR DISTANCE AND ONLINE EDUCATION (CDOE)

M.Sc. Mathematics - 2024 admission onwards

ELECTIVE – 1
Graph Theory and Applications

Prepared by:

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Periyar University
Salem 636011

Contents

Graphy Theory and Applcations

OBJECTIVES: The objective of the course is to introduce students with the fundamental concepts in graph theory, with a sense to know some of the new developments and its modern applications. They will be able to use these concepts/techniques in subsequent courses in the design and analysis of algorithms, software engineering and computer systems.

UNIT I: Graphs and Digraphs

Basic concepts – subgraphs – degrees of vertices – paths and connectedness – automorphism of a simple graphs – line graphs –operations on graphs –applications to social psychology - basic concepts in digraphs – tournaments.

UNIT II: Connectivity and trees

Vertex cuts and edge cuts - connectivity and edge connectivity - Cyclical edge connectivity of a graph - Definition, Characterization and simple properties of trees - centers and centraoids - counting spanning trees - Cayley's formula - Applications: Connector Problem - Kruskal's Algorithm.

UNIT III: Independent sets, Matchings and Cycles

Independents sets and coverings (both vertex & edge) - matchings and factors - matchings in bipartite graphs - Eularian graphs and Hamiltonian graphs - Introduction - Eulerian Graphs - Hamiltonian Graphs - 2-Factorable Graphs.

UNIT IV:Graph colorings

Vertex colorings – applications of graph coloring - critical graphs - Brooks Theorem - other coloring parameters - b-colorings; Edge colorings - the time table problem - Vizings theorem - Kirkman's Schoolgirl Problem - chromatic polynomials.

UNIT V: Planar Graphs Planar and non planar graphs – Euler formula and its consequences – K5 and K3,3 are non planar graphs – dual of a plane graph – The four color theorem and the Heawood five color theorem – Hamiltonian plane graphs – Tait coloring.

REFERENCE

R. Balakrishnan and K. Ranganathan, A Textbook of Graph Theory, Second Edition, Springer, New York, 2012.

SUGGESTING READINGS

1. J. Clark and D.A. Holton, A First look at Graph Theory, Allied Publishers, New Delhi, 1995.
2. R.J. Wilson and J.J. Watkins, Graphs: An Introductory Approach, John Wiley and Sons, New York, 1989.
3. S.A. Choudum, A First Course in Graph Theory, MacMillan India Ltd. 1987.
4. J.A. Bondy and U.S.R. Murty, Graph Theory and Applications, Macmillan, London, 1976.

Unit 1

Graphs and Digraphs

Objectives

1. To understand the basic concepts of graph theory, including definitions, diagrammatic representation of graphs, and types of graphs

2. To understand the significance of paths and cycles

3. To gain knowledge about graphs like Line graphs, and various operations on graphs

4. To understand the basic concepts of directed graphs

5. To learn about some important results on tournaments

1.1 Introduction

In mathematics, graph theory is the study of graphs, which are mathematical structures used to model pairwise relations between objects. A graph in this context is made up of vertices (also called **nodes or points**) which are connected by edges (also called **links or lines**).

Graphs serve as mathematical models to analyze many concrete real-world problems successfully. Some puzzles and several problems of a practical nature have been instrumental and played major role in the development of various topics in graph theory.

1.2 Basic Concepts

Definition 1. *A* **graph** *is an ordered triple* $G = \big(v(G), E(G), I(G)\big)$*, where* $V(G)$ *is a non-empty set,* $E(G)$ *is a set disjoint from* $V(G)$*, and* $I(G)$ *is an Incidence relation that associates with each element of* $E(G)$ *to an unordered pair of elements (same or distinct) of* $V(G)$*. Elements of* $V(G)$ *are called the* **vertices** *of* G *and the elements of* $E(G)$ *are called the* **edges** *of* G*. If, for the edge* e *of* G*,* $I_G(e) = \{u, v\}$*, we write* $I_G(e) = uv$*.*

Example 2. *If the vertex set* $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$*, the edge set* $E(G) = \{e_1, e_2,$
$e_3, e_4, e_5, e_6\}$ *and* I_G *is given by*

$$I_G(e_1) = \{v_1, v_5\}$$
$$I_G(e_2) = \{v_2, v_3\}$$
$$I_G(e_3) = \{v_2, v_4\}$$
$$I_G(e_4) = \{v_2, v_5\}$$
$$I_G(e_5) = \{v_2, v_5\}$$
$$I_G(e_6) = \{v_3, v_3\}$$

Then $G = \big(v(G), E(G), I(G)\big)$ *is a graph.*

Note: Diagrammatic representation of a graph. Each graph can be represented by a diagram in the plane. In this diagram, each vertex of the graph is represented by a point and each edge is represented by a line joining two vertices.

Definition 3.

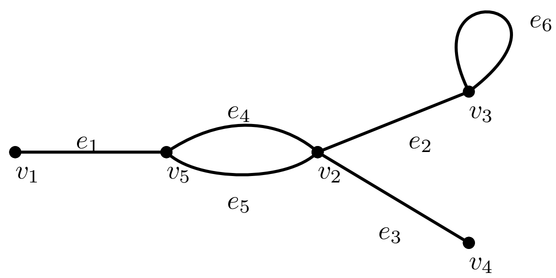


Figure 1.1: The graph $\big(v(G), E(G), I(G)\big)$ described in example 2

(i) If $I_G(e) = uv$ the the vertices u and v are called the **end vertices** *or* **ends** *of the edge e. The vertices u and v are then said to be* **incident** *with e.*

(ii) A set of two or more edges of a graph G is called a set of **multiple** *or* **parallel** *edges if they have the same pair of distinct ends.*

(iii) An edge for which the two ends are the same called a **loop** *at the common vertex.*

(iv) A vertex u is a **neighbor** *of v in G, if uv is an edge of G and $u \neq v$.*

(v) The set of all neighbors of v is the **open neighborhood** *of v or the* **neighbor set** *of v, and is denoted by $N_G(v)$.*

(vi) The set $N_G(v) = N_G(v) \cup \{v\}$ is the **closed neighborhood** *of v in G.*

(vii) **Vertices** *u and v are* **adjacent** *each other G if and only if there is an edge of G with u and v as its ends.*

(viii) Two **distinct edges** *e and f are said to be* **adjacent** *if and only if they have a common end vertex.*

(ix) A graph is **simple** *if it has no loops and no multiple edges. Thus, for a simple graph the incidence function I_n is one-to-one.*

Example 4. *In the graph of Fig 1.1,*

(i) The vertices v_1 and v_5 are end vertices of the edge e_1.

(ii) *The edges* e_4 *and* e_5 *are parallel edges.*

(iii) *The edge* e_6 *is a loop at* v_3.

(iv) *The vertex* v_4 *is a neighbor of* v_2.

(v) $N_G(v_2) = \{v_3, v_4, v_5\}$.

(vi) $N_G(v_2) = \{v_2, v_3, v_4, v_5\}$.

(vii) *The vertices* v_1 *and* v_5 *are adjacent, where as the vertices* v_1 *and* v_2 *are non-adjacent.*

(viii) *The edges* e_2 *and* e_3 *are adjacent, where as the edges* e_1 *and* e_2 *are non-adjacent.*

(ix) *As there are multiple edges and a loop, the graph is not a simple graph. Example of a simple graph is given in Fig.1.2.*

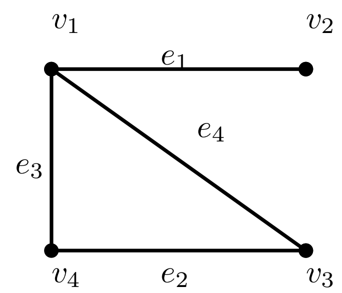


Figure 1.2: A simple graph

Definition 5. *A graph is called* **finite** *if both* $v(G)$ *and* $E(G)$ *are finite. A graph that is not finite is called* **infinite graph**. *The number of vertices of a graph G is denoted by* $n(G)$ *and the number of edges of G is denoted by* $m(G)$. *The number* $n(G)$ *is called the* **order** *of G and* $m(G)$ *is the* **size** *of* G.

Example 6. *In example 2, both $V(G)$ and $E(G)$ are finite, hence the graph G is finite.In this graph G, $n(G)$ = Order of $G = |V(G)| = 5$ $m(G)$ = Size of $G = |E(G)| = 6$. Example of an infinite graph is given in Fig.1.3.*

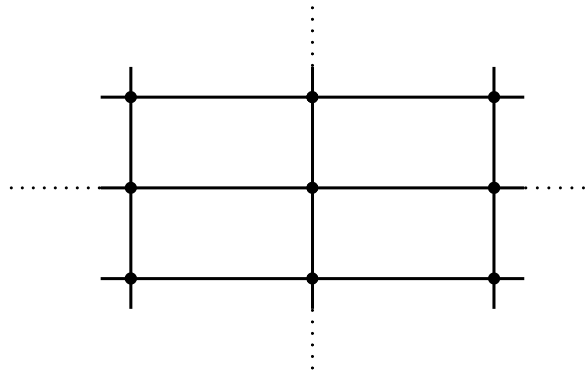


Figure 1.3: An infinite graph

Definition 7. *A graph is said to be* **labelled** *if its n vertices are distinguished from one another by labels such as $v_1, v_2, \ldots, v_n$.*

Example 8. *The graph given in Fig. 1.2 is a labelled graph and the graph given in Fig 1.3 is an unlabelled graph.*

Definition 9. *Let $G = \big(V(G), E(G), I_G\big)$ and $H = \big(V(H), E(H), I_H\big)$ be two graphs. A graph* **isomorphism** *from G to H is a pair (ϕ, θ) where $\phi : V(G) \to V(H)$ and $\theta : E(G) \to E(H)$ are bijections with the property that $I_G(e) = uv$ if and only if $I_H(\theta(e)) = \phi(u)\phi(v)$.*

Example 10. *Fig 1.4 exhibits two isomorphic graphs P and H, where P is the well-known Petersen Graph. We say that P is isomorphic to H and denote it by $P \cong H$.*

Definition 11.

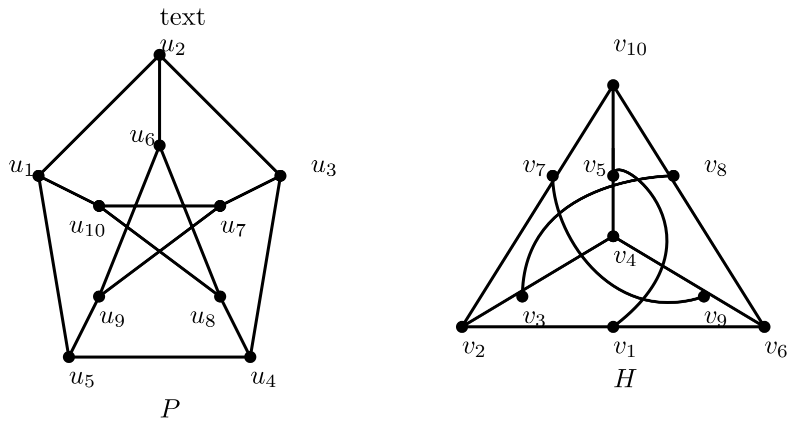


Figure 1.4: Isomorphic graphs

(i) A simple graph is said to be **complete** *if every pair of distinct vertices of G are adjacent in G. A* **complete graph** *on n vertices is denoted by K_n. The number of edges in K_n is* $\binom{n}{2} = \frac{n(n-1)}{2}$

(iii) A graph may possess no edge at all such a graph is called a **totally disconnected graph**. *Thus for a simple graph G,* $0 \leq m(G) = \frac{n(n-1)}{2}$

Example 12.

(i) In Fig.1.5, the complete graph K_1, K_2, K_3, K_4 and K_5 are given.

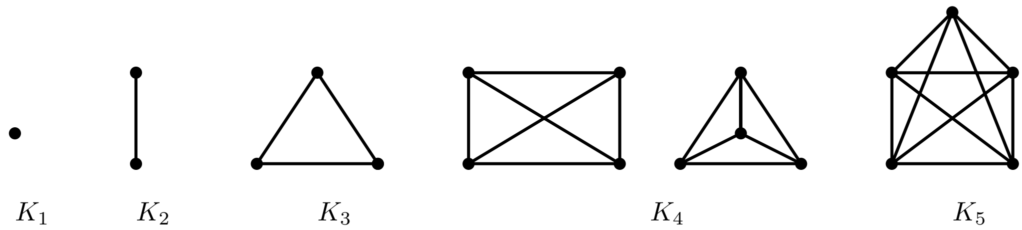


Figure 1.5: Some simple graphs

(ii) In Fig.1.6, A totally disconnected graph on five vertices is given.

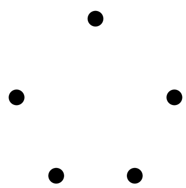


Figure 1.6: A totally disconnected graph on five vertices

Definition 13.

(i) A graph is **trivial** *if its vertex set is singleten and it contains no edges.*

(ii) A graph is **bipartite** *if its vertex set can be partitioned into two non-empty subsets X and Y such that each edge of G has one end in X and the other in Y. The pair (X, Y) is called a bipartitioned of the* **bipartite graph**. *The bipartite graph G with bipartition (X, Y) is denoted by $G(X, Y)$.*

(iii) A simple bipartite graph $G(X, Y)$ is complete if each vertex of X is adjacent to all the vertices of Y. If $G(X, Y)$ is complete with $|X| = p$ and $|Y| = q$ then $G(X, Y)$ is denoted by $K_{p,q}$.

(iv) A complete bipartite graph of the form $K_{1,q}$ is called a **star**.

Example 14.

(i) K_1 *is a trivial graph.*

(ii) A bipartite graph $G(X, Y)$ *is given in Fig.1.7*

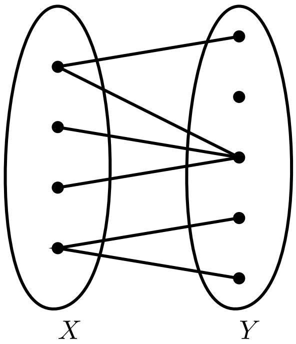


Figure 1.7: A bipartite graph

(iii) The complete bipartite graph $K_{2,3}$ *is given in Fig.1.8*

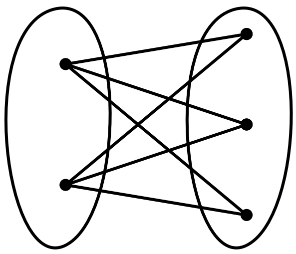


Figure 1.8: The graph $K_{2,3}$

(iv) A star graph is shown in Fig.1.9

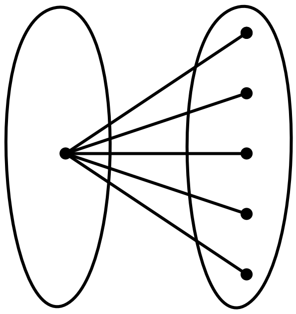


Figure 1.9: The Star graph $K_{1,5}$

Definition 15.

(i) Let G be a simple graph. The **complement** *G^c of G is defined by taking $V(G^c) = V(G)$ and two vertices u and v are adjacent in G^c if and only if they are non-adjacent in G.*

(ii) A simple graph G is called **self-complementary** *if $G \cong G^c$*

Example 16.

(i) A graph G and its complement G^c is shown in Fig.1.10

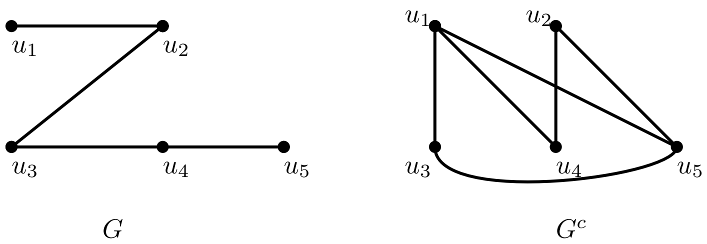


Figure 1.10: A graph G and its complement G^c

(ii) A self complementary graph is shown in Fig.1.11

Let us Sum Up:

In this section, we have studied definitions of graph, parallel edges, loop, neighborhood (open/closed), trivial representation of graph, order, labeled/

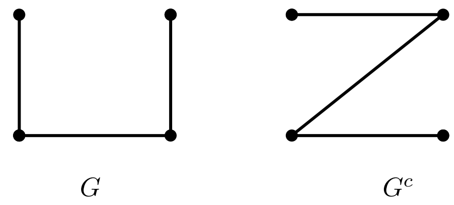


Figure 1.11: A self complementary graph

unlabeled graph, isomorphism of graphs, complete graph, bipartite graphs, self complementary graphs, etc. with relevant diagrams for better understanding.

Check your progress:

1. If G is disconnected, then G is ....

   (a) connected (b) disconnected

   (c) bipartite (d) complete bipartite

2. If G is self-complementary graph of order n, then ....

   (a) $x \equiv 0, 1 (mod 4)$ (b) $x \equiv 2 (mod 4)$

   (c) $x \equiv 3 (mod 4)$ (d) b & c

Illustration:

1. Every simple graph G is a subgraph of a complete graph, $m(G) \leq m(K_n)$.
2. If G is self complementary, then we define $m(G) + m(G^c) = m(K_n)$. Hence $m(K_n)$ must be even.

1.3 Subgraphs

Definition 17.

(i) A graph H is called a **subgraph** *of G if $V(H) \subseteq V(G), E(H) \subseteq E(G)$ and I_H is the restriction of I_G to $E(H)$. If H is a subgraph of G then G is said to be a* **super graph** *of H. A subgraph H of a graph G is a* **proper subgraph** *of G if either $V(H) \neq V(G)$ or $E(H) \neq E(G)$.*

(ii) A subgraph H of G is said to be an **induced subgraph** *of G if each edge of G having its ends in $V(H)$ is also an edge of H.*

(iii) A subgraph H of G is said to be a **spanning subgraph** *of G if $V(H) = V(G)$.*

(iv) The induced subgraph of G with vertex set $S \subseteq V(G)$ is called the subgraph of G **induced by** *S and is denoted by $G[S]$.*

(v) The edge induced subgraph of G with $E' \subseteq E(G)$ is called the subgraph of G **induced by the edge set** *E' and is denoted by $G[E']$.*

(vi) Let u and v be the vertices of a graph G. By $G + uv$, we mean the graph obtained by adding a new edge uv to G.

(vii) A **clique** *of G is a complete subgraph of G. A clique of G is a* **maximal clique** *of G if it is not properly contained in another clique of G.*

Example 18.

(i) In Fig.1.12 a subgraph of a graph G (given in example 1.2.2) is given.

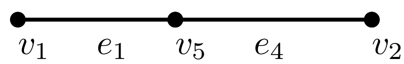


Figure 1.12: A subgraph of G

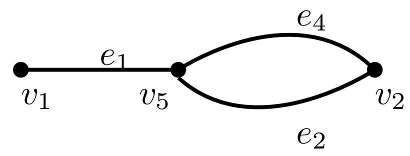


Figure 1.13: An induced subgraph of G

(ii) In Fig.1.13 an induced subgraph H of G is given, where, $V(H) = \{v_1, v_2, v_5\}$.

(iii) A spanning subgraph of G is given in Fig.1.14

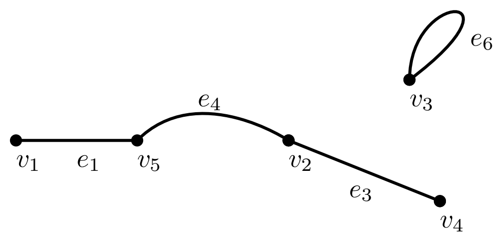


Figure 1.14: A spanning subgraph of G

(iv) Let $S = \{v_1, v_2, v_3\}$. The induced subgraph $G[S]$ is given in Fig.1.15

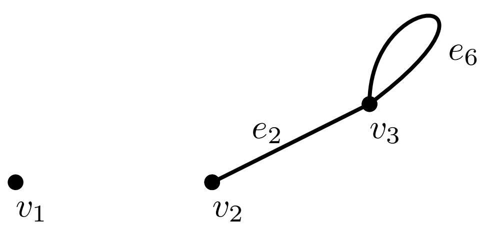


Figure 1.15: An induced subgraph of G induced by $S = \{v_1, v_2, v_3\}$

(v) Let $E' = \{e_1, e_2, e_3\}$. The edge induced subgraph $G[E']$ is given in Fig.1.16

(vi) In Fig. 1.17, the graph $G + v_3v_4$ is given.

(vii) A clique of G is given in Fig.1.18. Note that it is also a maximal clique. A clique of G, that is not maximal is given in Fig.1.19

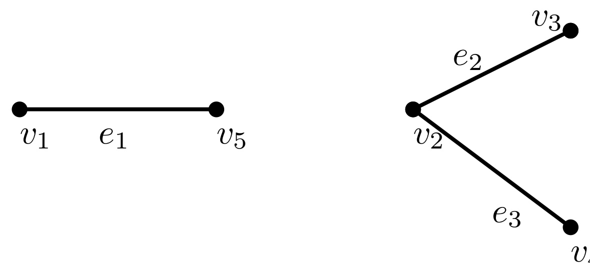


Figure 1.16: An edge-induced subgraph of G induced by $E' = \{e_1, e_2, e_3\}$

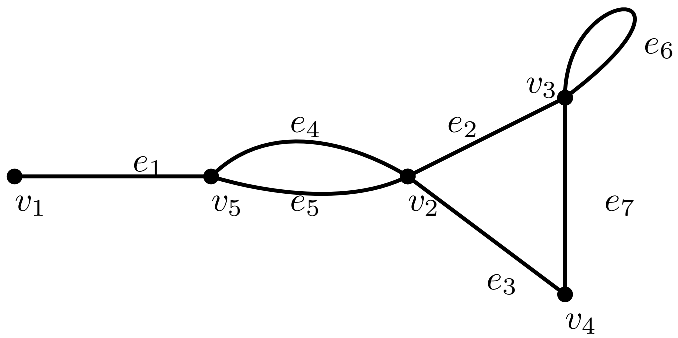


Figure 1.17: $G + v_3v_4$

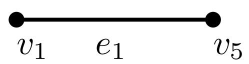


Figure 1.18: A clique of G

Figure 1.19: A clique of G that is not maximal

Definition 19. *Deletion of vertices and edges in a graph:*

(i) Let G be a graph and $S \subseteq V(G)$. The subgraph $G[V \setminus S]$ is said to be obtained from G by **deletion** *of S. This subgraph is denoted by $G - S$. If $S = \{v\}$, then $G - S$ is denoted by $G - v$*

(ii) Let $E' \subseteq E(G)$. The spanning subgraph of G with the edge set $E \setminus E'$ is the subgraph obtained from G by deleting the edge subset F'. This subgraph is denoted by $G - E'$. If $E' = \{e\}$, then $G - E'$ is denoted by

$G - e$.

Example 20.

(i) Let $S = \{v_1, v_3, v_4\}$. *The subgraph* $G - S$ *is given in Fig.1.20(a). The graph* $G - v_2$ *is given in Fig.1.20(b).*

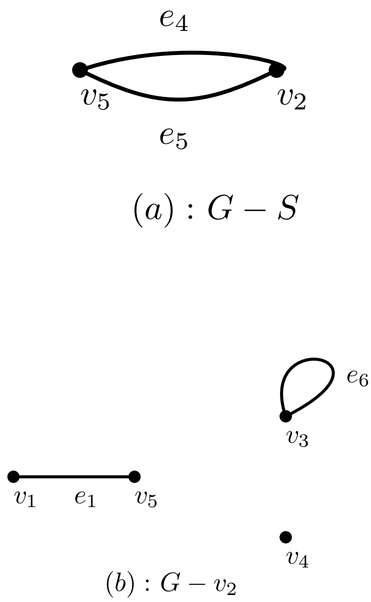
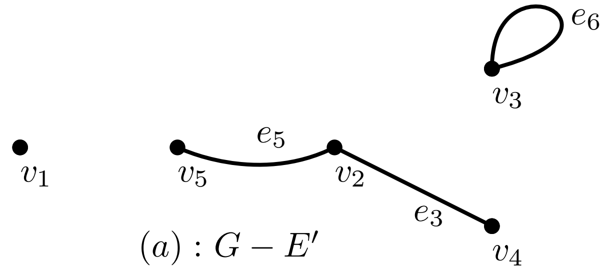
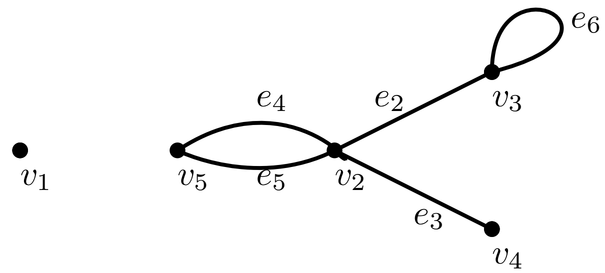


Figure 1.20: $G - S$ and $G - v_2$

(ii) Let $S = \{e_1, e_2, e_4\}$. *The subgraph* $G - E'$ *is given in Fig.1.21(a). The graph* $G - e_1$ *is given in Fig.1.21(b).*





$(b) : G - e_1$

Figure 1.21: $G - E'$ and $G - e_1$

Note: When a vertex deleted from G, all the edges incident to it are also deleted from G, whereas the deletion of an edge from G does not affect the vertices of G.

Let us Sum Up:

In this section, we have studied different types of subgraphs namely, induced subgraphs, spanning subgraphs, clique etc. with examples and diagrams for better understanding.

Answer: 1. (c)

Check your progress:

1. Whether the subgraphs containing all the edges of a given graph is spanning?

   (a) yes (b) no (c) not always (d) never

1.4 Degrees of Vertices

Definition 21.

(i) let G be a graph and $v \in V$. The number of edges incident at v in G

is called the **degree** *(or* **valency***) of the vertex v in G and is denoted by $d_G(v)$ or $d(v)$.*

(ii) A **loop** *at v is to be counted twice in computing the degree of v. The minimum of the degrees of the vertices of a graph G is denoted by $\delta(G)$ and the maximum degree of the vertices of a graph G is denoted by $\Delta(G)$.*

*(iii) A graph G is called k-***regular** *if every vertex of G has degree k.*

(iv) A graph is said to be **regular** *if it is k-regular for some non-negative integer k.*

(v) A 3- regular graph is called a **cubic graph**.

Example 22.

(i) For the graph G, given in example 1.2.2
$d(v_1) = 1;\quad d(v_2) = 4;\quad d(v_3) = 3;$
$d(v_4) = 1;\quad d(v_5) = 3$

(ii) In the same graph G, $\delta(G) = 1$; $\Delta(G) = 4$

(iii) The graph K_3 is 2-regular.

(iv) All complete graphs are regular graphs. The graph K_n is $(n-1)$-regular.

(v) The graph K_4 is cubic.

Definition 23.

(i) A spanning 1-regular subgraph of G is called a **1-factor** *or a* **perfect matching** *of G.*

(ii) A vertex of degree 0 is an **isolated vertex** *of G. A vertex of degree 1 is called a* **pendant vertex** *of G and the unique edge of G incident to such a vertex of G is a* **pendent edge** *of G.*

(iii) A sequence formed by the degree of the vertices of G, when the vertices are taken in the same order, is called a **degree sequence** *of G.*

Example 24.

(i) In Fig. 1.22, a graph G and its 1-factor is shown.

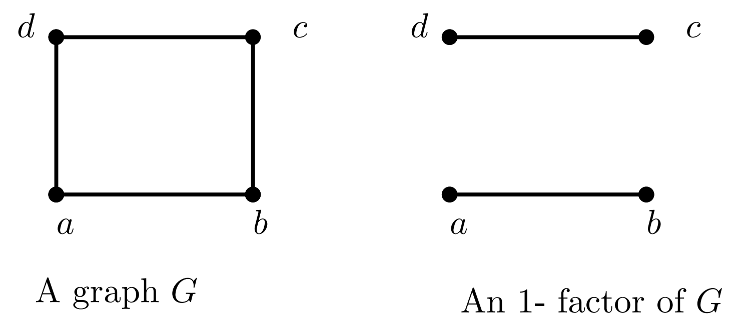


Figure 1.22: A graph G and its 1- factor

(ii) In the graph G given in Fig.1.23, the vertex v_7 is an isolated vertex, the vertex v_6 is a pendent vertex and v_5v_6 is a pendent edge.

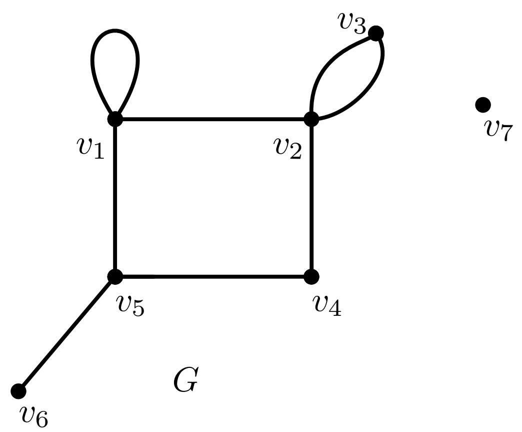


Figure 1.23: Isolated and pendent vertices in a graph

(iii) The degree sequence of G is $(0, 1, 2, 2, 4, 4, 5)$

Theorem 25. *(Euler) The sum of the degrees of the vertices of a graph is equal to twice the number of its edges.*

Proof. If $e = uv$ is an edge of G, then e is counted once while counting the degrees of each of u and v. Hence, each edge contributes 2 to the sum of the degrees of the vertices. Thus, the m edges of G contributes $2m$ to the degree sum. □

Corollary 26. *In any graph G, the number of vertices of odd degree is even.*

Proof. Let V_1 and V_2 be the subsets of vertices of G with odd and even degrees respectively. By Theorem 25, we have

$$\begin{aligned} 2m &= \sum_{v \in V} d_G(v) \\ 2m &= \sum_{v \in V_1} d_G(v) + \sum_{v \in V_2} d_G(v) \end{aligned}$$

we know that $2m$ and $\sum_{v \in V_2} d_G(v)$ are even. Hence, $\sum_{v \in V_2} d_G(v)$ is even. Since for each $v \in V_1$, $d_G(v)$ is odd, we have $|V_1|$ must be even. □

Definition 27. Graphical Sequences: *A sequence of non-negative integers $d = (d_1, d_2, \ldots, d_n)$ is called* **graphical** *if there exists a simple graph whose degree sequence is d. Clearly, a necessary condition for $d = (d_1, d_2, \ldots, d_n)$ to be graphical is that $\sum_{i=1}^{n} d_i$ is even and $d_i \geq 0, 1 \leq i \leq n$.*

These conditions, however, are not sufficient, as example 28. shows.

Example 28. *The sequence $d = (7, 6, 3, 3, 2, 1, 1, 1)$ is not graphical. Even though each term of d is a non-negative integers and the sum of the terms is even, d is not graphical. Suppose, if d is graphical, then there exists a simple graph G with eight vertices whose degree sequence is d. Let v_1 be the vertex of G with degree 7. Since G is simple, v_1 is adjacent to all the remaining vertices of G. Let v_2 be the vertex of G with degree 6.*

Then v_2 must be adjacent to another five vertices (already v_1 is adjacent to v_2. Continuing in this way, we observe that we can't get three pendent vertices.

Theorem 29. *In any group of n persons$(n \geq 2)$, there are at least two with the same number of friends.*

Proof. Denote the n persons by $v_1, v_2, v_3, \ldots, v_n$. Let G be the simple graph with vertex set $V = \{v_1, v_2, v_3, \ldots, v_n\}$ in which v_i and v_j are adjacent if and only if the corresponding persons are friends. Then the number of friends of v_1 is just the degree of v_i in G. Hence, to solve the problem, we must prove that there are vertices in G with the same degree. Suppose this is not true. Then the degree of the vertices of G must be distinct. i.e., $0, 1, 2, \ldots, (n-1)$. Vertex of degree $(n-1)$ must be adjacent to all the other vertices of G. Hence, there can not be a vertex of degree 0 in G. This contradiction shows that the degrees of the vertices of G can not all be distinct and hence at least two of them should have the same degree. □

Let us Sum Up:

Regular graph need not be a factor and factor need not be regular graph. No odd regular graph of odd order exist, since nowhere of odd degree vertex must always be even.

Check your progress:

1. If G is a regular bipartite graph with bipartite (X, Y), then ....
(a) $|X| \leq |Y|$ (b) $|X| < |Y|$ $|X| = |Y|$ (c) $|X| \neq |Y|$
Answer: 1. (c)

1.5 Paths and Connectedness

Definition 30.

(i) *A* **walk** *in a graph G is an alternating sequence $W = v_0e_1v_1e_2v_2 \ldots e_pv_p$ of vertices and edges beginning and ending with vertices in which v_{i-1} and v_i are the ends of e_i; v_0 is the* **origin** *and v_p is the* **terminus** *of W.*

(ii) *The walk W is said to* **join** *v_0 and v_p; it is also referred to as a $v_0 - v_p$ walk.*

(iii) *If the graph is simple, a walk is determined by the sequence of its vertices. The walk is* **closed** *if $v_0 = v_p$ and is* **open** *otherwise.*

(iv) *A walk is called a* **trail** *if all the edges appearing in the walk are distinct. It is called a* **path** *if all the vertices are distinct.*

(v) *A* **cycle** *is a closed trail in which the vertices are all distinct. The* **length** *of a walk is the number of edges in it. A walk of length 0 consists of just a single vertex.*

Example 31. *In the graph of Fig.1.24,*

(i) *$v_5e_7v_1e_1v_2e_4v_4e_5v_1e_7v_5e_9v_6$ is a walk but not a trail (as edge e_7 is repeated).*

(ii) *$v_1e_1v_2e_2v_3e_3v_2e_1v_1$ is a closed walk.*

(iii) *$v_1e_1v_2e_4v_4e_5v_1e_7v_5$ is a trail.*

(iv) *$v_6e_8v_1e_1v_2e_2v_3$ is a path.*

(v) *$v_1e_1v_2e_4v_4e_6v_5e_7v_1$ is a cycle.*

Definition 32.

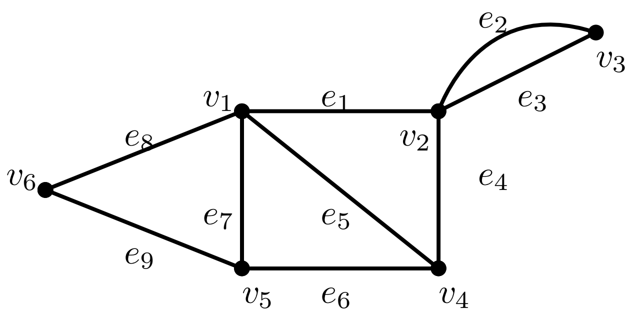


Figure 1.24: graph illustrating walks, trails, paths, and cycles

(i) A cycle of length k is denoted by C_k. Further, P_k denotes a path on k vertices. In particular, C_3 is often referred to as a **triangle***, C_4 as a* **square***, and C_5 as a* **pentagon***.*

(ii) If $P = v_0e_1v_1e_2v_2 \ldots e_kv_k$ is a path, then $P^{-1} = v_ke_kv_{k-1}e_{k-1}v_{k-2} \ldots v_1e_1v_0$ is also a path and P^{-1} is called the **inverse** *of the path P.*

(iii) The subsequence $v_ie_{i+1}v_{i+1} \ldots e_jv_j$ of P is called the $v_i - v_j$ **section** *of P.*

Definition 33.

(i) Let G be a graph. Two vertices u and v of G are said to be **connected** *if there is a $u - v$ path in G. The relation "connected" is an equivalence relation on $V(G)$.*

(ii) Let $V_1, V_2, \ldots, V_\omega$ be the equivalence classes. The subgraphs $G[V_1], G[V_2], \ldots, G[V_\omega]$ are called the **components** *of G.*

(iii) If $\omega = 1$; the graph G is connected; otherwise, the graph G is **disconnected** *with $\omega \geq 2$ components. (see Fig. 1.25).*

Definition 34. *The components of G are clearly the maximal connected subgraphs of G. We denote the number of components of G by $\omega(G)$. Let u and v be two vertices of G. If u and v are in the same component*

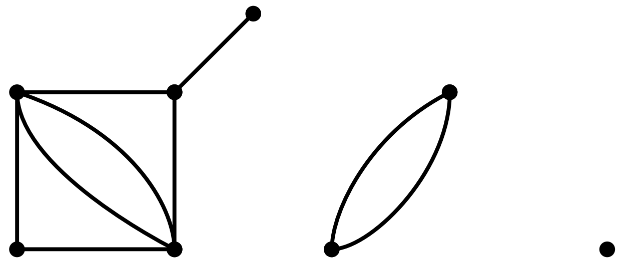


Figure 1.25: A graph G with three components

of G, we define $d(u, v)$ to be the length of a shortest $u - v$ path in G, otherwise, we define $d(u, v)$ to be ∞. If G is a connected graph, then d is a distance function or metric on $V(G)$ that is, $d(U, V)$ satisfies the following conditions:

(i) $d(u, v) \geq 0$, and $d(u, v) = 0$ if and only if $u = v$.

(ii) $d(u, v) = d(v, u)$.

(iii) $d(u, v) \leq d(u, w) + d(w, v)$, for every w in $V(G)$

Proposition 35. *If G is simple and $\delta \geq \frac{n-1}{2}$, then G is connected.*

Proof. Assume the contrary. Then G has at least two components, say G_1, G_2. Let v be any vertex of G_1, As $\delta \geq \frac{n-1}{2}$, $d(v) \geq \frac{n-1}{2}$. All the vertices adjacent to v in G must belong to G_1. Hence, G_1 contains at least $d(v) + 1 \geq \frac{n-1}{2} + 1 = \frac{n-1}{2}$ vertices. Similarly, G_2 contains at least $\frac{n+1}{2}$ vertices. Therefore G has at least $\frac{n+1}{2} + \frac{n+1}{2} = n + 1$ vertices, which is a contradiction. □

Theorem 36. *If a simple graph G is not connected, then G^c is connected.*

Proof. Let u and v be any two vertices of G^c (and therefore of G). If u and v belong to different components of G, then obviously u and v are nonadjacent in G and so they are adjacent in G^c. Thus u and v are connected in G^c. In case u and v belong to the same component of G, take a vertex w of G not belonging to this component of G. Then uw and vw are not edges of G and hence they are edges of G^c. Then uwv is a $u-v$ path in G^c. Thus G^c is connected. □

Theorem 37. *The number of edges of a simple graph of order n having ω components cannot exceed $\frac{(n-\omega)(n-\omega+1)}{2}$.*

Proof. Let $G_1, G_2, \ldots, G_\omega$ be the components of a simple graph G and let n_i be the number of vertices of G_i, $1 \leq i \leq \omega$. Then $m(G_i) \leq \frac{n_i(n_i-1)}{2}$, and hence $m(G_i) \leq \sum_{i=1}^{\omega} \frac{n_i(n_i-1)}{2}$ Since $n_i \geq 1$ for each i, $1 \leq i \leq \omega$. $n_1 = (n_1+n_2+\ldots+n_{i-1}+\ldots+n_{i+1}+\ldots+n_\omega)$. Hence, $\sum_{i=1}^{\omega} \frac{n_i(n_i-1)}{2} \leq \sum_{i=1}^{\omega} \frac{(n-\omega+1)(n_i-1)}{2} = \frac{(n-\omega+1)}{2} \sum_{i=1}^{\omega}(n_i-1) = \frac{(n-\omega+1)}{2}[(\sum_{i=1}^{\omega} n_i)-\omega] = \frac{(n-\omega+1)(n-\omega)}{2}$. □

Definition 38.

(i) *A graph G is called* **locally connected** *if, for every vertex v of G, the subgraph induced by the neighbor set $N_G(v)$ in G is connected.*

(ii) *A cycle is* **odd or even** *depending on whether its length is odd or even.*

Theorem 39. *A graph is bipartite if and only if it contains no odd cycles.*

Proof. Suppose that G is a bipartite graph with the bipartition (X, Y). Let $C = v_1e_1v_2e_2v_3e_3 \ldots v_ke_kv_1$ be a cycle in G. Without loss of generality, we can suppose that $v_1 \in X$. As v_2 is adjacent to v_1, $v_2 \in Y$. Similarly, v_3 belongs to X, v_4 to Y, and so on. Thus, $v_i \in X$ or Y according as i is odd or even,$1 \leq i \leq k$. Since v_kv_1 is an edge of G and $v_1 \in X, v_k \in Y$. Accordingly, k is even and C is an even cycle.

Conversely, let us suppose that G contains no odd cycles. We first assume that G is connected. Let u be a vertex of G. Define $X = \{v \in$

$V|d(u,v)$is even$\}$ and $Y = \{v \in V|d(u,v)$is even$\}$. We will prove that (X, Y) is a bipartition of G. To prove this we have only to show that no two vertices of X as well as no two vertices of Y are adjacent in G. Let v, w be two vertices of X. Then $p = d(u, v)$and $q = d(u, w)$ are even. Further, as $d(u, u) = 0$, $u \in X$. Let P be a $u - v$ shortest path of length p and Q. a $u - w$ shortest path of length q. (See Fig. 1.26.) Let w_1 be a vertex common to P and Q such that the $w_1 - v$ section of P and the $w_1 - w$ section of Q contain no vertices common to P and Q. Then the $u - w_1$ sections of both P and Q have the same length. Hence, the lengths

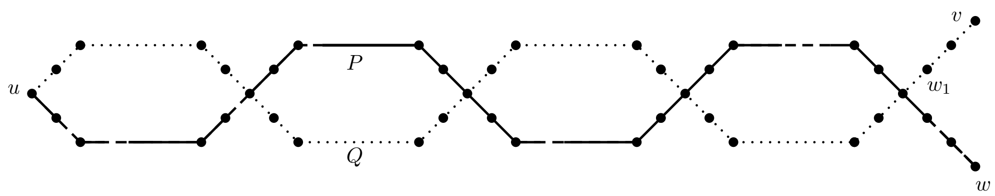


Figure 1.26: Graph for proof of Theorem 39

of the $w_1 - v$ section of $P$and the $w_1 - w$ section of Q are both even or both odd. Now if $e = vw$ is an edge of G, then the $w_1 - v$ section of P followed by the edge vw and the $w - w_1$ section of the $w - u$ path Q^{-1} is an odd cycle in G, contradicting the hypothesis. This contradiction proves that no two vertices of X are adjacent in G. Similarly, no two vertices of Y are adjacent in G. This proves the result when G is connected.

If G is not connected, let $G_1, G_2, \ldots, G_\omega$ be the components of G. By hypothesis, no component of G contains an odd cycle. Hence, by the previous paragraph, each component G_i, $1 \le i \le \omega$, is bipartite. Let (X_i, Y_i) be the bipartition of G_i. Then (X, Y), where $X = \bigcup_{i=1}^{\omega} X_i$ and $Y = \bigcup_{i=1}^{\omega} Y_i$ is a bipartition of G, and G is a bipartite graph. □

Example 40. *Prove that in a connected graph G with at least three vertices, any two longest paths have a vertex in common.*

Proof. Suppose $P = u_1u_2\ldots u_k$ and $Q = v_1v_2\ldots v_k$ are two longest paths in G having no vertex in common. As G is connected, there exists a $u_1 - v_1$ path P' in G. Certainly there exist vertices u_r and v_s of P', $1 \leq r \leq k, 1 \leq s \leq k$ such that the $u_r - v_s$ section P'' of P' has no internal vertex in common with P or Q.

Now, of the two sections $u_1 - u_r$ and $u_r - u_k$ of P, one must have length at least $\frac{k}{2}$. Similarly, of the two sections $v_1 - v_s$ and $v_s - v_k$ of Q, one must have length at least $\frac{k}{2}$. Let these sections be P_1 and Q_1, respectively. Then $P_1 \cup P'' \cup Q_1$ is a path of length at least $\frac{k}{2} + 1 + \frac{k}{2}$, contradicting that k is the length of a longest path in G. □

Let us sum pp:

1. Locally connected graph need not be connected and vice-versa.

Example: rK_s is locally connected but not connected. $K_{1,n}$ is connected but not locally connected.

2. Acyclic graphs are bipartite but the converse not true.

Check your Progress:

1. Does $K_{m,n}$ posses regular factor?

(a) yes (b) no (c) yes when $m = n$ (d) never

Answer: 1. (c)

1.6 Automorphism of a Simple Graph

Definition 41. *An* **automorphism** *of a graph G is an isomorphism of G onto itself.*

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Figure 1.27: Graph for the solution to Example 1.40

Theorem 42. *The set $Aut(G)$ of all automorphisms of a simple graph G is a group with respect to the composition $\circ$ of mappings as the group* *operation.*

Proof. We shall verify that the four axioms of a group are satisfied by the pair $(Aut(G), \circ)$.

(i) Let ϕ_1 and ϕ_2 be bijections on $V(G)$ preserving adjacency and nonadjacency. Clearly, the mapping $\phi_1 \circ \phi_2$ is a bijection on $V(G)$. If u and v are adjacent in G, then $\phi_2(u)$ and $\phi_2(v)$ are adjacent in G. But $(\phi_1 \circ \phi_2)(u) = \phi_1(\phi_2(u))$ and $(\phi_1 \circ \phi_2)(v) = \phi_1(\phi_2(v))$. Hence, $(\phi_1 \circ \phi_2)(u)$ and $(\phi_1 \circ \phi_2)(v)$ are adjacent in G; that is, $\phi_1 \circ \phi_2$ preserves adjacency. A similar argument shows that $\phi_1 \circ \phi_2$ preserves nonadjacency. Thus, $\phi_1 \circ \phi_2$ is an automorphism of G.

(ii) It is a well-known result that the composition of mappings of a set onto itself is associative

(iii) The identity mapping I of $V(G)$ onto itself is an automorphism of G, and it satisfies the condition $\phi \circ I = I \circ \phi$ for every $\phi \in Aut(G)$. Hence, I is the identity element of $Aut(G)$.

(iii) Finally, if ϕ is an automorphism of G, the inverse mapping ϕ^{-1} is also an automorphism of G. □

Theorem 43. *For any simple graph G,* $Aut(G) = Aut(G^c)$.

Since $V(G^c) = V(G)$, every bijection on $V(G)$ is also a bijection on $V(G^c)$. As an automorphism of G preserves the adjacency and nonadjacency of vertices of G, it also preserves the adjacency and nonadjacency of vertices of G^c. Hence, every element of $Aut(G)$ is also an element of $Aut(G^c)$, and vice versa.

Let us Sum Up:

Any automorphism of G is also an automorphism of G^c.

1.7 Line Graphs

Let G be a loopless graph. We construct a graph $L(G)$ in the following way: The vertex set of $L(G)$ is in $1-1$ correspondence with the edge set of G and two vertices of $L(G)$ are joined by an edge if and only if the corresponding edges of G are adjacent in G. The graph $L(G)$ (which is always a simple graph) is called the line graph or the edge graph of G. Fig.1.28, shows a graph and its line graph in which v_i of $L(G)$ corresponds to the edge e_i of G for each i. Some simple properties of the line

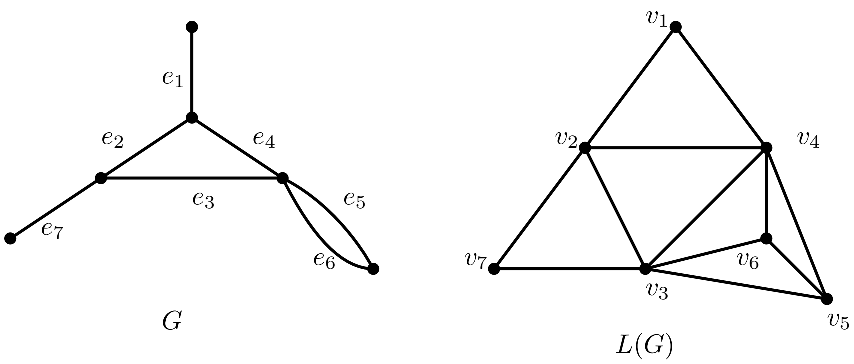


Figure 1.28: A graph G and its line graph $L(G)$

graph $L(G)$ of a graph G follow:

1. G is connected if and only if $L(G)$ is connected.

2. If H is a subgraph of G, then $L(H)$ is a subgraph of $L(G)$.

3. The edges incident at a vertex of G give rise to a maximal complete subgraph of $L(G)$.

4.If $e = uv$ is an edge of a simple graph G, the degree of e in $L(G)$ is the same as the number of edges of G adjacent to e in G. This number is $d_G(u) + d_G(v) - 2$. Hence, $d_{L(G)}(e) = d_G(u) + d_G(v) - 2$.

5. Finally, if G is a simple graph,

$$\begin{aligned}\sum_{e\in V(L(G))} d_{L(G)}(e) &= \sum_{uv\in E(G)} (d_G(u)+d_G(v)-2)\\ &= \left[\sum_{u\in V(G)} d_G(u)^2\right] - 2m(G)\\ &= \left[\sum_{i=1}^{n} d_i^2\right] - 2m.\end{aligned}$$

(since uv belongs to the stars at u and v.

where $(d_1, d_2, \ldots, d_n)$ is the degree sequence of G, and $m = m(G)$. By Euler's theorem (Theorem 25), it follows that the number of edges of $L(G)$ is given by

$m(L(G)) = \frac{1}{2}\left[\sum_{i=1}^{n} d_i^2\right] - m.$

Theorem 44. *The line graph of a simple graph G is a path if and only if G is a path.*

Proof. Let G be the path P_n on n vertices. Then clearly, $L(G)$ is the path P_{n-1} on $n-1$ vertices. Conversely, let $L(G)$ be a path. Then no vertex of G can have degree greater than 2 because if G has a vertex v of degree greater than 2, the edges incident to v would form a complete subgraph of $L(G)$ with at least three vertices. Hence, G must be either a cycle or a path. But G cannot be a cycle, because the line graph of a cycle is again a cycle. □

Theorem 45. *If the simple graphs G_1 and G_2 are isomorphic, then $L(G_1)$ and $L(G_2)$ are isomorphic.*

Proof. Let (ϕ, θ) be an isomorphism of G_1 onto G_2 Then θ is a bijection of $E(G_1)$ onto $E(G_2)$. We show that θ is an isomorphism of $L(G_1)$ to $L(G_2)$. We prove this by showing that θ preserves adjacency and nonadjacency. Let e_i and e_j be two adjacent vertices of $L(G_1)$. Then there exists a vertex v of G_1 incident with both e_i and e_j , and so $\phi(v)$ is a vertex incident with both $\theta(e_i)$ and $\theta(e_j)$. Hence, $\theta(e_i)$ and $\theta(e_j)$ are adjacent vertices in $L(G_2)$.

Now, let $\theta(e_i)$ and $\theta(e_j)$ be adjacent vertices in $L(G_2)$. This means that they are adjacent edges in G_2 and hence there exists a vertex v_0 of G_2 incident to both $\theta(e_i)$ and $\theta(e_j)$ in G_2. Then $\phi^{-1}(v')$ is a vertex of G_1 incident to both e_i and e_j, so that e_i and e_j are adjacent vertices of $L(G_1)$. Thus, e_i and e_j are adjacent vertices of $L(G_1)$ if and only if $\theta(e_i)$ and $\theta(e_j)$ are adjacent vertices of $L(G_2)$. Hence, θ is an isomorphism of $L(G_1)$ onto $L(G_2)$. (Recall that a line graph is always a simple graph.) □

Definition 46. *A graph H is called a* **forbidden** *subgraph for a property P of graphs if it satisfies the following condition: If a graph G has property P, then G cannot contain an induced subgraph isomorphic to H.*

Theorem 47. *If G is a line graph, then $K_{1,3}$ is a forbidden subgraph of G.*

Proof. Suppose that G is the line graph of graph H and that G contains a $K_{1,3}$ as an induced subgraph. If v is the vertex of degree 3 in $K_{1,3}$ and v_1, v_2, and v_3 are the neighbors of v in this $K_{1,3}$, then the edge e corresponding to v in H is adjacent to the three edges e_1, e_2, and e_3 corresponding to the vertices v_1, v_2, and v_3. Hence, one of the end vertices of e must be the end vertex of at least two of e_1, e_2, and e_3 in H, and hence v together with two of v_1, v_2, and v_3 form a triangle in G. This means that the $K_{1,3}$ subgraph of G considered above is not an induced subgraph of G, a contradiction. □

Let us Sum Up:

1. We have studied a type of trivial graph called line graph.
2. The graph G and $L(G)$ are isomorphic iff $G = C_a$.

Check your Progress:

1. Check the statement G has a cyclic iff $L(G)$ has a cyclic.
(a) true (b) false (c) never true (d) converse not true

Answer: 1. (d)

1.8 Operations on Graphs

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two simple graphs

Definition 48. Union of two graphs: *The graph $G = (V, E)$, where $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$ is called the* **union** *of G_1 and G_2 and is denoted by $G_1 \cup G_2$.*
When G_1 and G_2 are vertex disjoint, $G_1 \cup G_2$ is denoted by $G_1 + G_2$ and is called the sum of the graphs G_1 and G_2.

Definition 49. Intersection of two graphs: *If $V_1 \cap V_2 \neq \emptyset$, the graph $G = (V, E)$, where $V = V_1 \cap V_2$ and $E = E_1 \cap E_2$ is the intersection of G_1 and G_2 and is written as $G_1 \cap G_2$.*

Definition 50. Join of two graphs: *Let G_1 and G_2 be two vertex-disjoint graphs. Then the join $G_1 \vee G_2$ of G_1 and G_2 is the supergraph of $G_1 + G_2$ in which each vertex of G_1 is also adjacent to every vertex of G_2. Fig.1.29 illustrates the graph $G_1 \vee G_2$. If $G_1 = K_1$ and $G_2 = C_n$, then $G_1 \vee G_2$ is called the* **wheel** *W_n. W_5 is shown in Fig. 1.30. It is worthwhile to note*

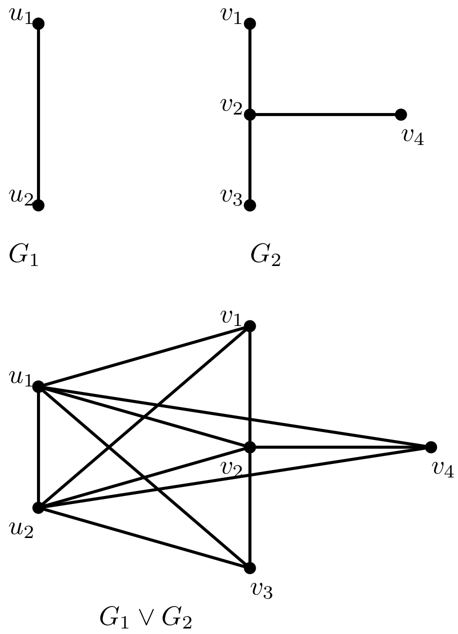


Figure 1.29: $G_1 \vee G_2$

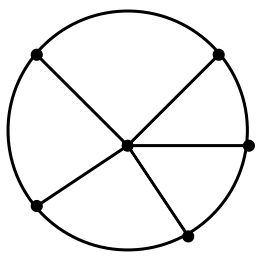


Figure 1.30: Wheel W_5

that $K_{m,n} = K_m^c \vee K_n^c$ and $K_n = K_1 \vee K_{n-1}$.

It follows from the above definitions that

(i) $n(G_1 \cup G_2) = n(G_1) + n(G_2) - n(G_1 \cap G_2), m(G_1 \cup G_2) = m(G_1) + m(G_2) - m(G_1 \cap G_2)$.

(ii) $n(G_1 + G_2) = n(G_1) + n(G_2), m(G_1 + G_2) = m(G_1) + m(G_2)$ and

(iii) $n(G_1 \vee G_2) = n(G_1) + n(G_2), m(G_1 \vee G_2) = m(G_1) + m(G_2) - n(G_1)n(G_2)$.

Let us Sum Up:

If G_1 and G_2 are disjoint, we can write $G_1 \vee G_2 = G_1 \cup G_2 \cup K_{m,n}$, where m_1 and m_2 are order G_1 and G_2 respectively.

1.9 Application to Social Psychology

Group dynamics is the study of social relationships between people within a particular group. The graphs that are commonly used to study these relationships are signed graphs. A **signed graph** is a graph G with sign $+$ or $-$ attached to each of its edges. An edge of G is positive (respectively, negative) if the sign attached to it is $+$ (respectively, $-$). A positive sign between two persons u and v would mean that u and v are "related", that is, they share the same social trait under consideration. A negative sign would indicate the opposite. The social trait may be "same political ideology", "friendship", "likes certain social customs", and so on. A group of people with such relations between them is called a **social system**. A social system is called **balanced** if any two of its people have a positive relation between them, or if it is possible to divide the group into two subgroups so that any two persons in the same subgroup have a positive relation between them while two persons of different subgroups have a negative relation between them. This of course means that if both u and v have negative relation to w, then u and v must have positive relation between them. In consonance with a balanced social system, a balanced signed graph G is defined as a graph in which the vertex set V can be partitioned into two subsets $V_i, i = 1, 2$, one of which may be empty, so that any edge in each $G[V_i]$ is positive, while any edge between V_1 and V_2 is negative.

1.10 Directed Graphs

Directed graphs arise in a natural way in many applications of graph theory. The street map of a city, an abstract representation of computer programs, and network flows can be represented only by directed graphs rather than by graphs. Directed graphs are also used in the study of sequential machines and system analysis in control theory.

Definition 51. *A* **directed** *graph D is an ordered triple $(V(D), A(D), I_D)$, where $V(D)$, is a nonempty set called the set of vertices of D; $A(D)$ is a set disjoint from $V(D)$, called the set of* **arcs** *of D; and I_D is an incidence map that associates with each arc of D an ordered pair of vertices of D. If a is an arc of D, and $I_D(a) = (u, v), u$ is called the* **tail** *of a, and v is the* **head** *of a. The arc a is said to join v with $u.u$ and v are called the* **ends** *of a. A directed graph is also called a* **digraph**.
With each digraph D, we can associate a graph G (written $G(D)$ when reference to D is needed) on the same vertex set as follows: Corresponding to each arc of D, there is an edge of G with the same ends. This graph G is called the **underlying graph** *of the digraph D. Thus, every digraph D defines a unique (up to isomorphism) graph G. Conversely, given any graph G, we can obtain a digraph from G by specifying for each edge of G an order of its ends. Such a specification is called an orientation of G. A digraph and its underlying graph are shown in Fig.2.1 Many of the concepts and terminology for graphs are also valid for digraphs. However, there are many concepts of digraphs involving the notion of orientation that apply only to digraphs.*

Definition 52. *If $a = (u, v)$ is an arc of D, a is said to be* **incident** *out of u and incident into $v.v$ is called an* **outneighbor** *of u, and u is called an*

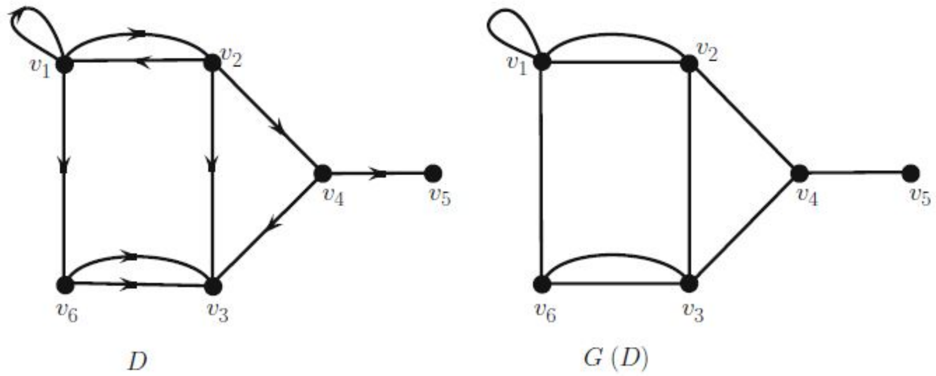


Figure 1.31: Diagraph D and its underlying graph $G(D)$

inneighbor of v. $N_D^+(u)$ *denotes the set of outneighbors of* u *in* D. *Similarly,* $N_D^-(u)$ *denotes the set of* **inneighbors** *of* u *in* D. *When no explicit reference to* D *is needed, we denote these sets by* $N^+(u)$ *and* $N^-(u)$, *respectively. An arc a is incident with* u *if it is either incident into or incident out of* u. *An arc having the same ends is called a* **loop** *of* D. *The number of arcs incident out of a vertex* v *is the* **outdegree** *of* v *and is denoted by* $d_D^+(v)$ *or* $d^+(v)$. *The number of arcs incident into* v *is its* **indegree** *and is denoted by* $d_D^-(v)$ *or* $d^-(v)$. *For the digraph D of Fig. 2.1, we have,* $d^+(v_1) = 3; d^+(v_2) = 3; d^+(v_3) = 0; d^+(v_4) = 2; d^+(v_5) = 0; d^+(v_6) = 2; d^-(v_1) = 2; d^-(v_2) = 1; d^-(v_3) = 4; d^-(v_4) = 1; d^-(v_5) = 1; d^-(v_6) = 1$. *(The loop at* v_1 *contributes* 1 *each to* $d^+(v_1)$ *and* $d^-(v_1)$*).*

The degree $d_D(v)$ *of a vertex* v *of a digraph* D *is the degree of* v *in* $G(D)$. *Thus,*$d(v) = d^+(v) + d^-(v)$. *As each arc of a digraph contributes* 1 *to the sum of the outdegrees and* 1 *to the sum of indegrees, we have*

$$\sum_{v \in V(D)} d^+(v) = \sum_{v \in V(D)} d^-(v) = m(D)$$

where $m(D)$ *is the number of arcs of* D.

A vertex of D *is isolated if its degree is* 0; *it is pendant if its degree is*

1. *Thus, for a pendant vertex v, either $d^{+}(v) = 1$ and $d^{-}(v) = 0$, or $d^{+}(v) = 0$ and $d^{-}(v) = 1$.*

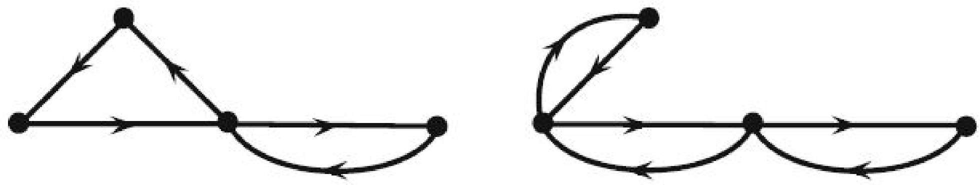


Figure 1.32: A strong diagraph (left) and a symmetric diagraph (right)

Definition 53.

1. *A digraph D' is a subdigraph of a digraph D if $V(D') \subseteq V(D), A(D') \subseteq A(D)$, and $I_{D'}$ is the restriction of I_D to $A(D')$.*

2. *A directed walk joining the vertex v_0 to the vertex v_k in D is an alternating sequence $W = v_0a_1v_1a_2v_2 \ldots a_kv_k, 1 \leq i \leq k$, with a_i incident out of v_{i-1} and incident into v_i. Directed trails, directed paths, directed cycles, and induced subdigraphs are defined analogously as for graphs.*

3. *A vertex v is reachable from a vertex u of D if there is a directed path in D from u to v.*

4. *Two vertices of D are diconnected if each is reachable from the other in D. Clearly, diconnection is an equivalence relation on the vertex set of D, and if the equivalence classes are $V_1, V_2, \ldots, V_\omega$, the subdigraphs of D induced by $V_1, V_2, \ldots, V_\omega$ are called the dicomponents of D.*

5. *A digraph is diconnected (also called strongly-connected) if it has exactly one dicomponent. A diconnected digraph is also called a strong digraph.*

6. *A digraph is strict if its underlying graph is simple. A digraph D is symmetric if, whenever (u, v) is an arc of D, then (u, v) is also an arc of D (see Fig. 2.3).*
7. *A directed spanning path is called directed Hamilton path.*
8. *A directed spanning cycle is called directed Hamilton cycle.*

1.11 Tournaments

A digraph D is a tournament if its underlying graph is a complete graph. Thus, in a tournament, for every pair of distinct vertices u and v, either (u, v) or (v, u), but not both, is an arc of D. Figures 2.3 a, b display all tournaments on three and four vertices, respectively.

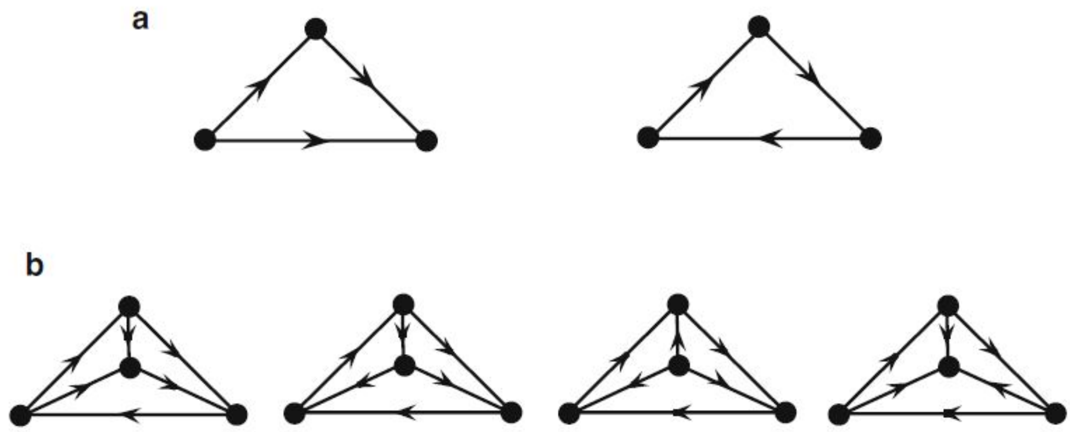


Figure 1.33: Tournaments on (a) three and (b) four vertices

Theorem 54. *(Rèdei). Every tournament contains a directed Hamilton path.*

Proof. (By induction on the number of vertices n of the tournament.) The result can be directly verified for all tournaments having two or three vertices. Hence, suppose that the result is true for all tournaments on $n \geq 3$

vertices. Let T be a tournament on $n+1$ vertices $v_1, v_2, \ldots, v_{n+1}$. Now, delete v_{n+1} from T. The resulting subdigraph T' of T is a tournament on n vertices and hence by the induction hypothesis contains a directed Hamilton path. Assume that the Hamilton path is $v_1v_2\ldots v_n$, relabeling the vertices, if necessary.

If the arc joining v_1 and v_{n+1} has v_{n+1} as its tail, then $v_{n+1}v_1v_2\ldots v_n$ is a directed Hamilton path in T and the result stands proved (see Fig. 2.4a).

If the arc joining v_n and v_{n+1} is directed from v_n to v_{n+1}, then $v_1v_2\ldots v_nv_{n+1}$ is a directed Hamilton path in T (see Fig. 2.4b). Now suppose that none

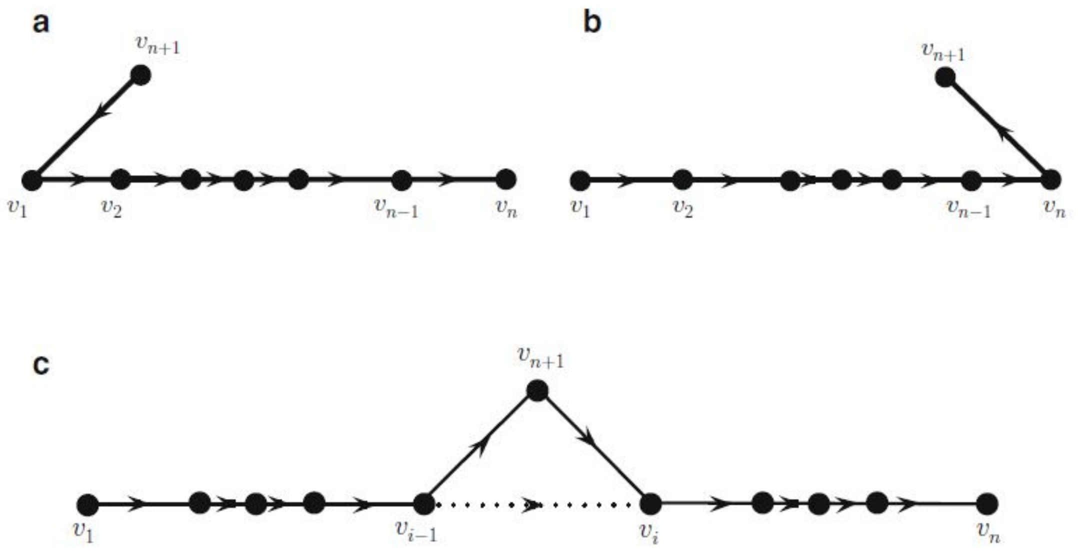


Figure 1.34: Digraphs for proof of Theorem 54

of (v_{n+1}, v_1) and v_n, v_{n+1} is an arc of T. Hence, (v_1, v_{n+1}) and (v_{n+1}, v_n) are arcs of T -the first arc incident into v_{n+1} and the second arc incident out of v_{n+1}. Thus, as we pass on from v_1 to v_n, we encounter a reversal of the orientation of edges incident with v_{n+1}. Let $v_i, 2 \leq i \leq n$, be the first vertex where this reversal takes place, so that (v_{i-1}, v_{n+1}) and (v_{n+1}, v_i) are arcs of T. Then $v_1v_2\ldots v_{i-1}v_{n+1}v_iv_{i+1}\ldots v_n$ is a directed Hamilton path of T (see Fig. 2.4c). $\square$

Theorem 55. *(Moon) Every vertex of a diconnected tournament T on n vertices with $n \geq 3$ is contained in a directed k-cycle, $3 \leq k \leq n$. (T is then said to be vertex-pancyclic.)*

Proof. Let T be a diconnected tournament with $n \geq 3$ and u, a vertex of T. Let $S = N^+(u)$, the set of all outneighbors of u in T, and $S' = N^-(u)$, the set of all inneighbors of u in T. As T is diconnected, none of S and S' is empty. If $[S, S']$ denotes the set of all arcs of T having their tails in S and heads in S', then $[S, S']$ is also nonempty for the same reason. If (v, w) is an arc of $[S, S']$, then (u, v, w, u) is a directed 3-cycle in T containing u. (see Fig. 2.5a.) Suppose that u belongs to directed cycles

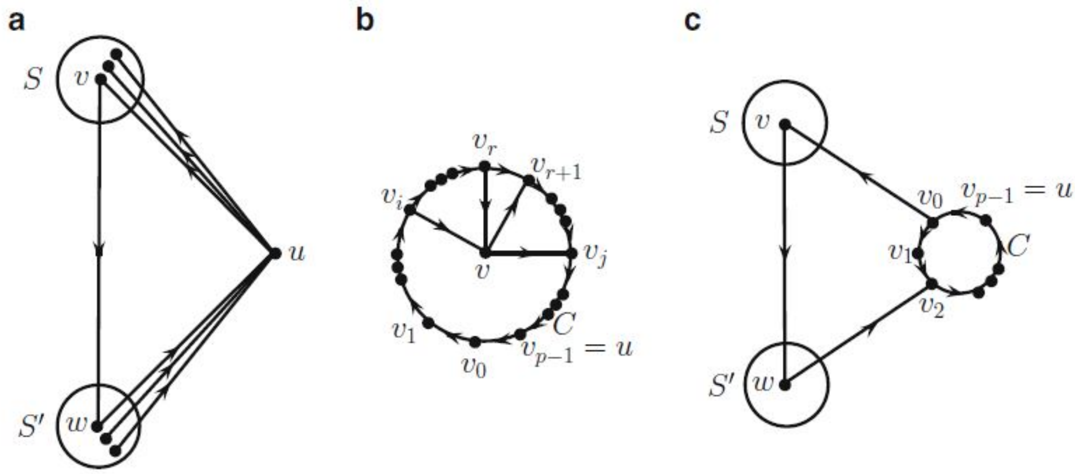


Figure 1.35: Digraphs for proof of Theorem 55

of T of all lengths $k, 3 \leq k \leq p$, where $p < n$. We shall prove that there is a directed $(p+1)$-cycle of T containing u.

Let $C : (v_0, v_1, \ldots, v_{p-1}, v_0)$ be a directed p-cycle containing u, where $v_{p-1} = u$. Suppose that v is a vertex of T not belonging to C such that for some i and j, $0 \leq i, j \leq p-1, i \neq j$, there exist arcs (v_i, v) and (v, v_j) of T (see Fig. 2.6b). Then there must exist arcs (v_r, v) and (v, v_{r+1}) of $A(T), i \leq r \leq j-1$ (suffixes taken modulo p), and hence

$(v_0, v_1, \ldots, v_r, v, v_{r+1}, \ldots, v_{p-1}; v_0)$ is a directed $(p+1)$- cycle containing u (see Fig. 2.5b).

If no such v exists, then for every vertex v of T not belonging to $V(C)$, either $(v_i, v) \in A(T)$ for every $i, 0 \leq i \leq p-1$, or $(v, v_i) \in A(T)$ for every $i, 0 \leq i \leq p-1$. Let $S = \{v \in V(T) \setminus V(C) : (v_i, v) \in A(T) \text{for each} i, 0 \leq i \leq p-1\}$ and $S' = \{w \in V(T) \setminus V(C) : (wv_i) \in A(T) \text{ for each} i; 0 \leq i \leq p-1\}$. The diconnectedness of T implies that none of S, S', and $[S, S']$, is empty. Let (v, w) be an arc of $[S, S']$, Then $(v_0, v, w, v_2, \ldots, v_{p-1}, v_0)$ is a directed $(p+1)$-cycle of T. containing $v_{p-1} = u$ (see Fig. 2.5c). □

Let us Sum Up

1. For a simple graph G, The incidence function I_G is one-to-one.
2. The complete graph K_n has the maximum number of edges among all simple graphs.
3. The totally disconnected graph has no edges at all.
4. For a simple graph G, $0 \leq m(G) \leq \frac{n(n-1)}{2}$.
5. For a simple graph G, we have $(G^c)^c = G$.
6. If $|V(G)| = n$, then $|E(G)| + |E(G^c)| = |E(K_n)| = \frac{n(n-1)}{2}$.
7.If $d = (d_1, d_2, \ldots, d_n)$ is the degree sequence of G, then $\sum_{i=1}^{n} d_i = 2m$, where n and m are the orders and size of G, respectively.
8. A graph is bipartite if and only if it contains no odd cycles.
9. If G is simple and $\delta \leq k$, then G contains a path of length at least k.
10. The automorphism group of K_n is isomorphic to the symmetric group S_n of degree n.
11. The graphs for which the automorphism groups consists of just the identity permutation are called identity graph.

12. The line graph $L(G)$ is always a simple graph.
13. A graph G is connected if and only if $L(G)$ is connected.
14. A digraph is strict if its underlying graph is simple.
15. A digraph is a tournament if its underlying graph is a complete graph.

Check your Progress

1. An edge for which the two ends are the same is called a

   a. loop b. pendent edge c. multiple edge d. parallel edge

2. The order of a graph G is

   a. number of paths in G

   b. number of components in G

   c. b. number of vertices in G

   d. number of edges in G

3. Complete bipartite of the form $K_{1,q}$ is called a

   a. complete graph b. cycle c. path d. star

4. A subgraph H of G is a spanning subgraph of G if

   a. $V(H) < V(G)$ b. $V(H) \leq V(G)$ c. $V(H) > V(G)$ d. $V(H) = V(G)$

5. A clique of G is a ____ of G.

   a. subgraph b. complete subgraph c. induced subgraph d. spanning subgraph

6. A spanning 1-regular subgraph of G is called a ____ of G.

   a. perfect matching b. walk c. cycle d. components

7. A walk is called a trail if

   a. all the vertices are distinct

b. all the edges are distinct

c. origin and terminus are the same

d. the components are distinct

8. If $w \geq 2$, then the graph is

a. connected b. disconnected c. simple d. complete

9. For any simple graph G, $Aut(G) =$

a. $Aut(K_n)$ b. $Aut(K_n^c)$ c. $Aut(G^c)$ d. $L(G)$

10. A graph G is connected if and only if

a. G contains a path b. G contains a cycle c. $L(G)$ is disconnected

d. $L(G)$ is connected

11. The graph $K_1 \vee C_n =$ a. W_n b. K_{n+1} c. K_n d. $K_{1,n}$

12. The notation $N_D^+(u)$ denote the set of

a. inneighbors of u in D

b. neighbors of u in D

c. outneighbors of u in D

d. arcs of D

13. A digraph is disconnected if it has exactly

a. one directed path

b. two directed path

c. one dicomponent

d. two dicomponent

15. A digraph D is a tournament if its underlying graph is

a. simple b. connected c. bipartite d. complete

Answers for Check your Progress

(1) a (2) c (3) d (4) d (5) b (6) a (7) b (8) b (9) c (10) d
(11) a (12) c (13) c (14) b (15) d

Exercises

1. Show that Herschel graph is bipartite.
2. Show that $K_{m,n}, m \neq n$ has no spanning cycle.

References

1. R. Balakrishnan and K. Ranganathan, A Text Book of Graph Theory, second ed., Springer, New York, 2012.
2. J.A. Bondy and U.S.R. Murty, Graph Theory with Application.

Suggested Readings

1. S. Arumugam Issac, introduction to Graph Theory.

♣♣♣♣♣

Unit 2

Connectivity and Trees

Objectives

1. To learn vertex cuts and edge cuts
2. To understand the significance of connectivity and edge connectivity.
3. To discuss the properties of tress and counting the number of spanning trees.
4. To apply the concept of trees in everyday life problems.
5. To introduce algorithms to find minimum-weight spanning trees.

2.1 Introduction:

The connectivity of a graph is a "measure "of its connectedness. Some connected graphs are connected rather "loosely "in the sense that the deletion of a vertex or an edge from the graph destroys the connectedness of the graph. There are graphs at the other extreme as well, such as the complete graphs K_n, $n \geq 2$, which remain connected after the removal of any k vertices, $1 \leq k \leq n-1$.

In this chapter, we study the two graph parameters, namely, vertex connectivity and edge connectivity.

Definition 56.

1. *A subset V' of the vertex set $V(G)$ of a connected graph G is a vertex cut of G if $G - V'$ is disconnected; it is a k-vertex cut if $|V'| = k.V'$ is then called a separating set of vertices of G. A vertex v of G is a cut vertex of G if $\{v\}$ is a vertex cut of G.*

2. *Let G be a nontrivial connected graph with vertex set $V(G)$ and let S be a nonempty subset of $V(G)$. For $\bar{S} = V \setminus S \neq \emptyset$, let $[S, \bar{S}]$ denote the set of all edges of G that have one end vertex in S and the other in $\bar{S}$. A set of edges of G of the form $[S, \bar{S}]$, is called an edge cut of G. An edge e is a cut edge of G if $\{e\}$ is an edge cut of G. An edge cut of cardinality k is called a k-edge cut of G.*

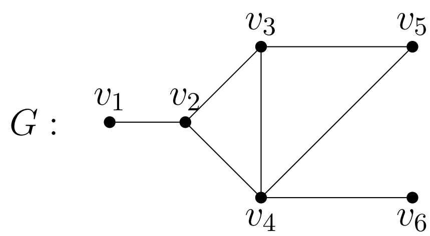


Figure 2.1: Graph illustrating vertex cuts and edge cuts

Example 57. *For the graph of Fig. 2.1, $\{v_2\}$ and $\{v_3, v_4\}$, are vertex cuts. The edge subsets $\{v_3v_5, v_4v_5, v_1v_2\}$, and $\{v_4v_6\}$ are all edge cuts. Of these, v_2 is a cut vertex,and v_1v_2 and v_4v_6 are both cut edges. For the edge cut $\{v_3v_5, v_4v_5\}$, we may take $S = \{v_5\}$ so that $\bar{S} = \{v_1, v_2, v_3, v_4, v_6\}$.*

Theorem 58. *An edge $e = xy$ of a connected graph G is a cut edge of G if and only if e belongs to no cycle of G.*

Proof. Let e be a cut edge of G and let $[S, \bar{S}] = e$ be the partition of V defined by $G - e$ so that one of x and y belongs to S, and the other to $\bar{S}$, say, $x \in S$ and $y \in \bar{S}$. If e belongs to a cycle of G, then $[S, \bar{S}]$ must contain at least one more edge, contradicting that $e = [S, \bar{S}]$. Hence, e cannot belong to a cycle.
Conversely, assume that e is not a cut edge of G. Then $G - e$ is connected, and hence there exists an $x - y$ path P in $G - e$. Then $P \cup \{e\}$ is a cycle in G containing e. □

Theorem 59. *An edge $e = xy$ is a cut edge of a connected graph G if and only if there exist vertices u and v such that e belongs to every $u - v$ path in G.*

Proof. Let $e = xy$ be a cut edge of G. Then $G - e$ has two components, say, G_1 and G_2. Let $u \in V(G_1)$ and $v \in V(G_2)$. Then, clearly, every $u - v$ path in G contains e.
Conversely, suppose that there exist vertices u and v satisfying the condition of the theorem. Then there exists no $u - v$ path in $G - e$ so that $G - e$ is disconnected. Hence, e is a cut edge of G. □

Theorem 60. *A connected graph G with at least two vertices contains at least two vertices that are not cut vertices.*

Proof. First, suppose that $n(G) \geq 3$. Let u and v be vertices of G such that $d(u, v)$ is maximum. Then neither u nor v is a cut vertex of G. For if u were a cut vertex of G, $G - u$ would be disconnected, having at least two components. The vertex v belongs to one of these components. Let w be any vertex belonging to a component of $G - u$ not containing v. Then every $v - w$ path in G must contain u (see Fig. 2.2). Consequently, $d(v, w) > d(v, u)$, contradicting the choice of u and v. Hence, u is not a

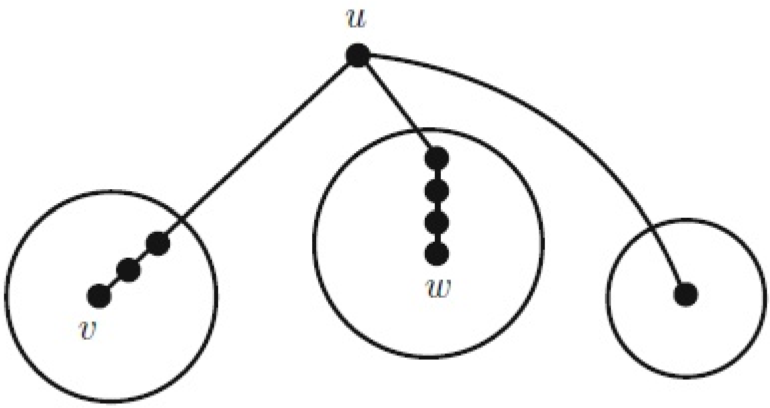


Figure 2.2: Graph for proof of Theorem 60

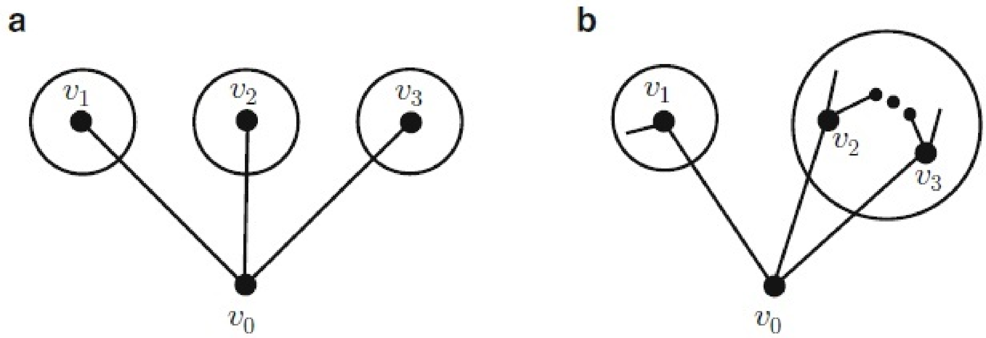


Figure 2.3: Graph for proof of Proposition 61

cut vertex of G. Similarly, v is not a cut vertex of G.

If $n(G) = 2$, then K_2 is a spanning subgraph of G, and so no vertex of G is a cut vertex of G. This completes the proof of the theorem. □

Proposition 61. *A simple cubic (i.e., 3-regular) connected graph G has a cut vertex if and only if it has a cut edge.*

Proof. Let G have a cut vertex v_0. Let v_1, v_2, v_3 be the vertices of G that are adjacent to v_0 in G. Consider $G - v_0$, which has either two or three components. If $G - v_0$ has three components, no two of v_1, v_2, and v_3 can belong to the same component of $G - v_0$. In this case, each of v_0v_1, v_0v_2, and v_0v_3 is a cut edge of G. (See Fig. 2.3a.) In the case when $G - v_0$ has only two components, one of the vertices, say v_1, belongs to

one component of $G - v_0$, and v_2 and v_3 belong to the other component. In this case, v_0v_1 is a cut edge. (See Fig. 2.3b.)
Conversely, suppose that $e = uv$ is a cut edge of G. Then the deletion of u results in the deletion of the edge uv. Since G is cubic, $G - u$ is disconnected. Accordingly, u is a cut vertex of G. □

Let us Sum Up:

In this section, we have studied definition and some important proposition of connectivity and edge-connectivity of graphs.
1. Note that an r - connected graph need not be $(r+1)$- connected where us it is $(r-1)$- connected.
2. We say that the graph G is r-connected if the removal of $(r-1)$ vertices does not disconnected. If does not removal of r vertices also connects the graph G. Similarly concepts holds for edge connectivity also.

Check your progress:

1. The graph $K_n, n \geq 3$ is .....
(a) r connected (b) $n-1$ connected (c) $r(\leq n-1)$ connected (d) n connected
2. The graph $K_{n,n}, n \geq 2$ is .....
(a) n-edge connected (b) $r(\leq n)$-edge connected
(d) r- connected (d) $(n-1)$-edge connected.
Answer: 1. (c) 2. (b)

2.2 Connectivity and Edge Connectivity

Definition 62. *For a nontrivial connected graph G having a pair of non-adjacent vertices, the minimum k for which there exists a k-vertex cut is called the vertex connectivity of simply the connectivity of G; it is denoted by $\kappa(G)$ or simply κ (kappa) when G is understood. If G is trivial or disconnected, $\kappa(G)$ is taken to be zero, whereas if G contains K_n as a spanning subgraph, $\kappa(G)$ is taken to be $n-1$.*

A set of vertices and/or edges of a connected graph G is said to be disconnect G if its deletion results in a disconnected graph.

Definition 63. *The edge connectivity of a connected graph G is the smallest k for which there exists a k-edge (i.e., an edge cut having k edges). The edge connectivity of a trial or disconnected graph is taken to be 0. The edge connectivity of G is denoted by $\lambda(G)$. If λ is the edge connectivity of a connected graph G, there exists a set of λ edges whose deletion results in a disconnected graph, and so subset of edges of G of size less than λ has this property.*

Definition 64. *A graph G is r-connected if $\kappa(G) \geq r$. Also, G is r-edge connected if $\lambda(G) \geq r$.*

For the graph G of Fig. 2.4, $\kappa(G) = 1$ and $\lambda(G) = 2$.

Theorem 65. *For any loopless connected graph G, $k(G) \leq \lambda(G) \leq \delta(G)$.*

Proof. We observe that $k = 0$ if and only if $\lambda = 0$. Also, $\delta = 0$ implies that $D = 0$ and $\lambda = 0$. Hence we may assume that k, λ, and δ are all at least 1. Let E be an edge cut of G with λ edges. Let u and v be the end vertices of an edge of E. For each edge of E that does not have both u

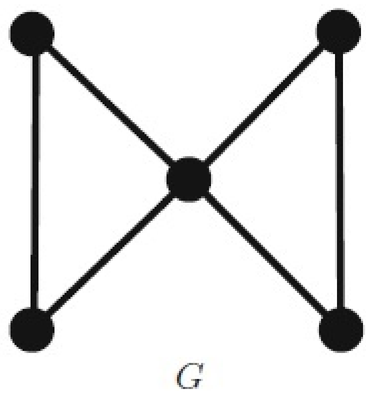


Figure 2.4: A 1-connected graph

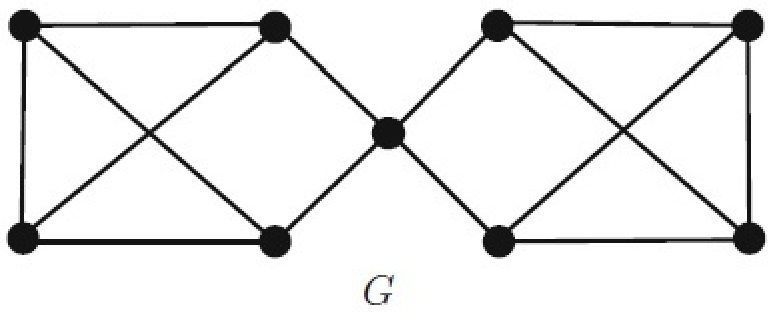


Figure 2.5: Graph G with $\kappa = 1$, $\lambda = 2$ and $\delta = 3$

and v as end vertices, remove an end vertex that is different from u and v. If there are t such edges, at most t vertices have been removed. If the resulting graph, say H, is disconnected, then $k \leq t < \lambda$. Otherwise, there will remain a subset of edges of E having u and v as end vertices, the removal of which from H would disconnect G. Hence, in addition to the already removed vertices, the removal of one of u and v will result in either a disconnected graph or a trivial graph. In the process, a set of at most $t + 1$ vertices has been removed and $k \leq t + 1 \leq \lambda$.

Finally, it is clear that $\lambda \leq \delta$. In fact, if v is a vertex of G with $d_G(v) = \delta$. then the set $[\{v\}, V \setminus \{v\}]$ of δ edges of G incident at v forms an edge cut of G. Thus, $\lambda \leq \delta$. $\square$

Note: It is possible that the inequlities in Theorem 65 can be strict. See the graph G of Fig. 2.5, for which $\kappa = 1,\ \lambda = 2$, and $\delta = 3$.

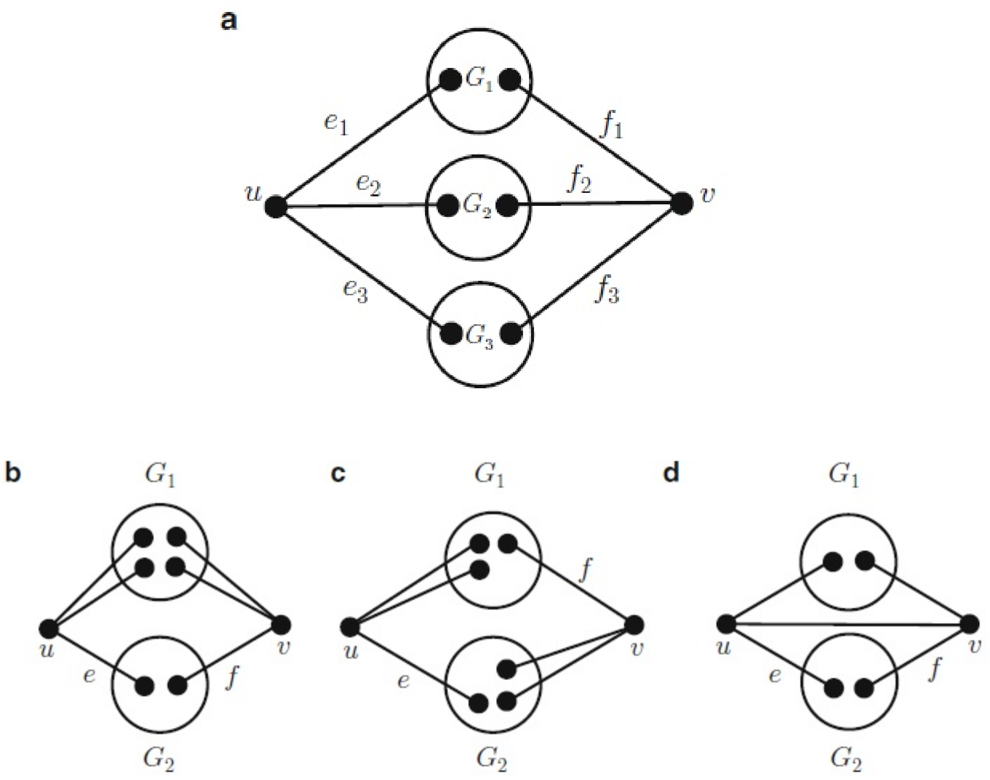


Figure 2.6: Connected cubic graph for proof of Theorem 66

Theorem 66. *The connectivity and edge connectivity of a simple cubic graph G are equal.*

Proof. We need only consider the case of a connected cubic graph. Again, since $\kappa \leq \lambda \leq \delta = 3$, we have only to consider the cases when $\kappa = 1,\ 2,$ or 3. Now, Proposition 61 implies that for a simple cubic graph G, $\kappa = 1$ if and only if $\lambda = 1$.

If $\kappa = 3$, then by Theorem 65, $3 = \kappa \leq \lambda \leq \delta = 3$, and hence $\lambda = 3$.

We shall now prove that $\kappa = 2$ implies that $\lambda = 2$.

Suppose $\kappa = 2$ and $\{u, v\}$ is a 2-vertex cut of G. The deletion of $\{u, v\}$ results in a disconnected subgraph G' of G. Since each of u and

v must be joined to each component of G', and since G is cubic, G' can have at most three components. If G' has three components, $G_1,\ G_2$, and G_3, and if e_i and $f_i,\ i = 1,\ 2,\ 3$, join, respectively, u and v with G_i, then each pair $\{e_i,\ f_i\}$ is an edge cut of G (see Fig. 2.6a).

If G' has only two components, G_1 and G_2, then each of u and v is joined to one of G_1 and G_2 by a single edge, say, e and f, respectively, so that $\{e, f\}$ is an edge cut G (see Fig. 2.6b-d).

Hence, in either case there exists an edge cut consisting of two edges. As such, $\lambda \geq 2$. But by Theorem 65, $\lambda \geq \kappa = 2$. Hence $\lambda = 2$. Finally, the above arguments show that if $\lambda = 3$, then $\kappa = 3$, and if $\lambda = 2$, then $\kappa = 2$. □

Definition 67. *A family of two or more paths in a graph G is said to be internally disjoint if no vertex of G is an internal vertex of more than one path in the family.*

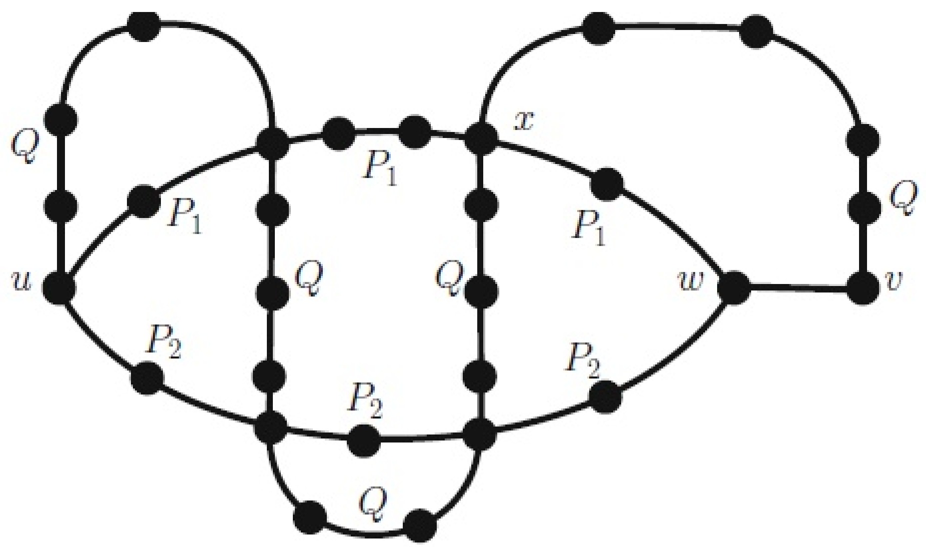


Figure 2.7: Graph for proof of Theorem 68

Theorem 68. *(Whitney [?]). A graph G with at least three vertices is* 2-*connected if and only if any two vertices of G are connected by at least* *two internally disjoint paths.*

Proof. Let G be 2-connected. Then G contains no cut vertex. Let u and v be two distinct vertices of G. We now use induction on $d(u, v)$ to prove that u and v are joined by two internally disjoint paths.

If $d(u, v) = 1$, let $e = uv$. As G is 2-connected and $n(G) \geq 3$, e cannot be a cut edge of G, since if e were a cut edge, at least one of u and v must be a cut vertex. By Theorem 58, e belongs to a cycle C in G. Then $C - e$ is a $u - v$ path in G, internally disjoint from the path uv.

Now assume that any two vertices x and y of G with $d(x, y) = k - 1,\ k \geq 2$, are joined by two internally disjoint $x - y$ paths in G. Let $d(u, v) = k$. Let P be a $u - v$ path of length k and w be the vertex of G just preceding v on P. Then $d(u, w) = k - 1$. By an induction hypothesis, there are two internally disjoint $u - w$ paths, say P_1 and P_2, in G. As G has no cut vertex, $G - w$ is connected and hence there exists a $u - v$ path Q in $G - w$. Q is clearly a $u - v$ path in G not containing w. Let x be the vertex of Q such that the $x - v$ section of Q contains only the vertex x in common with $P_1 \cup P_2$ (see Fig. 2.7).

We may suppose, without loss of generality, that x belongs to P_1. Then the union of the $u - x$ section of P_1 and $x - v$ section of Q and $P_2 \cup (wv)$ are two internally disjoint $u - v$ paths in G. This gives the proof in one direction.

In the other direction, assume that any two distinct vertices of G are connected by at least two internally disjoint paths. Then G is connected. Further, G cannot contain a cut vertex, since if v were a cut vertex of G, there must exist vertices u and w such that every $u - w$ path contains v, contradicting the hypothesis. Hence, G is 2-connected. □

Theorem 69. *A graph G with at least three vertices is 2-connected if and* *only if any two vertices of G lie on a common cycle.*

Proof. Let u and v be any two vertices of a 2-connected graph G. By Theorem 68, there exist two internally disjoint paths in G joining u and v. The union of these two paths is a cycle containing u and v.

Conversely, if any two vertices u and v lie on a cycle C, then C is the union of two internally disjoint $u - v$ paths. Again, by Theorem 68, G is 2-connected. □

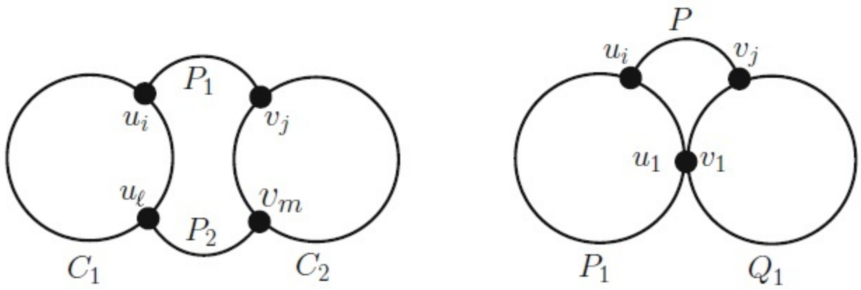


Figure 2.8: Graphs for proof of Theorem 70

Theorem 70. *In a* 2*-connected graph G, any two longest cycles have at least two vertices in common.*

Proof. Let $C_1 = u_1u_2 \cdots u_ku_1$ and $C_2 = v_1v_2 \cdots v_kv_1$ be two longest cycles in G. If C_1 and C_2 are disjoint, there exist (since G is 2-connected) two disjoint paths, say P_1 joining u_i and v_j and P_2 joining u_l and v_p, connecting C_1 and C_2 such that $u_i \neq u_l$ and $V_j \neq v_p$. u_i and u_l divide C_1 into two subpaths. Let L_1 be the longer of these subpaths. (If both subpaths are of equal length, we take either one of them to be L_1.) Let L_2 be defined in a similar manner in C_2. Then $L_1 \cup P_1 \cup L_2 \cup P_2$ is a cycle of length greater than that of C_1 (or C_2). Hence, C_1 and C_2 cannot be disjoint. (see Fig. 2.8).

Suppose that C_1 and C_2 have exactly one vertex, say $u_1 = v_1$, in common. Since G is 2-connected, u_1 is not a cut vertex of G, and so there exists a path P with one end vertex u_i in $C_1 - u_1$ and the other end vertex v_j in $C_1 - v_1$, which is internally disjoint from $C_1 \cup C_2$. Let P_1 denote the longer of the two $u_1 - u_i$ sections of C_1, and Q_1 denote the longer of the two $v_1 - v_j$ sections of C_2. If the two sections of C_1 or of C_2 are of equal length, take any one of then. Then $P_1 \cup P \cup Q_1$ is a cycle longer than C_1 (or C_2). But this is impossible. Thus, C_1 and C_2 must have at least two vertices in common. □

Theorem 71 gives a simple characterization of 3-edge connected graphs.

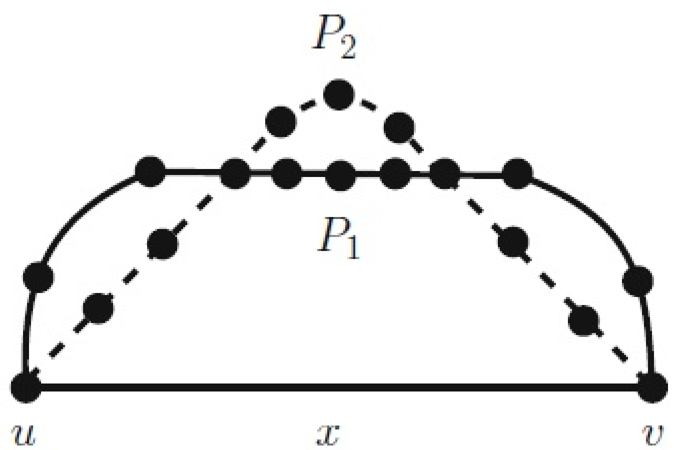


Figure 2.9: Graphs for proof of Theorem 71

Theorem 71. *A connected simple graph G is 3-edge connected if and only if every edge of G is the (exact) intersection of the edge sets of two cycles of G.*

Proof. Let G be 3-edge connected and let $x = uv$ be an edge of G. Since $G - x$ is 2-edge connected, there exist two edge-disjoint $u - v$ paths P_1 and P_2 in $G - x$. Now, $P_1 \cup \{x\}$ and $P_2 \cup \{x\}$ are two cycles of G, the intersection of whose edge sets is precisely $\{x\}$ (see Fig. 2.9).

Conversely, suppose that for each edge $x = uv$ there exit two cycles C and C' such that $\{x\} = E(C) \cap E(C')$. G cannot have a cut edge since, by hypothesis, each edge belongs to two cycles and no cut edge can belong to a cycle; nor can G contain an edge cut consisting of two edges x and y. (Since any cycles that contains x also contains y, the intersection of any two such cycles must contain both x and y, a contradiction.) Hence, $\lambda(G) \geq 3$, and G is 3-edge connected. □

2.3 Cyclical Edge Connectivity of a Graph

Definition 72. *Let G be a simple connected graph containing at least two disjoint cycles. Then the cyclical edge connectivity of G is defined to be the minimum number of edges of G whose deletion results in a graph having two components, each containing a cycle. It is denoted by $\lambda_c(G)$.*

It is clear that $\lambda \leq \lambda_c$. The graphs G and H of Fig. **??** *show that both $\lambda = \lambda_c$ and $\lambda < \lambda_c$ can happen.*

2.4 Definition, Characterization, and Simple Properties of Trees

A connected graph without cycles is defined as a *tree*. A graph without cycles is called an *acyclic graph or a forest.* So each component of a forest is a tree. Figure 2.10 displays two trees.

Theorem 73. *A simple graph is a tree if and only if any two distinct vertices are connected by a unique path.*

Proof. Let T be a tree. Suppose that two distinct vertices u and v are connected by two distinct $u - v$ paths. Then their union contains a cycle in T, contradicting that T is a tree.

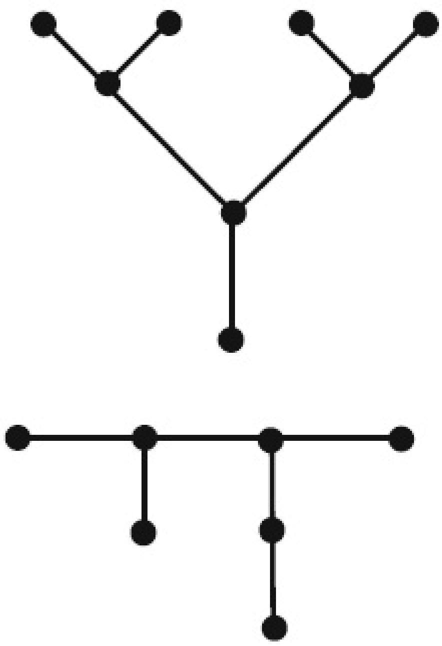


Figure 2.10: Examples of trees

Conversely, suppose that any two vertices of a graph G are connected by a unique path. Then G is obviously connected. Also, G cannot contain a cycle, since any two distinct vertices of a cycle are connected by two distinct paths. Hence G is a tree □

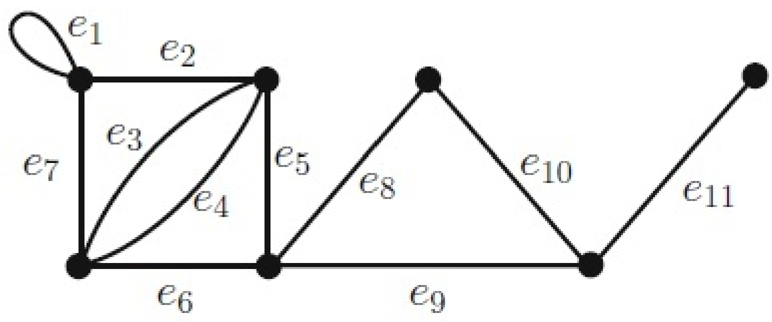
Definition 74. *A spanning subgraph of a graph G, which is also a tree, is called a spanning tree of G. A connected graph G and two of its spanning trees T_1 and T_2 are shown in Fig. 2.11.*

A loop cannot be an edge of any spanning tree, since such a loop constitutes a cycle (of length 1). On the other hand, a cut edge of G must be an edge of every spanning tree of G.

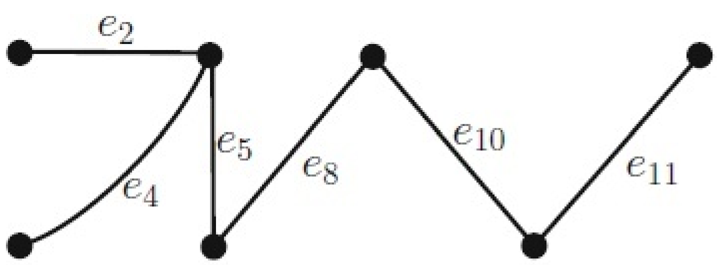
Theorem 75. *Every connected graph contains a spanning tree.*

Proof. Let G be a connected graph. Let $\mathcal{C}$ be the collection of all connected spanning subgraphs of G. $\mathcal{C}$ is nonempty as $G \in \mathcal{C}$. Let $T \in \mathcal{C}$

have the fewest number of edges. Then T must be a spanning tree of G. If not, T would contain a cycle of G, and the deletion of any edge of this cycle would give a (spanning) subgraph in $\mathcal{C}$ having one edge less than that of T. This contradicts the choice of T . Hence, T has no cycles and is therefore a spanning tree of G. $\square$



G

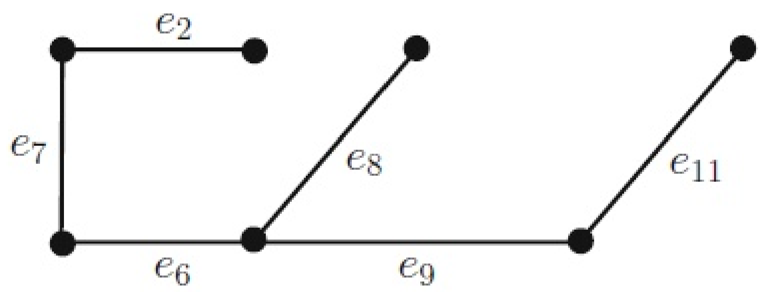


T_1

Theorem 76. *The number of edges in a tree on n vertices is $n - 1$. Conversely, a connected graph on n vertices and $n - 1$ edges is a tree.*

Proof. Let T be a tree.We use induction on n to prove that $m = n - 1$. When $n = 1$ or $n = 2$, the result is straightforward.

Now assume that the result is true for all trees on $n(n - 1)$ or fewer vertices, $n \geq 3$. Let T be a tree with n vertices. Let $e = uv$ be an edge of T. Then uv is the unique path in T joining u and v. Hence the deletion



T_2

Figure 2.11: Graph G and two of its spanning trees T_1 and T_2.

of e from T results in a disconnected graph having two components T_1 and T_2. Being connected subgraphs of a tree, T_1 and T_2 are themselves trees. As $n(T_1)$ and $n(T_2)$ are less than $n(T)$, by an induction hypothesis, $m(T_1) = n(T_1) - 1$ and $m(T_2) = n(T_2) - 1$. Therefore, $m(T) = m(T_1) + m(T_2) + 1 = n(T_1) - 1 + n(T_2) - 1 + 1 = n(T_1) + n(T_2) - 1 = n(T) - 1$. Hence, the result is true for T . By induction, the result follows in one direction.

Conversely, let G be a connected graph with n vertices and $n-1$ edges. By Theorem 75, there exists a spanning tree T of G. T has n vertices and being a tree has $(n-1)$ edges. Hence $G = T$, and G is a tree. □

Theorem 77. *A connected graph G is a tree if and only if every edge of G is a cut edge of G.*

Proof. If G is a tree, there are no cycles in G. Hence, no edge of G can belong to a cycle. By Theorem 58, each edge of G is a cut edge of G. Conversely, if every edge of a connected graph G is a cut edge of G, then G cannot contain a cycle, since no edge of a cycle is a cut edge of G. Hence, G is a tree. □

Theorem 78. *Prove that for a simple connected graph G, $L(G)$ is isomorphic to G if and only if G is a cycle.*

Proof. If G is a cycle, then clearly $L(G)$ is isomorphic to G. Conversely, let $G \simeq L(G)$. Then $n(G) = n(L(G))$, and $m(G) = m(L(G))$. But since $n(L(G)) = m(G)$, we have $m(G) = n(G)$. We know that G is unicyclic. Let $C = v_1v_2 \cdots v_kv_1$ be the unique cycle in G. If $G \neq C$, there must be an edge $e \notin E(C)$ incident with some vertex v_i of C (as G is connected). Thus, there is a star with at least three edges at v_i. This star induces a clique of size at least 3 in $L(G)(\simeq G)$. This shows that there exists at least one more cycle in $L(G)$ distinct from the cycle corresponding to C in G. This contradicts the fact that $L(G) \simeq G$ (as G is unicyclic). □

Let us Sum Up:

We studied definition of different types of trees and its interesting properties.

1. Note that disconnected graphs does not contain tree but contains factors.

Check your Progress:

1. Simple graph on n vertices and m edges with ω components have ...... cyclic.

(a) $m - n + \omega$ (b) $n - \omega$ (c) $m - \omega$ (d) $n - m - \omega$

Answer: 1. (a)

2.5 Centers and Centroids

Definition 79. *Let G be a connected graph.*

1. *The diameter of G is defined as* $max\{d(u, v) : u, v \in V(G)\}$ *and is denoted by* $diam(G)$.

2. *If* v *is a vertex of G, its eccentricity* $e(v)$ *is defined by* $e(v) = max\{d(v, u) : u \in V(G)\}$.

3. *The radius* $r(G)$ *of* G *is the minimum eccentricity of G; that is,* $r(G) = min\{e(v) : v \in V(G)\}$. *Note that* $diam(G) = max\{e(v) : v \in V(G)\}$.

4. *A vertex* v *of G is called a central vertex if* $e(v) = r(G)$. *The set of central vertices of G is called the center of G.*

Example 80. *Figure 2.12 displays two graphs* T *and* G *with the eccentricities of their vertices. We find that* $r(T) = 4$ *and* $diam(T) = 7$. *Each of* u *and* v *is a central vertex of* T. *Also,* $r(G) = 3$ *and* $diam(G) = 4$. *Further, G has five central vertices.*

Remark 81. *It is obvious that* $r(G) \leq diam(G)$. *For a complete graph,* $r(G) = diam(G) = 1$. *For a complete bipartite graph* $G(X, Y)$ *with* $|X| \geq 2$ *and* $|Y| \geq 2$, $r(G) = diam(G) = 2$. *For the graphs of Fig. 2.12,* $r(G) < diam(G)$.

Theorem 82. *(Jordan [**?**]) Every tree has a center consisting of either a single vertex or two adjacent vertices.*

Proof. The result is obvious for the trees K_1 and K_2. The vertices of K_1 and K_2 are central vertices. Now let T be a tree with $n(T) \geq 3$. Then T has at least two pendant vertices. Clearly, the pendant vertices of T cannot be central vertices. Delete all the pendant vertices from T. This results in a subtree T' of T. As any maximum-distance path in T from any vertex of T' ends at a pendant vertex of T, the eccentricity of each vertex of T' is

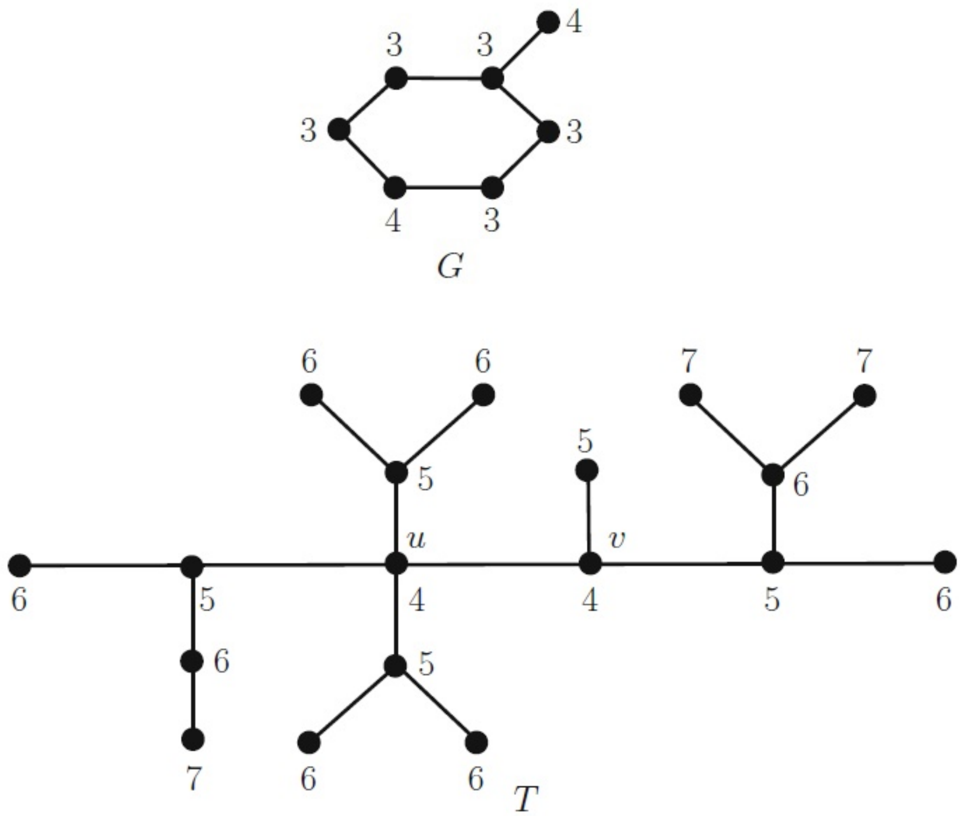


Figure 2.12: Eccentricities of vertices for graphs G and T

one less than the eccentricity of the same vertex in T. Hence the vertices of minimum eccentricity of T' are the same as those of T. In other words, T and T' have the same center. Now, if T'' is the tree obtained from T' by deleting all the pendant vertices of T', then T'' and T' have the same center. Hence the centers of T'' and T are the same. Repeat the process of deleting the pendant vertices in the successive subtrees of T until there results a K_1 or K_2. This will always be the case as T is finite. Hence the center of T is either a single vertex or a pair of adjacent vertices. □

Definition 83. *1. A branch at a vertex u of a tree T is a maximal subtree containing u as an end vertex. Hence the number of branches at u is $d(u)$.*

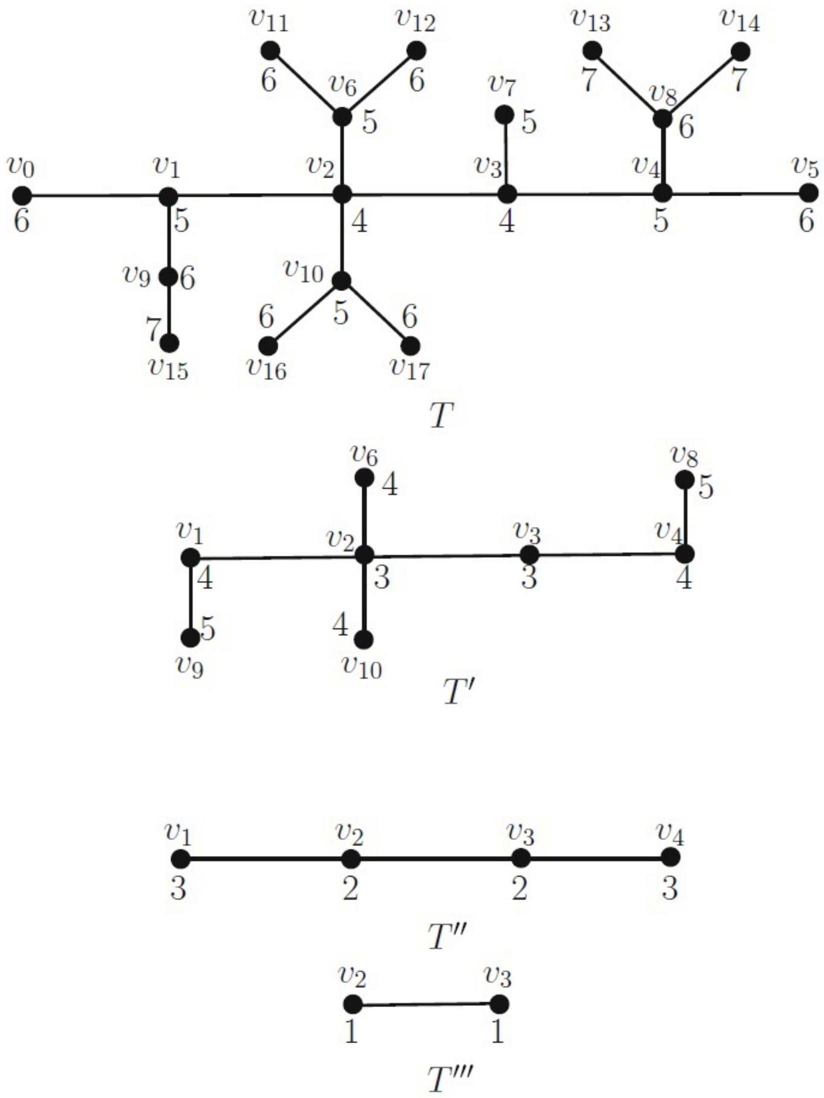


Figure 2.13: Determining the center of tree T

For instance, in Fig. 2.14, there are three branches of the tree at u.

2. *The weight of a vertex u of T is the maximum number of edges in any branch at u.*

3. *A vertex v is a centroid vertex of T if v has minimum weight. The set of all centroid vertices is called the centroid of T .*

In Fig. 2.15 the numbers in the parentheses indicate the weights of the

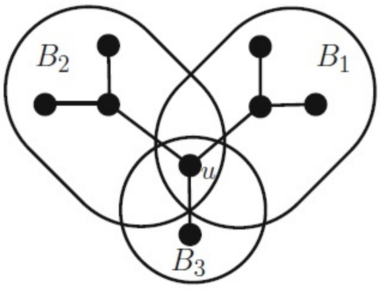


Figure 2.14: Tree showing three branches at u

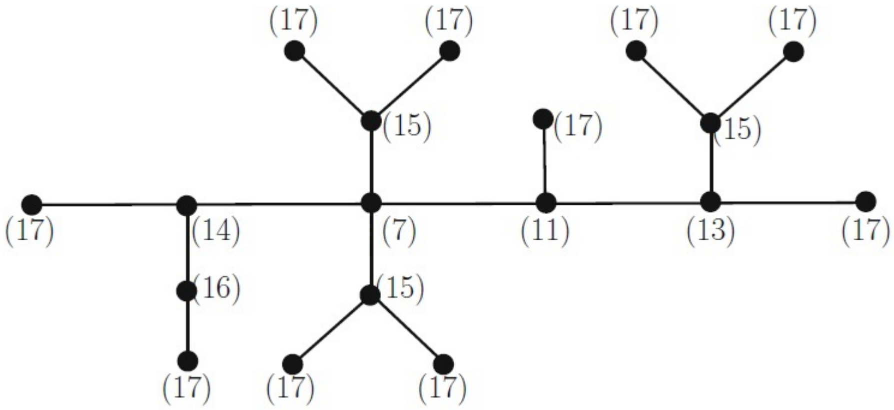


Figure 2.15: Weights of vertices of a tree

corresponding vertices. It is clear that all the end vertices of T *have the same weight, namely,* $m(T)$.

Let us Sum Up:

1 Note that the end vertices have largest weight.

2. Radius and diameter of complete graphs and complete bipartite graphs are equal.

Check your Progress:

1. Radius and diameter of the Peterson graph are .....

(a) $(3, 3)$ (b) $(2, 3)$ (c) $(2, 2)$ (d) $(1, 2)$

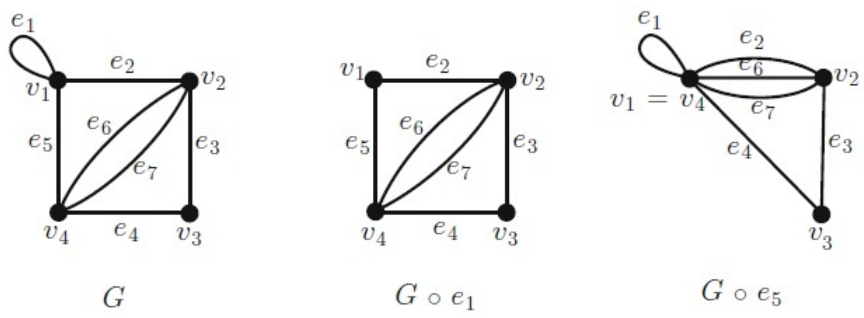


Figure 2.16: Edge contraction

2.6 Counting the Number of Spanning Tress

The number of spanning trees of a connected labeled graph G will be denoted by $\tau(G)$. If G is disconnected, we take $\tau(G) = 0$.

Definition 84. *An edge e of a graph G is said to be contracted if it is deleted from G and its ends are identified. The resulting graph is denoted by $G \circ e$.*

Edge contraction is illustrated in Fig. 2.16.

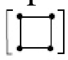
If e is not a loop of G, then $n(G \circ e) = n(G) - 1$, $m(G \circ e) = m(G) - 1$, and $w(G \circ e) = w(G)$. For a loop e, $n(G \circ e) = n(G)$, $m(G \circ e) = m(G) - 1$, and $w(G \circ e) = w(G)$.

Theorem 85. *If e is not a loop of a connected graph G, $\tau(G) = \tau(G - e) + \tau(G \circ e)$.*

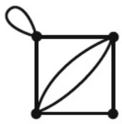
Proof. $\tau(G)$ is the sum of the number of spanning trees of G containing e and the number of spanning trees of G not containing e.

Since $V(G-e) = V(G)$, every spanning tree of $G-e$ is a spanning tree of G not containing e, and conversely, any spanning tree of G for which e is not an edge is also a spanning tree of $G - e$. Hence the number of spanning trees of G not containing e is precisely the number of spanning trees of $G - e$, that is, $\tau(G - e)$. If T is a spanning tree of G containing e, the contraction of e in both T and G results in a spanning tree $T \circ e$ of $G \circ e$.

Conversely, if T_0 is a spanning tree of $G \circ e$, there exists a unique spanning tree T of G containing e such that $T \circ e = T_0$. Thus, the number of spanning trees of G containing e is $\tau(G \circ e)$. Hence $\tau(G) = \tau(G - e) + \tau(G \circ e)$. □

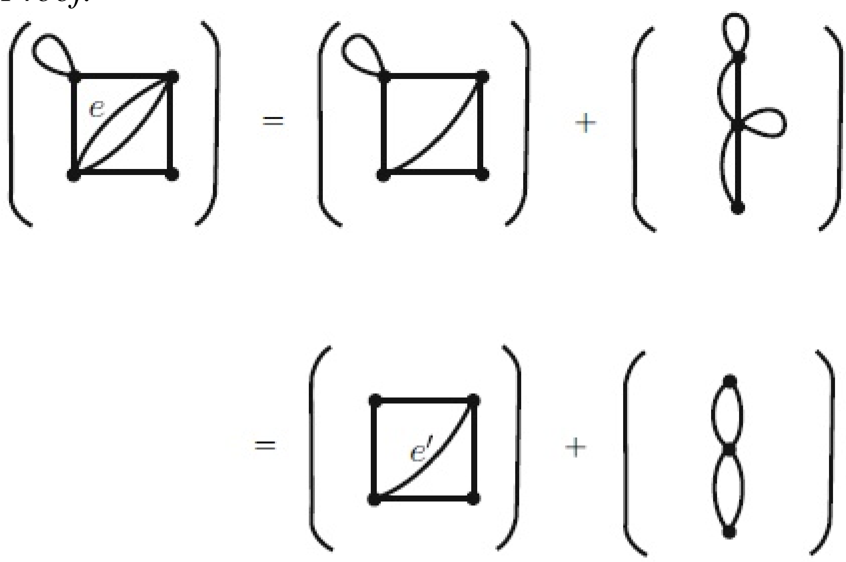
We illustrate below the use of Theorem 85 in calculating the number of spanning trees. In this illustration, each graph within parentheses stands for the number of its spanning trees. For example,  stands for the number of spanning trees of C_4.

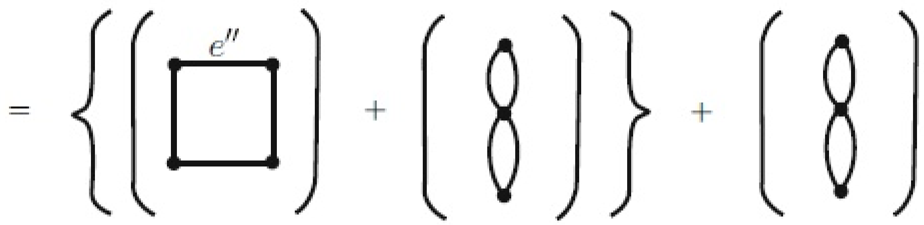
Example 86. *Find $\tau(G)$ for the following graph G:*

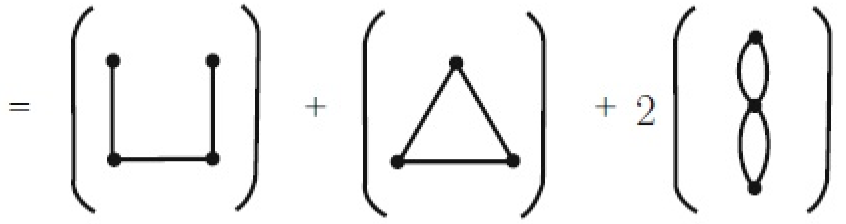


G

Proof.







$= 1 + 3 + 2(4)$

$= 12.$

Theorem 87. *A simple connected graph G contains k pairwise edge-disjoint spanning trees if and only if for each partition $\mathcal{P}$ of $V(G)$ into p parts, the number $m(\mathcal{P})$ of edges of G joining distinct parts is at least $k(p-1)$, $2 \le p \le |V(G)|$.*

Proof. We prove only the easier part of the theorem (necessity of the condition). Suppose G has k pairwise edge-disjoint spanning trees. If T is one of them and if $\mathcal{P} = \{V_1, V_2, \cdots V_p\}$ is a partition of $V(G)$ into p parts, then G must have at least $|\mathcal{P}| - 1$ edges of T . As this is true for each of the k pairwise edge-disjoint trees of G, the number of edges joining distinct parts of $\mathcal{P}$ is at least $k(p-1)$. □

2.7 Cayley's Formula

Cayley was the first mathematician to obtain a formula for the number of spanning trees of a labeled complete graph.

Before we prove Theorem 90, we establish two lemmas.

Lemma 88. *Let* $(d_1, \cdots, d_n)$ *be a sequence of positive integers with* $\sum_{i=1}^{n} d_i = 2(n-1)$. *Then there exists a tree* T *with vertex set* $\{v_1, v_2, \cdots, v_n\}$ *and* $d(v_i) = d_i$, $1 \leq i \leq n$.

Proof. It is easy to prove the result by induction on n. □

Lemma 89. *Let* $\{v_1, \cdots, v_n\}$, $n \geq 2$ *be given and let* $\{d_1, \cdots .d_n\}$ *be a sequence of positive integers such that* $\sum_{i=1}^{n} d_i = 2(n-1)$. *Then the number of tress with* $\{v_1, \cdots, v_n\}$ *as the vertex set in which* v_i *has degree* d_i, $1 \leq i \leq n$, *is* $\frac{(n-2)!}{(d_1-1)!\cdots(d_n-1)!}$.

Theorem 90. *(Cayley **??**)* $\tau(K_n) = n^{n-2}$, *where* K_n *is a labeled complete graph on* n *vertices,* $n \geq 2$.

Proof. The total number of tress T_n with vertex set $\{v_1, \cdots, v_n\}$ is obtained by summing over all possible sequences $(d_1, \cdots, d_n)$ with $\sum_{i=1}^{n} = 2n-2$. Hence,

$$\begin{aligned} \tau(K_n) &= \sum_{d_i \geq 1} \frac{(n-2)!}{(d_1-1)!\cdots(d_n-1)!} \text{ with} \sum_{i=1} d_i = 2n-2 \\ &= \sum_{k_i \geq 0} \frac{(n-2)!}{k_1!\cdots k_n!} \text{ with } \sum_{i=1}^{n} k_i = n-2, \\ &\qquad \text{where } k_i = d_i - 1,\ 1 \leq i \leq n. \end{aligned}$$

Putting $x_1 = x_2 = \cdots = x_n = 1$ and $m = n-2$ in the multinomial expansion $(x_1+x_2+\cdots+x_n)^m = \sum\limits_{k_i \geq 0} \frac{x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}}{k_1!k_2!\cdots k_n!} m!$ with $(k_1+k_2+\cdots+$

$k_n) = m$, we get $n^{n-2} = \sum\limits_{k_i \geq 0} \frac{x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}}{k_1! k_2! \cdots k_n!}$ with $(k_1 + k_2 + \cdots + k_n) = n - 2$. Thus, $\tau(K_n) = n^{n-2}$. □

2.8 The Connector Problem

Problem 91. *Various cities in a country are to be linked via roads. Given the various possibilities of connecting the cities and the costs involved, what is the most economical way of laying roads so that in the resulting road network, any two cities are connected by a chain of roads? Similar problems involve designing railroad networks and water-line transports.*

Problem 92. *layout for a housing settlement in a city is to be prepared. Various locations of the settlement are to be linked by roads. Given the various possibilities of linking the locations and their costs, what is the minimum-cost layout so that any two locations are connected by a chain of roads?*

Problem 93. *layout for the electrical wiring of a building is to be prepared. Given the costs of the various possibilities, what is the minimum-cost layout?*

These three problems are particular cases of a graph-theoretical problem known as the connector problem.

Definition 94. *Let G be a graph. To each edge e of G, we associate a nonnegative number $w(e)$ called its weight. The resulting graph is a weighted graph. If H is a subgraph of G, the sum of the weights of the edges of H is called the weight of H. In particular, the sum of the weights of the edges of a path is called the weight of the path.*

2.9 Kruskal's Algorithm

Let G be a simple connected weighted graph with edge set $E = \{e_1, e_2, \cdots, e_m\}$. The three steps of the algorithm are as follows:

Step 1: Choose an edge e_1 with its weight $w(e_1)$ as small as possible.

Step 2: If the edges $e_1, e_2, \cdots, e_i$, $i \geq 1$, have already been chosen, choose e_{i+1} from the set $E \setminus \{e_1, e_2, \cdots, e_i\}$ such that

(i) The subgraph induced by the edge set $\{e_1, e_2, \cdots, e_{i+1}\}$ is acyclic, and

(ii) $w(e_i + 1)$ is as small as possible subject to (i).

Step 3: Stop when step 2 cannot be implemented further.

We now show that Kruskal's algorithm does indeed produce a minimum-weight spanning tree.

Theorem 95. *Any spanning tree produced by Kruskal's algorithm is a minimum weight spanning tree.*

Proof. Let G be a simple connected graph of order n with edge set $E(G) = \{e_1, e_2, \cdots, e_m\}$. Let T^* be a spanning tree produced by Kruskal's algorithm and let $E(T^*) = \{e_1, e_2, \cdots, e_{n-1}\}$. For any spanning tree T of G, let $f(T)$ be the least value of i such that $e_i \notin E(T)$. Suppose T^* is not of minimum weight. Let T_0 be any minimum-weight spanning tree with $f(T_0)$ as large as possible.

Suppose $f(T_0) = k$. This means that $e_1, \cdots, e_{k-1}$ are in both T_0 and T^*, but $e_k \notin T_0$. Then $T_0 + e_k$ contains a unique cycle C. Since not every edge of C can be in T^*, C must contain an edge e'_k not belonging to T^*.

Let $T_0' = T_0 + e_k - e_k'$. Then T_0' is another spanning tree of G. Moreover,

$$w(T_0') = w(T_0) + w(e_k) - w(e_k'). \tag{2.1}$$

Now, in Kruskal's algorithm, e_k was chosen as an edge with the smaller weight such that $G[\{e_1, \cdots, e_{k-1}, e_k\}]$ was acyclic. Since is $G[\{e_1, \cdots, e_{k-1}, e_k'\}]$ a subgraph of the tree T_0, it is also acyclic. Hence,

$$w(e_k) \leq w(e_k'), \tag{2.2}$$

and therefore from (2.1) and (2.2),

$$\begin{aligned} w(T_0) &= w(T_0) + w(e_k) - w(e_k') \\ &\leq w(T_0). \end{aligned}$$

But T_0 is of minimum weight. Hence, $w(T_0') = w(T_0)$, and so T_0' is also of minimum weight. However, as $\{e_1, \cdots, e_k\} \subset E(T_0')$,

$$f(T_0') > k = f(T_0),$$

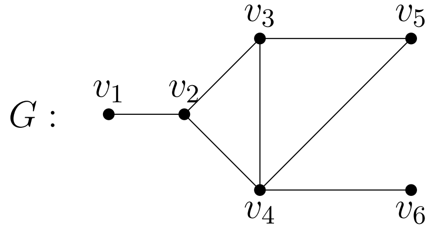
contradicting the choice of T_0. Thus, T^* is a minimum-weight spaniing tree of G. □

Let us sum up

1. If e is a cut edge of a connected graph G, then $G - e$ has exactly two components.
2. A simple cubic connected graph G has a cut vertex if and only if it has a cut edge.
3. For a loopless connected graph G, $\kappa(G) \leq \lambda(G) \leq \delta(G)$.
4. A connected graph G with at least two vertices contains at least two vertices that are not cut vertices.
5. If G is trivial or disconnected, then $\kappa(G) = 0$.
6. A graph G is r-connected, if $\kappa(G) \geq r$.
7. A graph G is r-edge connected, if $\lambda \geq r$.
8. Each component of a forest is a tree.
9. A simple graph is a tree if and only if any two distinct vertices are connected by a unique path.
10. Every connected graph contains a spanning tree.
11. The number of edges in a tree on n vertices is $n - 1$.
12. If $m(a) = n(a)$ for a simple connected graph G, then G is acyclic.
13. For a simple connected graph G, $L(a) \simeq G$ if and only if G is a cycle.
14. Every tree has a center consisting of either a single vertex or two adjacent vertices.
15. A vertex v is a centroid vertex of T if V has minimum weight.

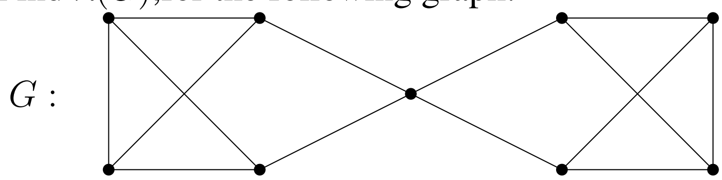
Check your progress

1. In a graph G, which is a cut vertex?



(a) $\{v_1\}$ (b) $\{v_2\}$ (c) $\{v_2, v_4\}$ (d) $\{v_4\}$

2. Find $\lambda(G)$,for the following graph:



(a) 1 (b) 2 (c) 3 (d) 4

3. Which one of the following is tree, regarding the cyclical edge connectivity λ_c?

(a) $\lambda \leq \lambda_c$ (b) $\lambda < \lambda_c$ (c) $\lambda = \lambda_c$ (d) $\lambda \geq \lambda_c$

4. For a simple cubic graph G,

(a) $\kappa(G) < \lambda(G)$ (b) $\kappa(G) > \lambda(G)$ (c) $\kappa(G) = \lambda(G)$

(d) $\kappa(G) \leq \lambda(G)$.

5. The edge connectivity of a graph is denoted by

(a) κ (b) λ (c) δ (d) λ_c

6. If v is a cut vertex of G, then $G - v$ is

(a) connected

(b) disconnected and has at least two components

(c) disconnected and has at least two components

(d) disconnected and has at least three components

7. A connected graph without cycles is called as a

   (a) simple graph (b) path (c) forest (d) tree

8. The number of edges is a tree on n vertices is

   (a) n (b) $n+1$ (c) $n-1$ (d) $n(n-1)$.

9. If $m(G) = n(G)$ for a simple connected graph G, then G is

   (a) unicyclic (b) acyclic (c) bipartite

   (d) a complete graph

10. The minimum eccentricity of G is called as

    (a) diameter of G (b) radius of G

    (c) degree of G (d) weight of G

11. If $e(v) = r(G)$, then v is called a

    (a) central vertex (b) isolated vertex

    (c) pendant vertex (d) branch

12. The number of branches at a vertex v is

    (a) weight of v (b) eccentricity of v

    (c) center of the graph (d) degree of v

13. The maximum number of edges in any branch at a vertex v is

    (a) weight of v (b) eccentricity of v

    (c) number of pendant edges at v (d) degree of v

14. If G is disconnected, then $\tau(G)$ is

    (a) 0 (b) 1 (c) 2 (d) ∞

15. For $n \geq 2,\ \tau(K_n) =$

    (a)n (b) $n-1$ (c) $n(n-1)$ (d) n^{n-2}

Answers

1. (b)	2. (b)	3. (a)	4. (c)	5. (b)
6. (c)	7. (d)	8. (c)	9. (a)	10. (b)
11. (a)	12. (d)	13. (a)	14. (a)	15. (d).

Exercises

1. Show that Herschel graph is bipartite.
2. Show that $K_{m,n}, m \neq n$ has no spanning cycle.

References

1. R. Balakrishnan and K. Ranganathan, A Text Book of Graph Theory, second ed., Springer, New York, 2012.
2. J.A. Bondy and U.S.R. Murty, Graph Theory with Application.

Suggested Readings

1. S. Arumugam Issac, introduction to Graph Theory.

♣♣♣♣♣

Unit 3

Independent Sets, Matchings and Cycles

3.1 Vertex-Independent Sets and Vertex Coverings

Objectives

1. To introduce the concepts of independent sets and coverings.
2. To improve the knowledge in matchings and factors.
3. To understand the concept of matchings in bipartite graphs.
4. To provide a foundation for Eulerian and Hamiltonian graphs.
5. To gain knowledge about 2-factorable graphs.

Definition 96. *A subset S of the vertex set V of a graph G is called* **independent** *if no two vertices of S are adjacent in G. $S \subseteq V$ is a* **maximum independent se***t of G if G has no independent set S' with $|S'| > |S|$. A* **maximal independent set** *of G is an independent set that is not a proper subset of another independent set of G.*

For example, in the graph of Fig. 3.1, $\{u, v, w\}$*is a maximum independent set and* $\{x, y\}$*; is a maximal independent set that is not maximum.*

Definition 97. *A subset* K *of* V *is called a covering of* G *if every edge of* G *is incident with at least one vertex of* K*. A covering* K *is* **minimum** *if there is no covering* K' *of* G *such that* $|K'| < |K|$*; it is* **minimal** *if there is no covering* K_1 *of* G *such that* K_1 *is a proper subset of* K*.*

In the graph W_5 *of Fig. 3.2,* $\{v_1, v_2, v_3, v4\}$ *is a covering of* W_5 *and* $\{v_1, v_3, v_4, v_6\}$ *is a minimal covering. Also, the set* $\{x, y\}$ *is a minimum covering of the graph of Fig. 3.1.*

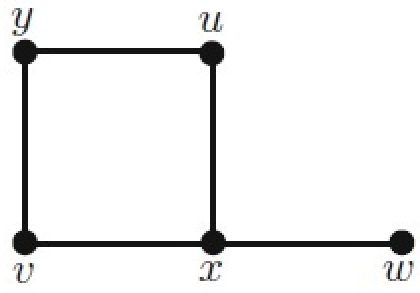


Figure 3.1: Graph with maximum independent set $\{u, v, w\}$ and maximal independent set $\{x, y\}$

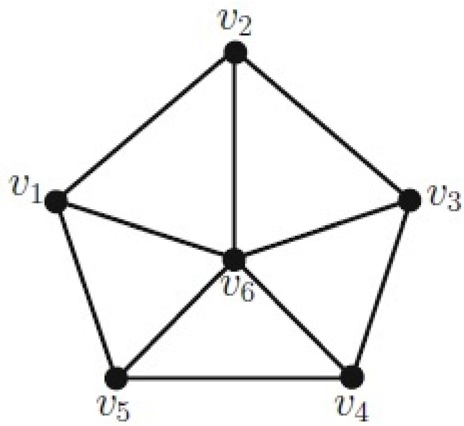


Figure 3.2: Wheel W_5

Theorem 98. *A subset S of V is independent if and only if $V \setminus S$ is a covering of G.*

Proof. S is independent if and only if no two vertices in S are adjacent in G. Hence, every edge of G must be incident to a vertex of $V \setminus S$. This is the case if and only if $V \setminus S$ is a covering of G. □

Definition 99. *The number of vertices in a maximum independent set of G is called the* **independence number** *(or the stability number) of G and is denoted by $\alpha(G)$. The number of vertices in a minimum covering of G is the* **covering number** *of G and is denoted by $\alpha(G)$. We denote these numbers simply by α and β when there is no confusion.*

Corollary 100. *For any graph G, $\alpha + \beta = n$.*

Proof. Let S be a maximum independent set of G. By Theorem 98, $V \setminus S$ is a covering of G and therefore $|V \setminus S| = n - \alpha \geq \beta$. Similarly, let K be a minimum covering of G. Then $V \setminus K$ is independent and so $|V \setminus K| = n - \beta \leq \alpha$. These two inequalities together imply that $n = \alpha + \beta$. □

Let us sum up:

In this section, we have studied definitions and some interesting relationships between independent sets and coverings.
Note that $\alpha(K_n) = 1$ and $\beta(K_n) = n - 1,\ \alpha(K_{m,n}) = n$ and $\beta(K_{m,n}) = m,\ \ m \leq n.$

Check your progess:

1. For the petersen graph, (α, β)=

   (a) (5,5) (b) (4,6) (c) (6,4) (d) (3,7)

Answer: 1. (b)

3.2 Edge-Independent Sets

Definition 101. *1. 1. A subset M of the edge set E of a loopless graph G is called* **independent** *if no two edges of M are adjacent in G.*

2. *A* **matching** *in G is a set of independent edges.*

3. *An* **edge covering** *of G is a subset L of E such that every vertex of G is incident to some edge of L. Hence, an edge covering of G exists if and only if $\delta > 0$.*

4. *A matching M of G is* **maximum** *if G has no matching M' with $|M'| > |M|$. M is maximal if G has no matching M' strictly containing M. $\alpha'(G)$ is the cardinality of a maximum matching and $\beta'(G)$ is the size of a minimum edge covering of G.*

5. *set S of vertices of G is said to be* **saturated** *by a matching M of G or M*-**saturated** *if every vertex of S is incident to some edge of M. A vertex v of G is M-saturated if $\{v\}$ is M-saturated. v is M-unsaturated if it is not M-saturated.*

For example, in the wheel W_5 (Fig. 3.2), $M = \{v_1v_2, v_4v_6\}$ is a maximal matching; $\{v_1v_5,\ v_2v_3,\ v_4v_6\}$ is a maximum matching and a minimum edge covering; the vertices $v_1,\ v_2,\ v_4$, and v_6 are M-saturated, whereas v_3 and v_5 are M-unsaturated.

Theorem 102. *For any graph G for which $\delta > 0$, $\alpha' + \beta' = n$.*

Proof. Let M be a maximum matching in G so that $|M| = \alpha'$. Let U be the set of M-unsaturated vertices in G. Since M is maximum, U is an independent set of vertices with $|U| = n - 2\alpha'$. Since $\delta > 0$, we can pick one edge for each vertex in U incident with it. Let F be the set of edges thus chosen. Then $M \cup F$ is an edge covering of G. Hence, $|M \cup F| = |M| + |F| = \alpha' + n - 2\alpha' \geq \beta'$, and therefore

$$n \geq \alpha' + \beta'. \tag{3.1}$$

Now let L be a minimum edge covering of G so that $|L| = \beta'$. Let $H = G[L]$ be the edge subgraph of G defined by L, and let M_H be a maximum matching in H. Denote the set of M_H-unsaturated vertices in H by U. As L is an edge covering of G, H is a spanning subgraph of G.

Consequently, $|L| - |M_H| = |L \setminus M_H| \geq |U| = n - 2|M_H|$ and so $|L| + |M_H| \geq n$. But since M_H is a matching in G, $|M_H| \leq \alpha'$.
Thus,

$$n \leq |L| + |M_H| \leq \beta' = \alpha'. \tag{3.2}$$

Inequalities (3.1) and (3.2) imply that $\alpha' + \beta' = n$. □

Let us Sum Up:

Edge independent set and edge covering set are not complement of one another. However, their sum is n.

Check your Progess:

1. For the petersen graph, (α', β')=

   (a) (4,6) (b) (5,5) (c) (6,4) (d) (3,7)

Answer: 1. (b)

3.3 Matchings and Factors

Definition 103. *A* **matching** *of a graph G is (as given in Definition 101) a set of independent edges of G. If $e = uv$ is an edge of a matching M of G, the end vertices u and v of e are said to be matched by M.*

If M_1 and M_2 are matchings of G, the edge subgraph defined by $M_1 \bigtriangleup M_2$, the symmetric difference of M_1 and M_2, is a subgraph H of G whose components are paths or even cycles of G in which the edges alternate between M_1 and M_2.

Definition 104. *An M-augmenting path in G is a path in which the edges alternate between $E \setminus M$ and M and its end vertices are M-unsaturated. An Malternating path in G is a path whose edges alternate between $E \setminus M$ and M.*

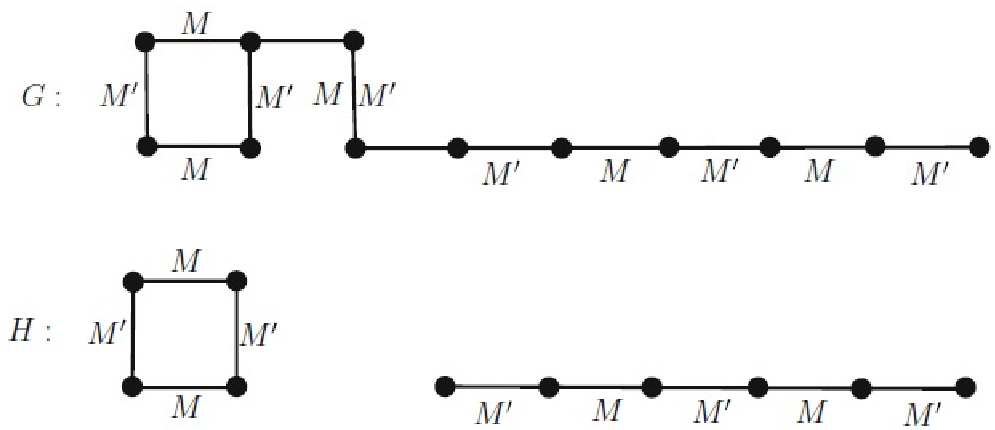


Figure 3.3: Graphs for proof of Theorem 106

Example 105. *In the graph G of Fig. 3.2, $M_1 = \{v_1v_2,\ v_3v_4,\ v_5v_6\}$, $M_2 = \{v_1v_2,\ v_3v_6,\ v_5v_5\}$, and $M_3 = \{v_3v_4,\ v_5v_6\}$ are matchings of G. Moreover, $G[M_1 \bigtriangleup M_2]$ is the even cycle $(v_3v_4v_5v_6v_3)$. The path $v_2v_3v_4v_6v_5v_1$ is an M_3-augmenting path in G.*

Maximum matching have been characterized by Berge [**?**].

Theorem 106. *A matching M of a graph G is maximum if and only if G has no M-augmenting path.*

Proof. Assume first that M is maximum. If G has an M-augmenting path $P : v_0v_1v_2 \cdots v_{2t+1}$ in which the edges alternate between $E \setminus M$ and M, then P has one edge of $E \setminus M$ more than that of M. Define

$$M' = (M \cup \{v_0v_1,\ v_2v_3,\ \cdots,v_{2t}v_{2t+1}\}) \setminus \{v_1v_2,\ v_3v_4, \cdots, v_{2t-1}v_{2t}\}.$$

Clearly, M' is a matching of G with $|M'| = |M| + 1$, which is a contradiction since M is a maximum matching of G.

Conversely, assume that G has no M-augmenting path. Then M must be maximum. If not, there exists a matching M' of G with $|M'| > |M|$. Let H be the edge subgraph $G[M \triangle M']$ defined by the symmetric difference of M and M'. Then the components of H are paths or even cycles in which the edges alternate between M and M'. Since $|M'| > |M|$, at least one of the components of H must be a path starting and ending with edges of M'. But then such a path is an M-augmenting path of G, contradicting the assumption (see Fig. 3.3). □

Definition 107. *A factor of a graph G is a spanning subgraph of G. A k-factor of G is a factor of G that is k-regular. Thus, a 1-factor of G is a matching that saturates all the vertices of G. For this reason, a 1-factor of G is called a perfect matching of G. A 2-factor of G is a factor of G that is a disjoint union of cycles of G. A graph G is k-factorable if G is an edge-disjoint union of k-factors of G.*

Example 108. *In Fig. 3.4, G_1 is 1-factorable and G_2 is 2-factorable, whereas G_3 has neither a 1-factor nor a 2-factor. The dotted, solid, and* *ordinary lines of G_1 give the three distinct 1-factors, and the dotted and ordinary lines of G_2 give its two distinct 2-factors.*

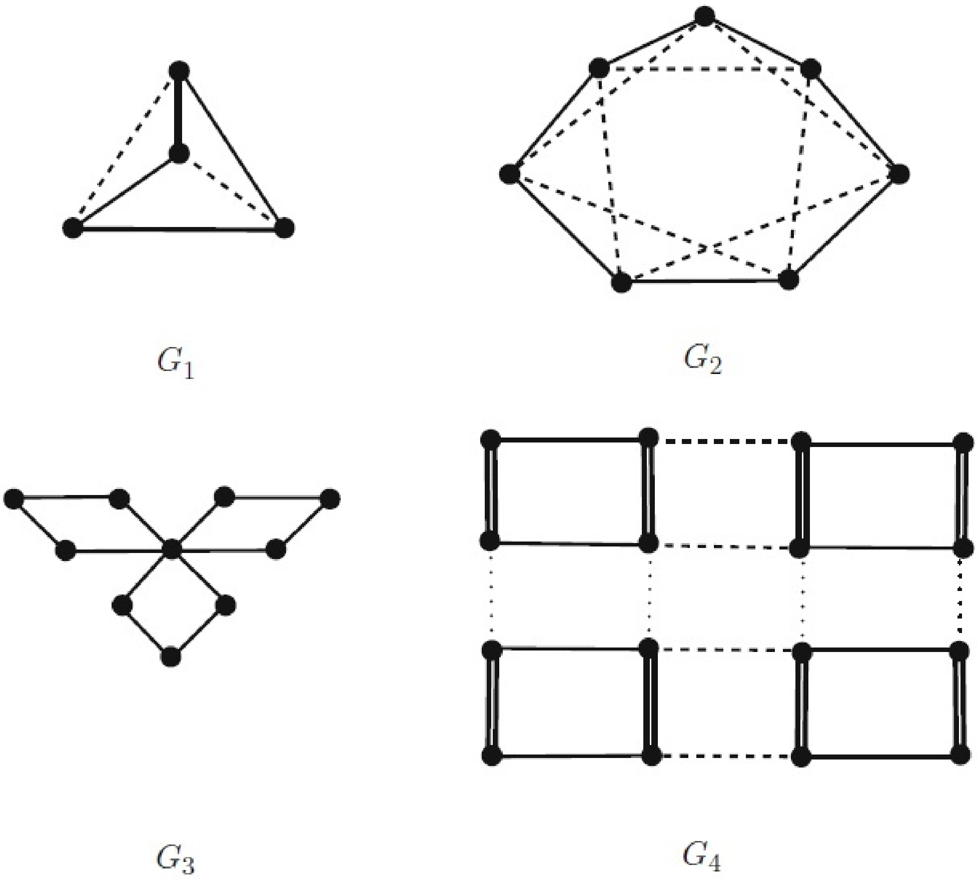


Figure 3.4: Graphs illustrating facrorability

3.4 Matchings in Bipartite Graphs

For a subset $S \subseteq V$ in a graph G, $N(S)$ denotes the neighbor set of S, that is, the set of all vertices each of which is adjacent to at least one vertex in S.

Theorem 109. *(Hall) Let G be a bipartite graph with bipartition (X, Y).*

Then G has a matching that saturates all the vertices of X if and only if

$$|N(S)| \geq |S| \tag{3.3}$$

for every subset S of X.

Proof. If G has a matching that saturates all the vertices of X, then distinct vertices of X are matched to distinct vertices of Y . Hence, trivially, $|N(S)| \geq |S|$ for every subset $S \subseteq X$.

Conversely, assume that the condition (3.3) above holds but that G has no matching that saturates all the vertices of X. Let M be a maximum matching of G. As M does not saturate all the vertices of X, there exists a vertex $x_0 \in X$ that is M-unsaturated. Let Z denote the set of all vertices of G connected to x0 by Malternating paths. Since M is a maximum matching, by Theorem 106, G has no M-augmenting path. As x_0 is M-unsaturated, x_0 is the only vertex of Z that is M-unsaturated. Let $A = Z \cap X$ and $B = Z \cap Y$. Then the vertices of $A \setminus \{x0\}$ get matched under M to the vertices of B, and $N(A) = B$. Thus, since $|B| = A - 1$, $|N(A)| = |B| = |A| - 1 < |A|$, and this contradicts the assumption (3.3) (see Fig. 3.5). □

We now give some important consequences of Hall's theorem

Theorem 110. *A $k(\geq 1)$-regular bipartite graph is 1-factorable.*

Proof. Let G be k-regular with bipartition (X, Y). Then $E(G)$= the set of edges incident to the vertices of X= the set of edges incident to the vertices of Y. Hence, $k|X| = |E(G)| = k|Y|$; and therefore $|X| = |Y|$. If $S \subseteq X$, then $N(S) \subseteq Y$, and $N(N(S))$ contains S. Let E_1 and E_2 be the sets of edges of G incident to S and $N(S)$, respectively. Then $E_1 \subseteq E_2$, $|E_1| = k|S|$, and $|E_2| = k|N(S)|$. Hence, as $E_2| \geq E_1$,

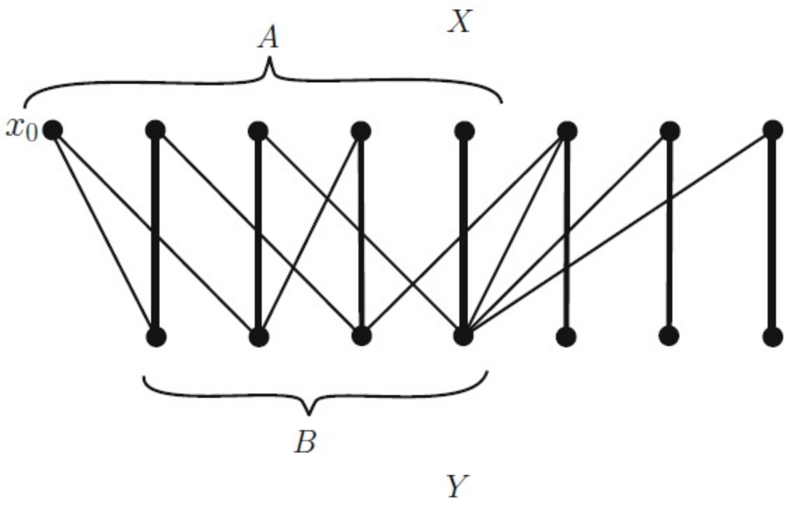


Figure 3.5: Figure for proof of Theorem 109 (matching edges are bold-faced)

$|N(S)| \geq |S|$. So by Hall's theorem (Theorem 109), G has a matching that saturates all the vertices of X; that is, G has a perfect matching M. Deletion of the edges of M from G results in a $k-1$-regular bipartite graph. Repeated application of the above argument shows that G is 1-factorable □

Lemma 111. *Let K be any covering and M any matching of a graph G with $|K| = |M|$. Then K is a minimum covering and M is a maximum matching.*

Proof. let M^* be a maximum matching and K^* a minimum covering of G. Then $|M| \leq |M^*|$ and $|K| \geq |K^*|$. Hence, we have $|M| \leq |M^*| \leq |K^*| \leq |K|$. Since $|M| = |K|$, we must have $|M| = |M^*| = |K^*| = |K|$, proving the lemma. □

Theorem 112. *In a bipartite graph the minimum number of vertices that cover all the edges of G is equal to the maximum number of independent edges; that is, $\alpha'(G) = \beta(G)$.*

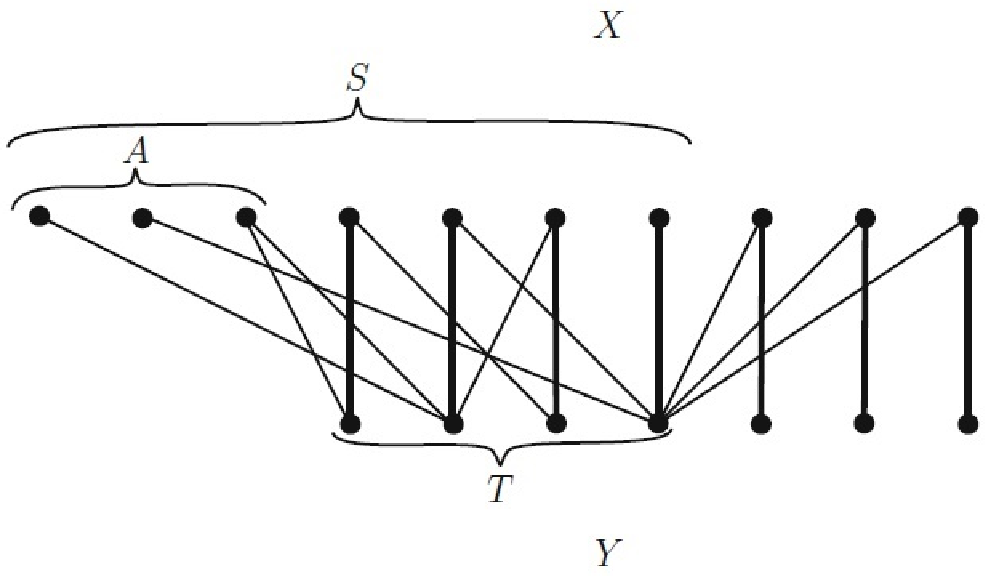


Figure 3.6: Graph for proof of Theorem 112

Proof. Let G be a bipartite graph with bipartition (X, Y). Let M be a maximum matching in G. Denote by A the set of vertices of X unsaturated by M (see Fig. 3.6). As in the proof of Theorem 109, let Z stand for the set of vertices connected to A by M-alternating paths starting in A. Let $S = X \cap Z$ and $T = Y \cap Z$. Then clearly, $T = N(S)$ and $K = T \cup (X \setminus S)$ is a covering of G, because if there is an edge e not incident to any vertex in K, then one of the end vertices of e must be in S and the other in $Y \setminus T$, contradicting the fact that $N(S) = T$. Clearly, $|K| = |M|$, and so by Lemma 111, M is a maximum matching and K a minimum covering of G. □

Theorem 113. *(Matrix version of KRonig's theorem) In a binary matrix, the minimum number of lines that cover all the 1's is equal to the maximum number of independent 1's.*

Proof. Let $A = (a_{ij})$ be a binary matrix of size p by q. Form a bipartite graph G with bipartition (X, Y), where X and Y are sets of cardinality p and q, respectively, say, $X = \{v_1, v_2, \cdots, v_p\}$ and $Y = \{w_1, w_2, \cdots, w_q\}$.

Make v_i adjacent to w_j in G if and only if $a_{ij} = 1$. Then an entry 1 in A corresponds to an edge of G, and two independent 1's in A correspond to two independent edges of G. Further, each vertex of G corresponds to a line of A. Thus, the matrix version of Konig's theorem is actually a restatement of Konig's theorem. □

A consequence of Theorem 109 is the theorem on the existence of a system of distinct representatives (SDR) for a family of subsets of a given finite set.

Definition 114. *Let $\mathcal{F} = \{A_\alpha : \alpha \in J\}$ be a family of sets. An SDR for the family $\mathcal{F}$ is a family of elements $\{x_\alpha : \alpha \in J\}$ such that $x_\alpha \in A_\alpha$ for every $\alpha \in J$ and $x_\alpha \neq x_\beta$ whenever $\alpha \neq \beta$.*

Example 115. *For instance, if $A_1 = \{1\}$, $A_2 = \{2, 3\}$, $A_3 = \{3, 4\}$, $A_4 = \{1, 2, 3, 4\}$, and $A_5 = \{2, 3, 4\}$, then the family $\{A_1, A_2, A_3, A_4\}$ has $\{1, 2, 3, 4\}$ as an SDR, whereas the family $\{A_1, A_2, A_3, A_4, A_5\}$ has no SDR. It is clear that for $\mathcal{F}$ to have an SDR, it is necessary that for any positive integer k, the union of any k sets of $\mathcal{F}$ must contain at least k elements. That this condition is also sufficient when F is a finite family of finite sets is the assertion of Hall's theorem on the existence of an SDR*

Theorem 116. *(Hall's theorem on the existence of an SDR [**?**]). Let $\mathcal{F} = \{A_i : 1 \leq i \leq r\}$ be a family of finite sets. Then $\mathcal{F}$ has an SDR if and only if the union of any k members of $\mathcal{F}$, $1 \leq k \leq r$, contains at least k elements.*

Definition 117. *A component of a graph is odd or even according to whether it has an odd or even number of vertices. Let $o(G)$ denote the number of odd components of G.*

Theorem 118. *(Tutte's 1-factor theorem [?]). A graph G has a 1-factor if and only if*

$$o(G - S) \leq |S|, \tag{3.4}$$

for all $S \subseteq V$.

Proof. While considering matchings in graphs, we are interested only in the adjacency of pairs of vertices. Hence, we may assume without loss of generality that G is simple. If G has a 1-factor M, each of the odd components of $G - S$ must have at least one vertex, which is to be matched only to a vertex of S under M. Hence, for each odd component of $G - S$, there exists an edge of the matching with one end in S. Hence, the number of vertices in S should be at least as large as the number of odd components in $G - S$; that is, $o(G - S) \leq |S|$. Conversely, assume that

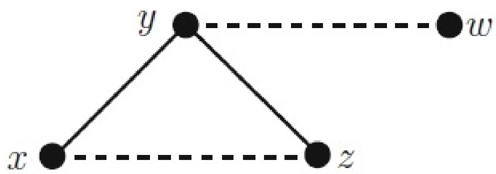


Figure 3.7: Supergraph G^* for proof of Theorem 118. Unbroken lines correspond to edges of G^* and broken lines correspond to edges not belonging to G^*

condition (3.4) holds. If G has no 1-factor, we join pairs of non adjacent vertices of G until we get a maximal supergraph G^* of G with G^* having no 1-factor. Condition (3.4) holds clearly for G^* as

$$o(G^* - S) \leq o(G - S). \tag{3.5}$$

(When two odd components are joined by an edge, the result is an even component.)

Taking $S = \phi$ in (3.4), we see that $o(G) = 0$, and so $n(G^*)(= n(G)) = n$ is even. Further, for every pair of non adjacent vertices u and v of G^*, $G^* + uv$ has a 1-factor, and any such 1-factor must necessarily contain the edge uv.

Let K be the set of vertices of G^* of degree $(n-1)$. $K \neq V$, since otherwise $G^* = K_n$ has a perfect matching. We claim that each component of $G^* - K$ is complete. Suppose to the contrary that some component G_1 of $G^* - K$ is not complete. Then in G_1 there are vertices x, y and z such that $xy \in E(G^*)$, $yz \in E(G^*)$, but xz does not belong to $E(G^*)$. Moreover, since $y \in V(G_1)$, $d_{G^*}(y) < n - 1$ and hence there exists a vertex w of G^* with $yw \notin E(G^*)$. Necessarily, w does not belong to K. (See Fig. 3.7.)

By the choice of G^*, each of $G^* \in xz$ and $G^* \in yw$ has a 1-factor, say M_1 and M_2, respectively. Necessarily, $xz \in M_1$ and $yw \in M_2$. Let H be the subgraph of $G^* + \{xz, yw\}$ induced by the edges in the symmetric difference $M_1 \bigtriangleup M_2$ of M_1 and M_2. Since M_1 and M_2 are 1-factors, each vertex of G^* is saturated by both M_1 and M_2, and H is a disjoint union of even cycles in which the edges alternate between M_1 and M_2. There arise two cases:

Case 1. xz and yw belong to different components of H (Fig. 3.8a). If yw belongs to the even cycle C, then the edges of M_1 in C together with the edges of M_2 not belonging to C form a 1-factor in G^*, contradicting the choice of G^*.

Case 2. xz and yw belong to the same component C of H. Since each component of H is a cycle, C is a cycle (Fig. 3.8b). By the symmetry of x and z, we may suppose that the vertices $x,\ y,$ w, and z occur in that order on C. Then the edges of M_1 belonging to the $yw \cdots z$ section of C together with the edge yz and the edges of M_2 not in the $yw \cdots z$ section of C form a 1-factor of G^*, again contradicting the choice of G^*. Thus, each component of $G^* - K$ is complete.

By condition (3.5), $o(G^* - K) \leq |K|$. Hence, a vertex of each of the odd components of $G^* - K$ is matched to a vertex of K. (This is possible since each vertex of K is adjacent to evry other vertex of G^*). Also, the remaining vertices in each of the odd and even components of $G^* - K$ can be matched among themselves (see Fig. 3.9). The total number of vertices thus matched is even. Since $|V(G^*)|$ is even, the remaining vertices, if any, of K can be matched among themselves. This gives a 1-factor of G^*. Note that if $K = \phi$, $0(G^*) = 0$, and the existence of a 1-factor in G^* is trivially true. But by choice, G^* has no 1-factor. This contradiction proves that G has a 1-factor. □

Corollary 119. *(Petersen [**?**]). Every connected* 3*-regular graph having no cut edges has a* 1*-factor.*

Proof. Let G be a connected 3-regular graph without cut edges. Let $S \subseteq V$. Denote by $G_1,\ G_2, \cdots, G_k$ the odd components of $G - S$. Let m_i be the number of edges of G having one end in $V(G_i)$ and the other end in

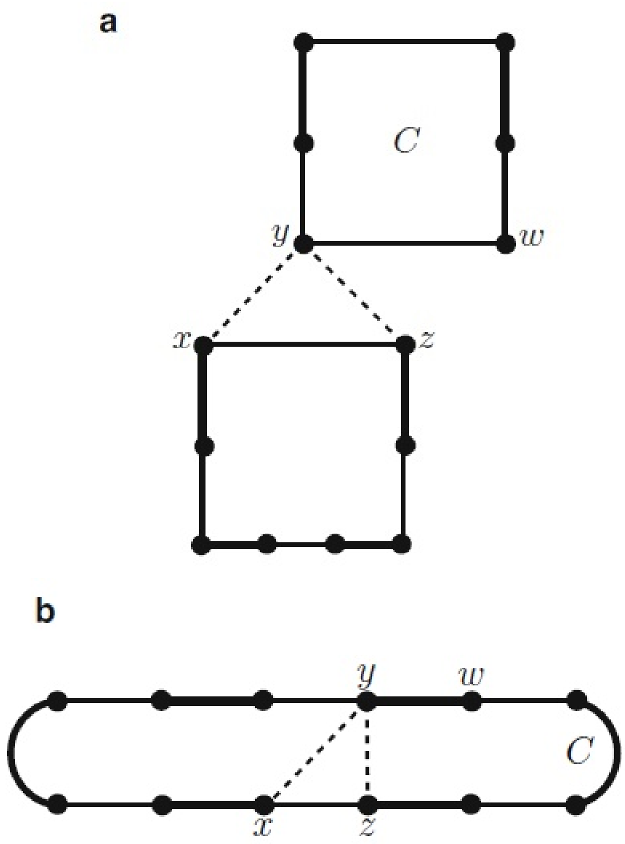


Figure 3.8: 1-factors M_1 and M_2 for (a) case 1 and (b) case 2 in proof of Theorem 118. Ordinary lines correspond to edges of M_1 and bold lines correspond to edges of M_2

S. Since G is a cubic graph,

$$\sum_{v \in V(G_i)} d(v) = 3n(G_i), \text{ and} \tag{3.6}$$

$$\sum_{v \in S} = 3|S|. \tag{3.7}$$

Now $E(G_i) = [V(G_i), V(G_i) \cup S] \setminus [V(G_i), S]$, where $[A, B]$ denotes the set of edges having one end in A and the other end in B, $A \subseteq V$, $B \subseteq V$. Hence, $m_i = |[V(G_i), S] = \sum_{v \in V(G_i)} d(v) - 2m(G_i)$, and since $d(v)$ is 3 for each v and $V(G_i)$ is an odd component, m_i is odd for each i. Further, as G has no cut edges, $m_i \geq 3$. Thus, $o(G - S) = k \leq \frac{1}{3}\sum_{i=1}^{k} m_i \leq$

$\frac{1}{3}\sum_{v\in s} d(v) = \frac{1}{3}3|S| = |S|$. Therefore, by Tutte's theorem (Theorem 118), G has a 1-factor. □

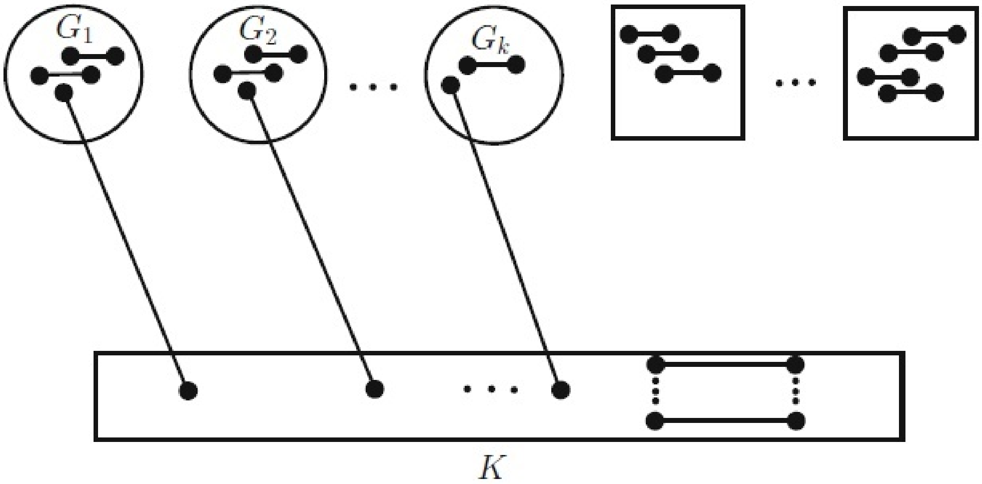


Figure 3.9: Components of $G^* - K$ for proof of Theorem 118

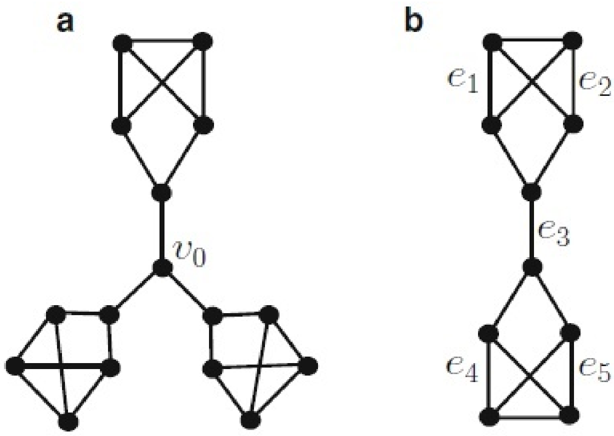


Figure 3.10: (a) 3-regular graph with cut edges having no 1-factor; (b) cubic graph with a 1-factor having a cut edge

Example 120. *A* 3*-regular graph with cut edges may not have a* 1*-factor (see Fig. 3.10a). Again, a cubic graph with a* 1*-factor may have cut edges (see Fig. 3.10b).*

In Fig. 3.10a, if $S = \{v_0\}$, $o(G - S) = 3 > 1 = |S|$, and so G has no 1-factor. In Fig. 3.10b, $\{e_1, e_2, e_3, e_4, e_5\}$ is a 1-factor, and e_3 is a cut edge of G.

If G has no 1-factor, by Theorem 118 there exists $S \subset V(G)$ with $o(G-S) > |S|$. Such a set S is called an antifactor set of G; clearly, S is a proper subset of $V(G)$.

Let G be a graph of even order n and let S be an antifactor set of G. Then $|S|$ and $o(G-S)$ have the same parity, and therefore $o(G-S) \equiv |S| \pmod 2$. Thus, we make the following observation

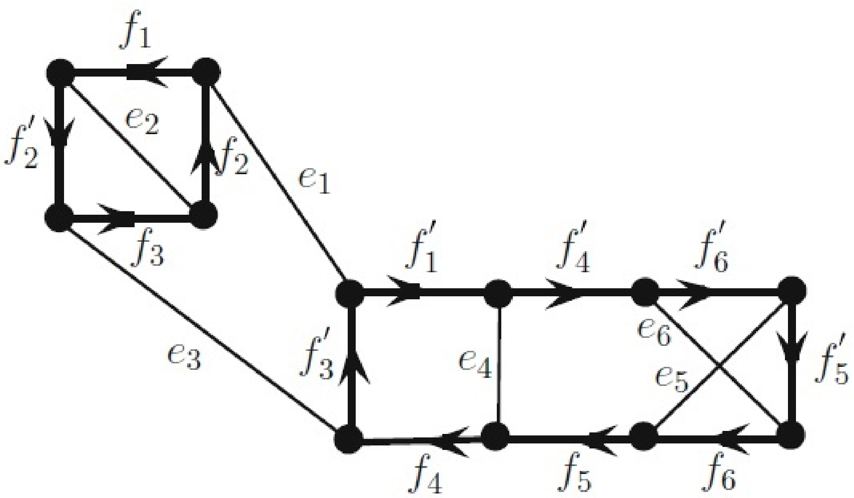


Figure 3.11: Figure for the proof of Corollary 122

Observation 121. *If S is an antifactor set of a graph G of even order, then* $0(G-S) \geq |S| + 2$.

Corollary 122. *(W. H. Cunnigham; see [**?**]). The edge set of a simple 2-edge-connected cubic graph G can be partitioned into paths of length 3.*

Proof. By Corollary 119, G is a union of a 1-factor and a 2-factor. Orient the edges of each cycle of the above 2-factor in any manner so that each cycle becomes a directed cycle. Then if e_1 is any edge of the 1-factor, and $f_1,\ f_1'$ are the two arcs of G having their tails at the end vertices of e_1, then $\{e_1, f_1, f_1'\}$ forms a typical 3-path of the edge partition of G (see Fig. 3.11). □

Corollary 123. *A $(p-1)$-regular connected simple graph on $2p$ vertices has a 1-factor.*

Proof. Proof is by contradiction. Let G be a $(p-1)$-regular connected simple graph on $2p$ vertices having no 1-factor. Then G has an antifactor set S. By Observation 121, $o(G-S) \geq |S|+2$. Hence, $|S|+(|S|+2) \leq 2p$, and therefore $|S| \leq p-1$. Let $|S| = p-r$. Then $r \neq 1$ since if $r = 1$. $|S|+(|S|+2) \leq 2p$,, and therefore $o(G-S) = p+1$. (Recall that G has $2p$ vertices.) Hence, each odd component of $(G-S)$ is a singleton, and therefore each such vertex must be adjacent to all the $p-1$ vertices of S [as G is $(p-1)$-regular]. But thismeans that every vertex of S is of degree at least $p+1$, a contradiction. Thus, $|S| = p-r$, $2 \leq r \leq p-1$. If G' is any component of $G-S$ and $v \in V(G')$, then v can be adjacent to at most $|S|$ vertices of S. Therefore, as G is $(p-1)$-regular, v must be adjacent to at least $(p-1)-(p-r) = r-1$ vertices of G'. Thus, $|V(G')| \geq r$. Counting the vertices of all the odd components of $G-S$ and the vertices of S, we get $(|S|+2)r+|S| \leq 2p$, or $(p-r+2)r+(p-r) \leq 2p$. This gives $(r-1)(r-p) \geq 0$, violating the condition on r. ☐

Theorem 124. *(D. P. Sumner [**?**]). Let G be a connected graph of even order n. If G is claw-free (i.e., contains no $K_{1,3}$ as an induced subgraph), then G has a 1-factor.*

Proof. If G has no 1-factor, G contains a minimal antifactor set S of G. There must be an edge between S and each odd component of $G-S$. Since $(G-S) > |S|$ and G is of even order, by Observation 121, $o(G-S) \geq |S|+2$. Hence, there are two possibilities: (i) There exists $v \in S$, and $vx,\ vy,\ vz$ are edges of G with $x,\ y$ and z belonging to distinct odd components of $G-S$. This cannot occur since by hypothesis G is $K_{1,3}$-free. (ii) There exist a vertex v of S, and edges vu and vw of G with u and

w in distinct odd components of $G - S$. Suppose G_u and G_w are the odd components containing u and w, respectively. Then $< G_u \cup G_w\{v\} >$ is an odd component of $G-S_1$, where $S_1 = S-\{v\}$. Further $o(G-S)-1 > |S|-1 = |S_1|$, and hence S_1 is an antifactor set of G with $|S_1| = |S|-1$, a contradiction to the choice of S. Thus, G must have a 1-factor. [Note that by Observation 121, the case $|S| = 1$ and $o(G-S) = 2$ cannot arise.] □

Let us Sum Up:

In this section, we have studied factors and some interesting properties like Hall's theorem, Tutte's theorem, Petersen theorem, etc.
Note that K_{2n} and $K_{m,n}$ are 1-factorable. Further, by the application of Hall's theorem, every regular bipartate graph is 1-factorable.

Check your Progess:

1. Does $K_{m,n}$ posses a 1-factor

   (a) Yes (b) No (c) Yes, when $m = n$ (d) Never

2. How many edge-disjoint 1-factor petersen graph have?

   (a) 1 (b) 2 (c) 3 (d) 0

Answers: 1. (c) 2. (a)

3.5 Eulerian Graphs

Definition 125. *An Euler trail in a graph G is a spanning trail in G that contains all the edges of G. An Euler tour of G is a closed Euler trail of G. G is called Eulerian (Fig. 3.12a) if G has an Euler tour. It was Euler who first considered these graphs, and hence their name.*

Euler showed in 1736 that the celebrated Konigsberg bridge problem has no solution. The city of Konigsberg (now called Kaliningrad) has seven bridges linking two islands A and B and the banks C and D of the Pregel (now called Pregalya) River, as shown in Fig. 3.13.

The problem was to start from any one of the four land areas, take a stroll across the seven bridges, and get back to the starting point without crossing any bridge a second time. This problem can be converted into one concerning the graph obtained by representing each land area by a vertex and each bridge by an edge. The resulting graph H is the graph of Fig. 3.12b. The Konigsberg bridge problem will have a solution provided that this graph H is Eulerian. But this is not the case since it has vertices of odd degrees.

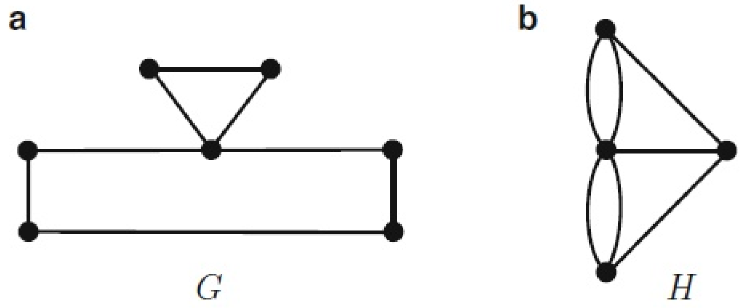


Figure 3.12: (a) Eulerian graph G; (b) non- Eulerian graph H

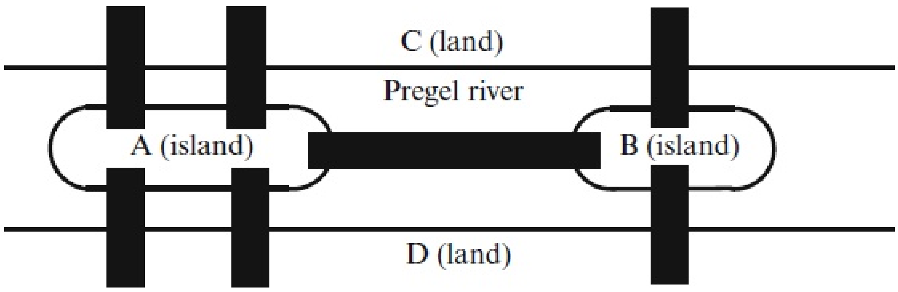


Figure 3.13: Konigsberg bridge problem

Theorem 126. *For a nontrivial connected graph G, the following statements are equivalent:*

(i) G is Eulerian.

(ii) The degree of each vertex of G is an even positive integer.

(iii) G is an edge-disjoint union of cycles.

Proof. $(i) \Rightarrow (ii)$: Let T be an Euler tour of G described from some vertex $v_0 \in V(G)$. If $v \in V(G)$, and $v \neq v_0$, then every time T enters v, it must move out of v to get back to v_0. Hence two edges incident with v are used during a visit to v, and therefore, $d(v)$ is even. At v_0, every time T moves out of v_0, it must get back to v_0. Consequently, $d(v_0)$ is also even. Thus, the degree of each vertex of G is even.

$(ii) \Rightarrow (iii)$: As $\delta(G) \geq 2$, G contains a cycle C_1. In $G \setminus E(C_1)$, remove the isolated vertices if there are any. Let the resulting subgraph of G be G_1. If G_1 is nonempty, each vertex of G_1 is again of even positive degree. Hence $\delta(G_1) \geq 2$, and so G_1 contains a cycle C_2. It follows that after a finite number, say r, of steps, $G \setminus E(C_1 \cup \cdots \cup C_r)$ is totally disconnected. Then G is the edge disjoint union of the cycles $C_1, C_2, \cdots, C_r$

$(iii) \Rightarrow (i)$: Assume that G is an edge-disjoint union of cycles. Since any cycle is Eulerian, G certainly contains an Eulerian subgraph. Let G_1 be a longest closed trail in G. Then G_1 must be G. If not, let $G_2 = G \setminus (G_1)$. Since G is an edge disjoint union of cycles, every vertex of G is of even degree ≥ 2. Further, since G_1 is Eulerian, each vertex of G_1 is of even degree ≥ 2. Hence each vertex of G_2 is of even degree. Since G_2 is not totally disconnected and G is connected, G_2 contains a cycle C having a vertex v in common with G_1. Describe the Euler tour of G_1. starting and ending at v and follow it by C. Then $G_1 \cup C$ is a closed trail in G longer than G_1. This contradicts the choice of G_1, and so G_1 must be G. Hence G is Eulerian. □

If $G_1, \cdots, G_r$ are subgraphs of a graph G that are pairwise edge-disjoint and their union is G, then this fact is denoted by writing $G = G_1 \oplus \cdots \oplus G_r$. In the above equation, if $G_i = C_i$, a cycle of G for each i, then $G = C_1 \oplus \cdots \oplus C_r$. The set of cycles $S = \{C_1, \cdots, C_r\}$ is then called a cycle decomposition of G. Thus, Theorem 126 implies that a connected graph is Eulerian if and only if it admits a cycle decomposition.

Let us Sum Up:

In this section, we have studied the historical development of graph theory and some interesting equivalent conditions on Eulerian graphs.
Note that complete graphs are not Eulerian always. K_n is Eulerian only when n is odd. Similarly, $K_{m,n}$ is Eulerian only when both m and n are even.

Check your Progess:

1. Whether $G = P - I$ is Eulerian? when P is the Petersen graph and I is a 1-fractor.

   (a) Yes (b) No (c) Depends on I (d) Never

Answer: 1. (d)

3.6 Hamiltonian Graphs

Definition 127. *A graph is called Hamiltonian if it has a spanning cycle (see Fig. 3.14a). These graphs were first studied by Sir William Hamilton,* *a mathematician. A spanning cycle of a graph G, when it exists, is often called a Hamilton cycle (or Hamiltonian cycle) of G.*

Definition 128. *A graph G is called traceable if it has a spanning path of G (see Fig. 3.14b). A spanning path of G is also called a Hamilton path (or Hamiltonian path) of G.*

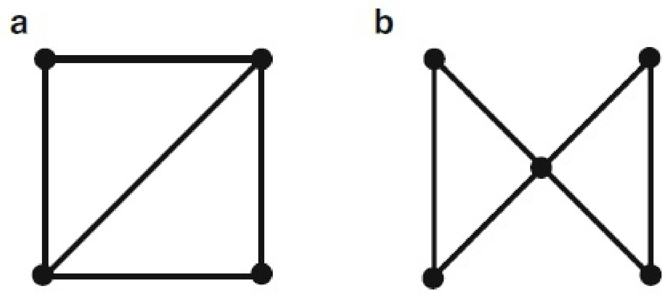


Figure 3.14: (a) Hamiltonian graph; (b) non-Hamiltonian but traceable graph

Theorem 129. *If G is Hamiltonian, then for every nonempty proper subset S of V, $w(G-s) \leq |S|$.*

Proof. Let C be a Hamilton cycle in G. Then, since C is a spanning subgraph of G, $w(G-S) \leq w(C-S)$. If $|S| = 1$, $C-S$ is a path, and therefore $w(C-S) = 1 = |S|$. The removal of a vertex from a path P results in one or two components, according to whether the removed vertex is an end vertex or an internal vertex of P. Hence, by induction, the number of components in $C-S$ cannot exceed $|S|$. This proves that $w(G-s) \leq w(C-S) \leq |S|$. □

Theorem 130. *(Ore [**?**]). Let G be a simple graph with $n \geq 3$ vertices. If, for every pair of nonadjacent vertices u, v of G, $d(u) + d(v) \geq n$, then G is Hamiltonian.*

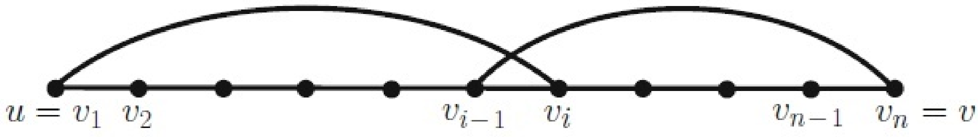


Figure 3.15: Hamilton path for proof of Theorem 130

Proof. Suppose that G satisfies the condition of the theorem, but G is not Hamiltonian. Add edges to G (without adding vertices) and get a supergraph G^* of G such that G^* is a maximal simple graph that satisfies the condition of the theorem, but G^* is non-Hamiltonian. Such a graph G^* must exist since G is non-Hamiltonian while the complete graph on $V(G)$ is Hamiltonian. Hence, for any pair u and v of nonadjacent vertices of G^*, $G^* + uv$ must contain a Hamilton cycle C. This cycle C would certainly contain the edge $e = uv$. Then $C - e$ is a Hamilton path $u = v_1v_2v_3 \cdots v_n = v$ of G^* (see Fig. 3.15).

Now, if $v_i \in N(u),\ v_{i-1} \notin N(v)$; otherwise, $v_1v_2 \cdots v_{i-1}v_nv_{n-1}v_{n-2} \cdots v_{i+1}v_iv_1$ would be a Hamilton cycle in G^*. Hence, for each vertex v_i adjacent to u, the vertex v_{i-1} of $V - \{v\}$ is nonadjacent to v. But then

$$d_{G^*}(v) \leq (n-1) - d_{G^*}(u).$$

This gives that $d_{G^*}(u) + d_{G^*}(v) \leq n - 1$, and therefore $d_G(u) + d_G(v) \leq n - 1$, a contradiction. □

Corollary 131. *(Direc [**?**]). If G is a simple graph with $n \geq 3$ and $\delta \geq \frac{n}{2}$, then G is Hamiltonian.*

Corollary 132. *Let G be a simple graph with $n \geq 3$ vertices. If $d(u) + d(v) \geq n - 1$ for every pair of nonadjacent vertices u and v of G, then G is traceable.*

Proof. Choose a new vertex w and let G_0 be the graph $G \vee \{w\}$. Then each vertex of G has its degree increased by one, and therefore in G', $d(u) + d(v) \geq n+1$ for every pair of nonadjacent vertices. Since $|V(G')| = n+1$, by Theorem 130, G' is Hamiltonian. If C' is a Hamilton cycle of G', then $C' - w$ is a Hamilton path of G. Thus, G is traceable. □

Definition 133. *The closure of a graph G, denoted $cl(G)$, is defined to be that supergraph of G obtained from G by recursively joining pairs of nonadjacent vertices whose degree sum is at least n until no such pair exists.*

Theorem 134. *The closure $cl(G)$ of a graph G is well defined.*

Proof. Let G_1 and G_2 be two graphs obtained from G by recursively joining pairs of nonadjacent vertices whose degree sum is at least n until no such pair exists. We have to prove that $G_1 = G_2$.

Let $\{e_1, \cdots, e_p\}$ and $\{f_1, \cdots, f_p\}$ be the sets of new edges added to G in these sequential orderings to get G_1 and G_2, respectively. We want to show that each e_i is some f_j (and therefore belongs to G_1) and that each f_k is some e_l (and therefore belongs to G_1). Let e_i be the first edge in $\{e_1, e_2, \cdots, e_p\}$ not belonging to G_2. Then $\{e_1, \cdots, e_{i-1}\}$ are all in both G_1 and G_2, and $uve_i \notin E(G_2)$. Let $H = G + \{e_1, \cdots, e_{i-1}\}$. Then H is a subgraph of both G_1 and G_2. By the way $cl(G)$ is defined,

$$\begin{aligned} d_H(u) + d_H(v) &\geq n, \quad \text{and hence} \\ d_{G_2}(u) + d_{G_2}(v) &\geq n. \end{aligned} \tag{3.8}$$

But this is a contradiction since u and v are nonadjacent vertices of G_2, and G_2 is a closure of G. Thus $e_i \in E(G_2)$ for each i and similarly, $f_k \in E(G_1)$ for each k. □

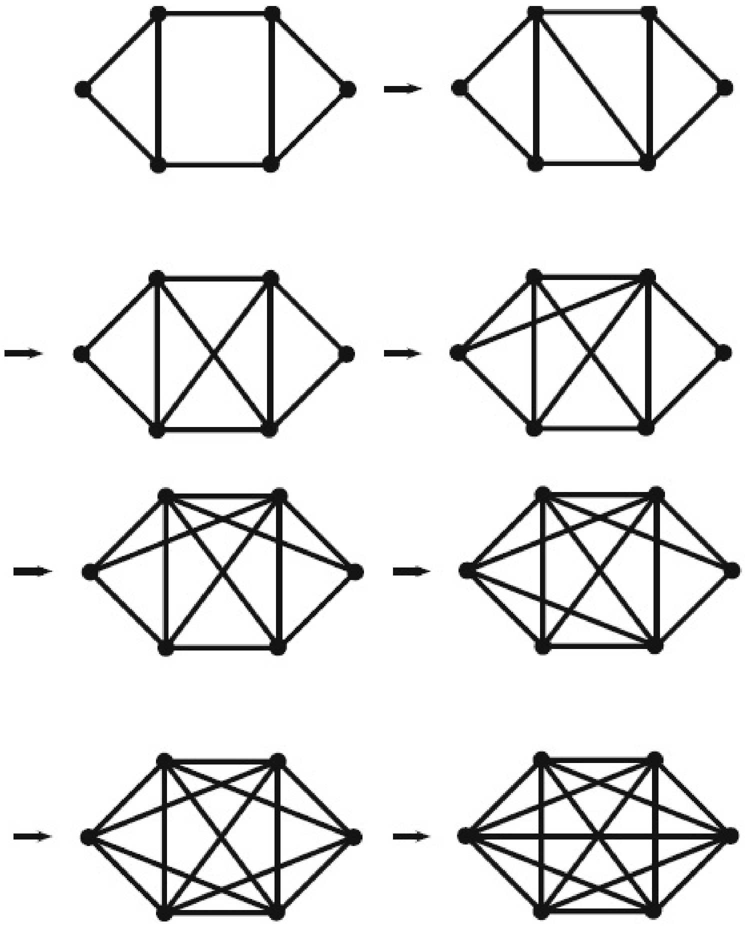


Figure 3.16: Closure of a graph

Theorem 135. *(Chvatal and Erdos). If, for a simple* 2*-connected graph* G*,* $\alpha \leq \kappa$*, then* G *is Hamiltonian. (*α *is the independence number of* G *and* κ *is the connectivity of* G*.)*

Proof. Suppose $\alpha \leq \kappa$ but G is not Hamiltonian. Let $C : v_0v_1 \cdots v_(p-1)$ be a longest cycle of G. We fix this orientation on C. By Dirac's theorem, $p \geq \kappa$. Let $v \in V(G) \setminus V(C)$. Then by Menger's theorem, there exist κ internally disjoint paths $P_1, \cdots, P_\kappa$ from v to C. Let $v_{i_1}, v_{i_2}, \cdots, v_{i_\kappa}$ be the end vertices (with suffixes in the increasing order) of these paths on C. No two of the consecutive vertices $v_{i_1}, v_{i_2}, \cdots, v_{i_\kappa}, v_{i_1}$ can be adjacent vertices of C, since otherwise we get a cycle of G longer than C. Hence, between any two consecutive vertices of $\{v_{i_1}, v_{i_2}, \cdots, v_{i_\kappa}, v_{i_1}\}$ there exists at least one vertex of G. Let u_{i_j} be the vertex next to v_{i_j} in the $v_{i_j} - v_{i_{j+1}}$ path along C (see Fig. 3.17a).

We claim that $\{u_{i_1}, \cdots, u_{i_\kappa}\}$ is an independent set of G. Suppose u_{i_j} is adjacent to $u_{i_m},\ m > j$ (suffixes taken modulo κ); then

$$u_{i_j} \cdots v_{i_{j+1}} \cdots v_{i_m} P_m^{-1} v P_j v_{i_j} \cdots v_{i_{j-1}} \cdots u_{i_m} u_{i_j}$$

is a cycle of G longer than C, a contradiction.

Further, $\{v, u_{i_1}, \cdots, u_{i_k}\}$ is also an independent set of G. [Otherwise, $vu_{i_m} \in E(G)$ for some m. See Fig. 3.17b. Then

$$vu_{i_m} \cdots v_{i_{m+1}} \cdots v_{i_k} \cdots v_{i_1} \cdots v_{i_m} P_m^{-1} v$$

is a cycle longer than C, a contradiction.] But this implies that $\alpha > \kappa$, a contradiction to our hypothesis. Thus G is Hamiltonian. □

Theorem 136. *If G is a simple graph with $n \geq 3$ vertices such that $d(u) + d(v) \geq n+1$ for every pair of nonadjacent vertices of G, then G is Hamiltonian-connected.*

Proof. Let u and v be any two vertices of G. Our aim is to show that a Hamilton path exists from u to v in G.

Choose a new vertex w, and let $G^* = G \cup \{wu, wv\}$. We claim that $cl(G^*) = K_{n+1}$. First, the recursive addition of the pairs of nonadjacent vertices u and v of G with $d(u) + d(v) \geq n+1$ gives K_n. Further, each vertex of K_n is of degree $n-1$ in K_n and $d(G^*)(w) = 2$. Hence, $cl(G^*) = K_{n+1}$. So G^* is Hamiltonian. Let C be a Hamilton cycle in G^*. Then $C - w$ is a Hamilton path in G from u to v. □

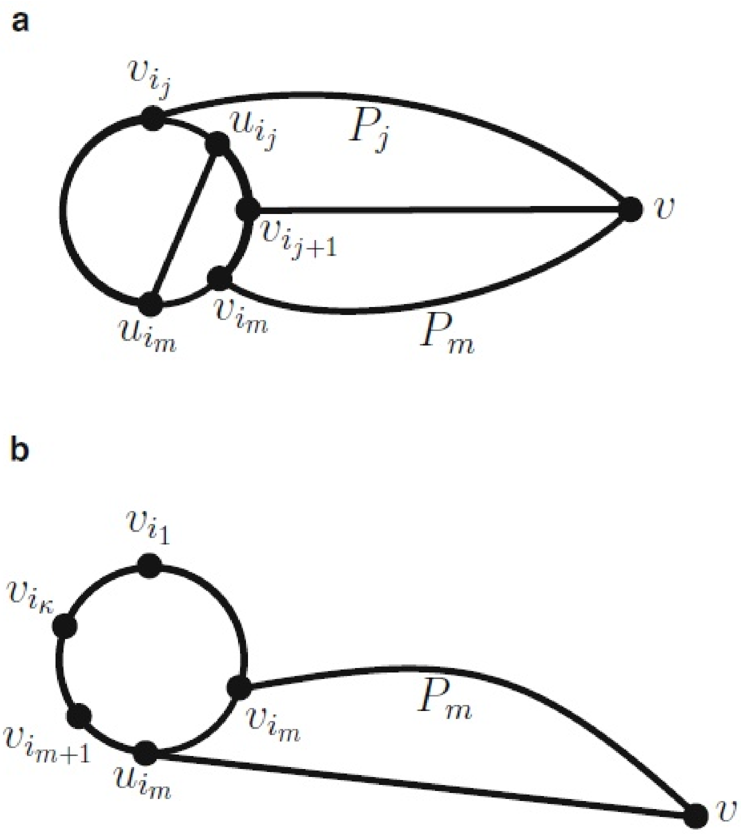


Figure 3.17: Graphs for proof of Theorem 135

Let us Sum Up:

In this section, we have studied the Hamiltonian graph and some necessary and sufficient conditions.
Note that there is no relationship between Eulerian and Hamiltonian graphs. For example, all the complete graphs are Hamiltonian, whereas they are not Eulerian. On the other hand, $K_{m,n}$ are mn Hamiltonian, whereas if is Eulerian when both m and n are even.

Check your Progress:

1. G is a Hamiltonian graph. Then G is
   (a) 1-connected (b) 2-connected

(c) 3-connected (d) 4-connected

Answer: 1. (b)

3.7 2-Factorable Graphs

Theorem 137. *(Peterson). Every $2k$-regular graph, $k \geq 1$, is 2-factorable.*

Proof. Let G be a $2k$-regular graph with $V = \{v_1, v_2, \cdots, v_n\}$. We may assume without loss of generality that G is connected. (Otherwise, we can consider the components of G separately.) Since each vertex of G is of even degree, by Theorem 126, G is Eulerian. Let T be an Euler tour of G. Form a bipartite graph H with bipartition (V, W), where $V\{v_1, v_2, \cdots, v_n\}$ and $W = \{w_1, w_2, \cdots, w_n\}$ and in which v_i is made adjacent to w_j if and only if v_j follows v_i immediately in T . Since at every vertex of G there are k incoming edges and k outgoing edges along T, H is k-regular. Hence, by Theorem 110, H is 1-factorable. Let the k 1-factors be $M_1, \cdots, M_k$. Label the edges of M_i with the label i; $1 \leq i \leq k$. Then the k edges incident at each v_i of H receive the k labels $1, 2, \cdots, k$, and hence if the edges v_iw_j and v_jw_r are in M_p, $1 \leq p \leq k$, identifying the vertex w_j with the vertex v_j for each j in M_p gives an edge labeling to G in which the edges v_iv_j and v_jv_r receive the label p. It is then clear that the edges of M_p yield a 2-factor of G with label p. Note that v_i is nonadjacent to w_i in H, $1 \leq i \leq k$. Since this is true for each of the 1-factors M_p, $1 \leq p \leq k$, we get a 2-factorization of G into k 2-factors. □

Theorem 138. *K_{2p+1} is 2-factorable into p Hamilton cycles.*

Proof. Label the vertics K_{2p+1} as $v_0, v_1, \cdots, v_{2p}$. For $i = 0, 1, \cdots, p$, let P be the path $v_iv_{i-1}v_{i+1}v_{i-2}v_{i+2}\cdots v_{i+p-1}v_{i-(p-1)}$ (suffixes taken modulo $2p$) and let C_i ba the Hamilton cycle obtained from P_i by joining

v_{2p} to the end vertices of P_i. The cycles C_i are edge-disjoint. This may seen by placing the $2p$ vertices $v_0, v_1, \cdots, v_{2p-1}$ symmertically on a circle and placing v_{2p} at the center of the circle and nothing that the edges $v_i v_{i-1}, v_{i+1} v_{i-2}, \cdots, v_{i+p-1} v_{i-p}$ from a set of p parallel chords of this circle. □

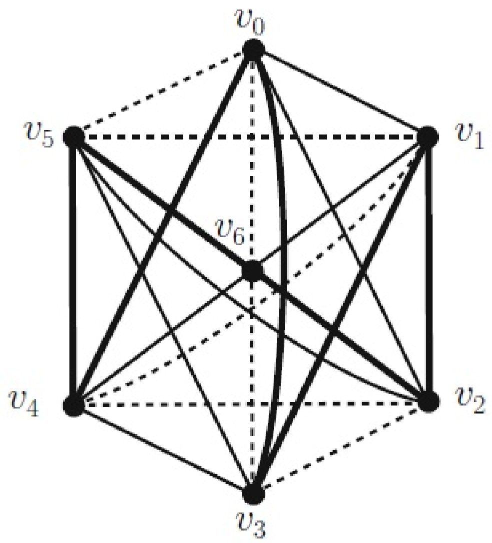


Figure 3.18: Parallel chords and edge-disjoint Hamilton cycles in K_7

Figure 3.18 displays the three sets of parallel chords and three edge-disjoint Hamilton cycles in K_7. The 2-factors are

$$F_1 : v_6 v_0 v_5 v_1 v_4 v_2 v_3 v_6,$$

$$F_2 : v_6 v_1 v_0 v_2 v_5 v_3 v_4 v_6,$$

$$F_3 : v_6 v_2 v_1 v_3 v_0 v_4 v_5 v_6$$

.

Let us sum up

1. A subset S of V is independent if and only if $V \setminus S$ is a covering of G.
2. The number of vertices in a maximum independent set of G is called the independence number of G.
3. The number of vertices in a minimum covering of G is the covering number of G.
4. A matching of G is a set of independent edges.
5. A matching M of a graph G is maximum if and only if G has no M-augmenting path.
6. A k-factor of G is a factor of G that is k-regular.
7. A 1-factor of G is called a perfect matching of G.
8. Every connected 3-regular graph having no cut edges has a 1-regular.
9. An Euler trail in a graph G is a spanning trial in G that contains all the edges of G.
10. A graph is called Hamiltonian if it has a spanning cycle.
11. A graph is called traceable if it has a spanning path.
12. The graph K_{2p+1} is 2-factorable into p Hamilton cycles.
13. Every $2k$-regular graph is 2-factorable.
14. A graph G with at least three vertices is Hamilton-connected if any two vertices of G are connected by Hamilton path in G.
15. For $n \geq 4$, C_n is not Hamilton-connected.

Check your progress

1. The independent number is denoted by

   (a) α (b) β (c) α' (d) β'

2. For any graph for which $\delta > 0$, $\alpha' + \beta' =$

   (a) δ (b) Δ (c) n (d) m

3. A matching in G is a set of

   (a) independent vertices (b) independent edges

   (c) parallel edges (d) loops

4. The notation $\beta'(G)$ denotes the size of

   (a) minimum covering of G

   (b) minimum edge covering of G

   (c) matching in G

   (d) maximum matching on G

5. A 1-factor of G is a ____ of a

   (a) matching (b) subgraph (c) spanning (d) perfect matching

6. A component of a graph is odd if it has an odd number of

   (a) vertices (b) edges (c) cycles (d) paths

7. A spanning cycle is called

   (a) an Euler trial (b) an Euler tour

   (c) a Hamilton cycle (d) a Hamilton path

8. If G is Hamiltonian, then for every non-empty proper subset S of V

   (a) $w(G - s) \leq |S|$ (b) $w(G - s) < |S|$

   (c) $w(G - S) = |S|$ (d) $w(G - S) > |S|$

9. If G is a simple graph with $n \geq 3$ and $\delta \geq \frac{n}{2}$, then G is

   (a) Eulerian (b) traceable

   (c) Hamiltonian (d) Hamiltonian- connected

10. In a graph G, if any two vertices are connected by a spanning path then G is

    (a) Eulerian (b) traceable

    (c) Hamiltonian (d) Hamiltonian- connected

11. The graph K_{2p+1} is 2-factorable is _______ Hamilton cycles.

    (a) $p+1$ (b) p (c) $p-1$ (d) $\frac{p}{2}$

12. For a simple 2-connected graph G, if $\alpha \leq k$, then G is

    (a) factorable (b) traceable

    (c) Hamiltonian (d) Eulerian

13. if $cl(G)$ is complete, then G is

    (a) Hamiltonian (b) Eulerian

    (c) traceable (d) complete

14. A spanning path is called a

    (a) Euler trial (b) Euler tour

    (b) trial (d) Hamilton path

15. A graph having a spanning path is called

    (a) Eulerian (b) Hamiltonian (c) Traceable (d) factor

Answers

1. (a) 2. (c) 3. (b) 4. (b) 5. (d)

6. (a) 7. (c) 8. (a) 9. (c) 10. (d)

11. (b) 12. (c) 13. (a) 14. (d) 15. (c)

Exercises

1. Show that Herschel graph is bipartite.
2. Show that $K_{m,n}, m \neq n$ has no spanning cycle.

References

1. R. Balakrishnan and K. Ranganathan, A Text Book of Graph Theory, second ed., Springer, New York, 2012.
2. J.A. Bondy and U.S.R. Murty, Graph Theory with Application.

Suggested Readings

1. S. Arumugam Issac, introduction to Graph Theory.

Unit 4

Graph Colorings

Objectives

1. To learn the significance of vertex colorings.
2. To gain knowledge about the applications of graph coloring concepts.
3. To introduce various coloring parameters.
4. To apply the concept of edge coloring to solve time table problem.
5. To compute chromatic polynomials.

4.1 Introduction

Graph theory would not be what it is today if there had been no coloring problems. In fact, a major portion of the 20th-century research in graph theory has its origin in the four-color problem.

In this chapter, we present basic concepts and important results concerning vertex colorings and edge colorings of graphs.

4.2 Vertex Colorings

Definition 139. *The* **chromatic number** $\chi(G)$ *of a graph* G *is the minimum number of independent subsets that partition the vertex set of* G*. Any such minimum partition is called a chromatic partition of* $V(G)$*.*

Definition 140. *A* **vertex coloring** *of* G *is a map* $f : V \to S$*, where* S *is a set of distinct colors; it is proper if adjacent vertices of* G *receive distinct colors of* S*. This means that if* $uv \in E(G)$*, then* $f(u) \neq f(v)$*. Thus,* $\chi(G)$ *is the minimum cardinality of* S *for which there exists a proper vertex coloring of* G *by colors of* S*. Clearly, in any proper vertex coloring of* G*, the vertices that receive the same color are independent. The vertices that receive a particular color make up a color class.*

Definition 141. *The* **chromatic number** *of a graph* G *is the minimum number of colors needed for a proper vertex coloring of* G*.* G *is* k*-chromatic if* $\chi(G) = k$*.*

Definition 142. *A* k**-coloring** *of a graph* G *is a vertex coloring of* G *that uses at most* k *colors.*

Definition 143. *A graph* G *is said to be* **k-colorable** *if* G *admits a proper vertex coloring using at most* k *colors.*

It is clear that $\chi(K_n) = n$. Further, $\chi(G) = 2$ if and only if G is bipartite having at least one edge. In particular, $\chi(T) = 2$ for any tree T with at least one edge (since any tree is bipartite).

$$\chi(C_n) = \begin{cases} 2 & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd.} \end{cases} \tag{4.1}$$

Theorem 144. *For any graph G with n vertices and independent number α,*

$$\frac{n}{\alpha} \leq \chi \leq n - \alpha + 1.$$

Proof. There exists a chromatic partition $\{V_1, V_2, \cdots, V_\chi\}$ of V. Since each V_i is independent, $|V_i| \leq \alpha,\ 1 \leq i \leq \chi$. Hence, $n = \sum_{i=1}^{\chi} |V_i| \leq \alpha\chi$, and this gives the inequality on the left.

To prove the inequality on the right, consider a maximum independent set S of α vertices. Then the subsets of $V \setminus S$ of cardinality 1 together with S yield a partition of V into $(n-\alpha)+1$ independent subsets. □

For a simple graph G, the number $\chi^c = \chi^c(G) = \chi(G^c)$ the chromatic number of G^c is the minimum number of subsets in a partition of $V(G)$ into subsets each inducing a complete subgraph of G.

Theorem 145. *(Nordhaus and Gaddum [**?**]). For any simple graph G,*

$$2\sqrt{n} \leq \chi + \chi^c \leq n+1, \textit{ and } n \leq \chi\chi^c \leq \left(\frac{n+1}{2}\right)^2.$$

Proof. Let $\chi(G)$ and let $V_1, V_2, \cdots, V_k$ be the k color classes in a chromatic partition of G. Then $\sum_{i=1}^{k} |V_i|$, and so $max_{1\leq i\leq k}|V_i| \geq \frac{n}{k}$. Since each V_i is an independent set of G, it induces a complete subgraph in G^c. Hence, $\chi^c \geq max_{1\leq i\leq k}|V_i|$, and so $\chi\chi^c = k\chi^c \geq k \circ max_{1\leq i\leq k}|V_i| \geq k \circ \frac{n}{k} = n$. Further, since the arithmetic mean of χ and χ^c is greater than or equal to their geometric mean, $\frac{\chi+\chi^c}{2} \geq \sqrt{\chi\chi^c} \geq \sqrt{n}$. Hence, $\chi + \chi^c \geq 2\sqrt{n}$. This establishes both the lower bounds.

To show that $\chi + \chi^c \leq n+1$, we use induction on n. When $n = 1$, $\chi = \chi^c = 1$, and so we have equality in this case. So assume that $\chi + \chi^c \leq (n-1)+1 = n$ for all graphs G having $n-1$ vertices, $n \geq 2$.

Let H be any graph with n vertices, and let v be any vertex of H. Then $G = H - v$ is a graph with $n - 1$ vertices and $G^c = (H - v)^c = H^c - v$. By the induction assumption, $\chi(G) + \chi(G^c) \leq n$.

Now $\chi(H) \leq \chi(G) + 1$ and $\chi(H^c) \leq \chi(G^c) + 1$. If either $\chi(H) \leq \chi(G)$ or $\chi(H^c) \leq \chi(G^c)$, then $\chi(H)+\chi(H^c) \leq \chi(G)+\chi(G^c)+1 \leq n+1$. Suppose then $\chi(H) = \chi(G)+1$ and $\chi(H^c) = \chi(G^c)+1$. $\chi(H) = \chi(G)+1$ implies that removal of v from H decreases the chromatic number, and hence $d_{H(v)} \geq \chi(G)$. [if $d_{H(v)} < \chi(G)$,then in any proper coloring of G with $\chi(G)$ colors at most $\chi(G) - 1$ colors would have been used to color the neighbors of v in G, and hence v can be given one of the left-out colors, and therefore we have a coloring of H with $\chi(G)$ colors. Hence, $\chi(H) = \chi(G)$, a contradiction.] For a similar reason, $\chi(H^c) = \chi(G^c)+1$ implies that $n - 1 - d_H(v) = d_{H^c}(v) \geq \chi(G^c)$; thus, $\chi(G) + \chi(G^c) \leq d_H(v) + n - 1 - d_H(v) = n - 1$. This implies, however, that $\chi(H) + \chi(H^c) = \chi(G) + \chi(G^c) + 2 \leq n + 1$. Finally, applying the inequality $\sqrt{\chi\chi^c} \leq \frac{\chi+\chi^c}{2}$, we get $\chi\chi^c \leq \left(\frac{\chi+\chi^c}{2}\right)^2 \leq \left(\frac{n+1}{2}\right)^2$. □

Let us Sum Up:

We studied definition of proper coloring, chromatic number and some interesting results. Note that $\chi(h) \geq 3$ of G contains a odd cycle. If G is a $k-$ partite graph, then $\chi(h) \leq k$.

4.3 Critical Graphs

Definition 146. *A graph G is called* **critical** *if for every proper subgraph H of G, $\chi(H) < \chi(G)$. Equivalently, $\chi(G - e) < \chi(G)$ for each edge e of G. Also, G is k-critical if it is k-chromatic and critical.*

Theorem 147. *If G is k-critical, then $\delta(G) \geq k-1$.*

Proof. Suppose $\delta(G) \leq k-2$. Let v be a vertex of minimum degree in G. Since G is k-critical, $\chi(G-v) = \chi(G) - 1 = k-1$. Hence, in any proper $(k-1)$-coloring of $G-v$, at most $(k-2)$ colors would have been used to color the neighbors of v in G. Thus, there is at least one color, say c, that is left out of these $k-1$ colors. If v is given the color c, a proper $(k-1)$-coloring of G is obtained. This is impossible since G is k-chromatic. Hence, $\delta(G) \geq (k-1)$. □

Corollary 148. *For any graph G, $\chi(G) \leq 1 + \Delta(G)$.*

Proof. Let G be a k-chromatic graph, and let H be a k-critical subgraph of G. Then $\chi(H) = \chi(G) = k$. By Theorem 147, $\delta(H) \geq k-1$, and hence $k \leq 1 + \delta(H) \leq 1 + \Delta(H) \leq 1 + \Delta(G)$. □

Theorem 149. *In a critical graph G, no vertex cut is a clique.*

Proof. Suppose G is a k-critical graph and S is a vertex cut of G that is a clique of G (i.e., a complete subgraph of G). Let $H_i,\ 1 \leq i \leq r$, be the components of $G \setminus S$, and let $G_i = G[V(H_i) \cup S]$. Then each G_i is a proper subgraph of G and hence admits a proper $(k-1)$-coloring. Since S is a clique, its vertices must receive distinct colors in any proper $(k-1)$-coloring of G_i. Hence, by fixing the colors for the vertices of S, and coloring for each i the remaining vertices of G_i so as to give a proper $(k-1)$-coloring of G_i, we obtain a proper $(k-1)$-coloring of G. This contradicts the fact that G is k-chromatic. □

Theorem 150. *(Brook's theorem) If a connected graph G is neither an odd cycle nor a complete graph, then $\chi(G) \leq \Delta(G)$.*

Proof. If $\Delta(G) \leq 2$, then G is either a path or a cycle. For a path G (other than K_1 and K_2), and for an even cycle G, $\chi(G) = 2 = \Delta(G)$. According to our assumption, G is not an odd cycle. So let $\Delta(G) \geq 3$.

The proof is by contradiction. Suppose the result is not true. Then there exists a minimal graph G of maximum degree $\Delta(G) = \Delta \geq 3$ such that G is not Δ- colorable, but for any vertex v of G, $G - v$ is Δ-colorable.

Claim 1. Let v be any vertex of G. Then in any proper Δ-coloring of $G - v$, all the Δ colors must be used for coloring the neighbors v in G. Otherwise, if some color i is not represented in $N_G(v)$, then v could be colored using i, and this would give a Δ-coloring of G, a contradiction to the choice of G. Thus, G is a Δ-regular graph satisfying Claim 1.

For $v \in V(G)$, let $N(v) = \{v_1, v_2, \cdots, v_g\}$. In a proper Δ-coloring of $G = v = H$, let v_i receive color $i,\ 1 \leq i \leq \Delta$. For $i \neq j$, let H_{ij} be the subgraph of H induced by the vertices receiving the ith and jth colors.

Claim 2. v_i and v_j belong to the same component of H_{ij}. Otherwise, the colors i and j can be interchanged in the component of H_{ij} that contains the vertex v_j. Such an interchange of colors once again yields a proper Δ-coloring of H. In this new coloring, both v_i and v_j receive the same color, namely, i, a contradiction to Claim 1. This proves Claim 2.

Claim 3. If C_{ij} is the component of H_{ij} containing v_i and v_j, then C_{ij} is a path in H_{ij}. As before, $N_H(v_i)$ contains exactly one vertex of color j. Further, C_{ij} cannot contain a vertex, say y, of degree at least 3; for, if y is the first such vertex on a $v_i - v_j$ path in C_{ij} that has been colored, say, with i, then at least three neighbors of y in C_{ij} have the color j. Hence, we can recolor y in H with a color different from both i and j, and in this new coloring of H, v_i and v_j would belong to distinct components of H_{ij}

(see Fig. 4.1a). (Note that by our choice of y; any $v_i - v_j$ path in H_{ij} must contain y.) But this contradicts Claim 3.

Claim 4. $C_{ij} \cap C_{ik} = \{v_i\}$ for $j \neq k$. Indeed, if $w \in C_{ij} \cap C_{ik},\ w \neq v_i$, then w is adjacent to two vertices of color j on C_{ij} and two vertices of color k on C_{ik} (see Fig. 4.1b). Again, we can recolor w in H by giving a color different from the colors of the neighbors of w in H. In this new coloring of H, v_i and v_j belong to distinct components of H_{ij}, a contradiction to Claim 2. This completes the proof of Claim 4.

We are now in a position to complete the proof of the theorem. By hypothesis, G is not complete. Hence,G has a vertex v, and a pair of nonadjacent vertices v_1 and v_2 in $N_G(v)$. Then the $v_1 - v_2$ path C_{12} in H_{12} of $H = G - v$ contains a vertex $y(\neq v_2)$ adjacent to v_1. Naturally, y would receive color 2. Since $\Delta \geq 3$, by Claim 1, there exists a vertex $v_3 \in N_G(v)$. Now interchange colors 1 and 3 in the path C_{13} of H_{13}. This would result in a new coloring of $H = G - v$. Denote the $v_i - v_j$ path in H under this new coloring by C'_{ij} (see Fig. 4.1c). Then $y \in C'_{23}$ since v_1 receives color 3 in the new coloring (whereas y retains color 2). Also, $y \in C_{12} - v_1 - C'_{12}$. Thus, $y \in C'_{23} \cap C'_{12}$. This contradicts Claim 4 (since $y \neq v_2$), and the proof is complete. □

Definition 151. *Let f be a k-coloring (not necessarily proper) of G, and let $(V_1, V_2, \cdots, V_k)$ be the color classes of G induced by f. Coloring f is* **pseudocomplete** *if between any two distinct color classes, there is at least one edge of G. f is complete if it is pseudocomplete and each $V_i,\ 1 \leq i \leq k$, is an independent set of G. Thus, $\chi(G)$ is the minimum k for which G has a complete k-coloring f.*

Definition 152. *The* **achromatic number** *$a(G)$ of a graph G is the maximum k for which G has a complete k-coloring.*

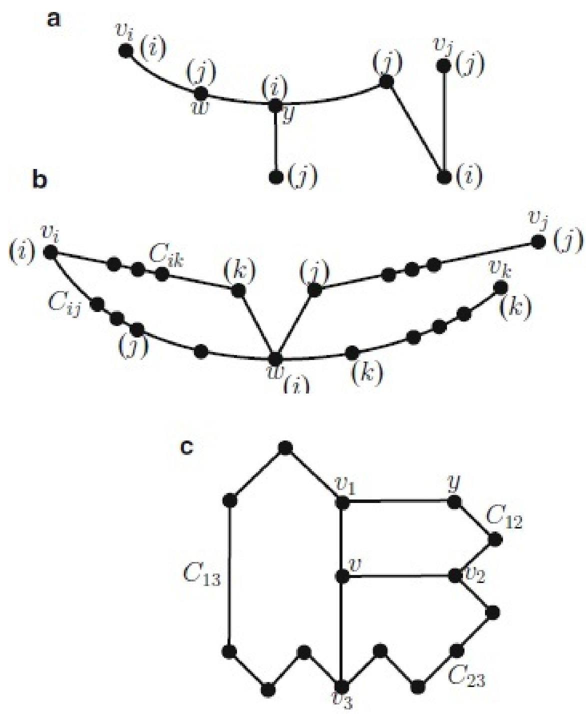


Figure 4.1: Graphs for proof of Theorem 150 (The numbers inside the parentheses denote the vertex colors)

Definition 153. *The* **pseudoachromatic number** $\psi(G)$ *of* G *is the maximum* k *for which* G *has a pseudocomplete* k*-coloring.*

Example 154. *Figure 4.2 gives (a) a chromatic, (b) an achromatic, and (c) a pseudoachromatic coloring of* $K_{3,3} - e$.

It is clear that for any graph G, $\chi(G) \leq a(G) \leq \psi(G)$.

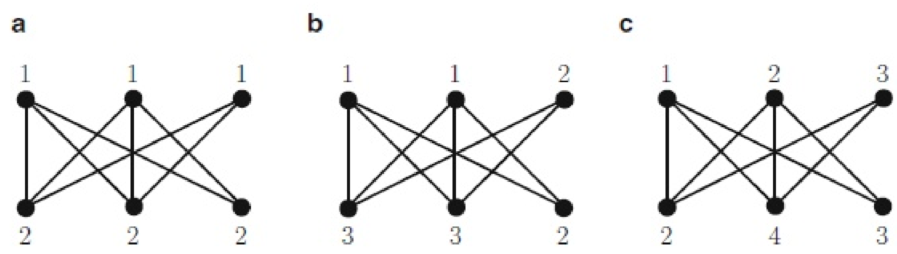


Figure 4.2: Different colorings of $K_{3,3} - e$

Definition 155. *A b-**coloring** of a graph G is a proper coloring with the additional property that each color class contains a color-dominating vertex (c.d.v.), that is, a vertex that has a neighbor in all the other color classes. The b-chromatic number of G is the largest k such that G has a b-coloring using k colors, it is denoted by $b(G)$.*

For any graph G and shows that $\chi(G) \leq b(G)$. Note that $b(K_n) = n$ while $b(K_{m,n}) = 2$.

Let G be a bipartite graph with bipartition (X, Y). A vertex $x \in X$ (respectively, $y \in Y$) is called a full vertex (or a charismatic vertex) of X (respectively, Y) if it is adjacent to all the vertices of Y (respectively, X).

Theorem 156. *[**?**] Let G be a nontrivial connected graph. Then $b(G) = 2$ if and only if G is bipartite and has a full vertex in each part of the bipartition.*

Proof. Suppose G is bipartite and has a full vertex in each part, say $x \in X$ and $y \in Y$. Naturally, in any b-coloring, the color class containing x, say W_1, is a subset of X and that containing y, say W_2, is a subset of Y. If G has a third color class W_3 disjoint from W_1 and W_2, then W_3 must have a c.d.v. adjacent to a vertex of W_1 and a vertex of W_2. This is impossible, as G is bipartite. Therefore, $b(G) = 2$.

Conversely, let $b(G) = 2$. Then $\chi(G) = 2$ and therefore G is bipartite. Let (X, Y) be the bipartition of G. Assume that G does not have a full vertex in at least one part, say, X. Let $x_1 \in X$. As x_1 is not a full vertex, there exists a vertex $y_1 \in Y$ to which it is not adjacent. Let X_1 be the maximal subset of X such that $V1 = X_1\{y_1\}$ is independent in G. Now choose a new vertex $x_2 \in X \setminus X_1$. Again, as X has no full vertex, we can find a $y_2 \in Y \setminus \{y_1\}$ to which x_2 is not adjacent. Let X_2 be the maximal subset of $X \setminus X_1$ such that $V_2X_2 \cup \{y_1\}$ is independent in G. In this way, all the vertices of X would be exhausted and let $V_1, V_2, \cdots, V_k$ be the independent sets thus formed. Also, let Y_0 denote the set of uncovered vertices of Y, if any. Since G is connected, $G \neq \langle V_1 \cup Y_0 \rangle,\ i,\ j,\ l \in \{1, 2, \cdots, k\}$. Hence, $k \geq 2$ when $Y_0 \neq \phi$; and $k \geq 3$ when $Y_0 = \phi$. Thus, the partition $V = V_1 \cup V_2 \cup \cdots \cup V_k \cup \{V_{k+1} = Y_0\}$ has at least 3 parts. If each of these parts has a c.d.v., we get a contradiction to the fact that $b(G) = 2$. If not, assume that the class V_l has no c.d.v. Then for each vertex x of V_l, there exists a color class $V_j,\ j \neq l$, having no neighbor of x. Then x could be moved to the class V_j. In this way, the vertices in V_l can be moved to the other V_i's without disturbing independence. Let us call the new classes $V'_1, V'_2, \cdots, V'_{l-1}, V'_{l+1}, \cdots, V'_{k+1}$, If each of these color classes contains a c.d.v., we get a contradiction as $k \geq 3$. Otherwise, argue as before and reduce the number of color classes. As G is connected, successive reductions should end up in at least three classes, contradicting the hypothesis that $b(G) = 2$. □

Let us Sum Up:

Brook's theorem stats that $\chi(G) \leq \Delta$, where G is neither an odd cycle nor a complete graph, where as $\chi(G) \leq \Delta + 1$ for any G.

Check your progress:

Chromatic number of the Petersan graph is _____

a) 2   b) 3   (c) 4   d) ≤ 3

Answer: (b).

4.4 Edge Colorings of Graphs

Definition 157. *An* **edge coloring** *of a loopless graph G is a function $\pi : E(G) \to S$, where S is a set of distinct colors; it is proper if no two adjacent edges receive the same color. Thus, a proper edge coloring π of G is a function $\pi : E(G) \to S$ such that $\pi(e) \neq \pi(e')$ whenever edges e and e' are adjacent in G, and it is a proper k-edge coloring of G if $|S| = k$.*

Definition 158. *The minimum k for which a loopless graph G has a proper k*-**edge coloring** *is called the edge-chromatic number or chromatic index of G. It is denoted by $\chi'(G)$. G is k-edge-chromatic if $\chi'(G) = k$.*

Theorem 159. *If G is a bipartite graph $\chi'(G) = \Delta(G)$.*

Proof. The proof is by induction on the size (i.e., number of edges) m of G. The result is true for $m = 1$. Assume the result for bipartite graphs of size at most $m - 1$. Let G have m edges. Let $e = uv \in E(G)$. Then $G - e$ has [since $\Delta(G - e) \leq \Delta(G)$] a proper Δ-edge coloring, say c. Out of these Δ colors, suppose that one particular color is not represented at both u and v. Then in this coloring the edge uv can be colored with this color, and a proper Δ-edge coloring of G is obtained.

In the other case (that is, in the case in which each of the Δ colors is represented either at u or at v in $G - e$), since the degrees of u and v in $G - e$ are at most $\Delta - 1$, there exists a color out of the Δ colors that is not represented in $G - e$ at u, and similarly there exists a color not represented at v. Thus, if color j is not represented at u in c, then j is represented at v in c, and if color i is not represented at v in c, then i is represented at u in c. Since G is bipartite and u and v are not in the same parts of the bipartition, there can exist no $u - v$ path in G in which the colors alternate between i and j.

Let P be a maximal path in $G - e$ starting from u in which the colors of the edges alternate between i and j. Interchange the colors i and j in P. This would still yield a proper edge coloring of $G - e$ using the Δ colors in which color i is not represented at both u and v. Now color the edge uv by the color i. This results in a proper Δ-edge coloring of G. □

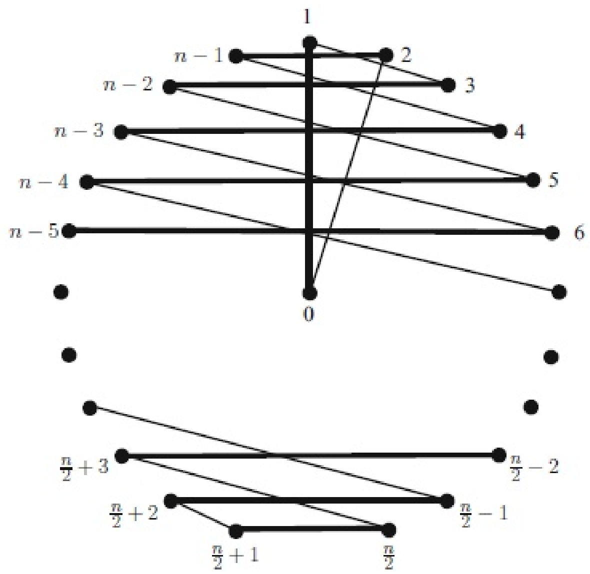


Figure 4.3: Graph for proof of Theorem 160

Theorem 160. $\chi'(K_n) = \begin{cases} n-1 & \textit{if } n \textit{ is even} \\ n & \textit{if } n \textit{ is odd} \end{cases}$

Proof. (Berge) Since K_n is regular of degree $n-1$, $\chi'(K_n) \geq n-1$.

Case 1. n is even. We show that $\chi'(K_n) \leq n-1$ by exhibiting a proper $(n-1)$- edge coloring of K_n. Label the n vertices of K_n as $0, 1, \cdots, n-1$. Draw a circle with center at 0 and place the remaining $n-1$ numbers on the circumference of the circle so that they form a regular $(n-1)$-gon (Fig. 4.3). Then the $\frac{n}{2}$ edges $(0,1)$, $(2, n-1)$, $(3, n-2)$, $\cdots$, $\left(\frac{n}{2}, \frac{n}{2}+1\right)$ form a 1-factor of K_n. These $\frac{n}{2}$ edges are the thick edges of Fig.4.3. Rotation of these edges through the angle $\frac{2\pi}{n-1}$ in succession gives $(n-1)$

edge-disjoint 1-factors of K_n. This would account for $\frac{n}{2}(n-1)$ edges and hence all the edges of K_n. (Actually, the above construction displays a 1-factorization of K_n when n is even.) Each 1-factor can be assigned a distinct color. Thus, $\chi'(K_n) \leq n - 1$. This proves the result in Case 1.

Case 2. n is odd. Take a new vertex and make it adjacent to all the n vertices of K_n. This gives K_{n+1}. By Case 1, $\chi'(K_{n+1}) = n$. The restriction of this edge coloring to K_n yields a proper n-edge coloring of K_n. Hence, $\chi'(K_n) \leq n$. However, K_n cannot be edge colored properly with $n - 1$ colors. This is because the size of any matching of K_n can contain no more than $\frac{n-1}{2}$ edges, and hence $n - 1$ matchings of K_n can contain no more than $\frac{(n-1)^2}{2}$ edges. But K_n has $\frac{n(n-1)}{2}$ edges. Thus, $\chi'(K_n) \geq n$, and hence $\chi'(K_n) = n$. □

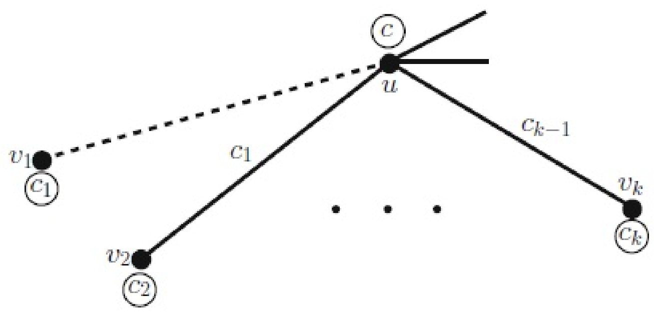


Figure 4.4: Graph for proof of Theorem 161

Theorem 161. *(Vizing-Gupta). For any simple graph* G, $\Delta(G) \leq \chi'(G) \leq 1 + \Delta(G)$.

Proof. In a proper edge coloring of G, $\Delta(G)$, colors are to be used for the edges incident at a vertex of maximum degree in G. Hence, $\chi'(G) \geq \Delta(G)$.

We now prove that $\chi'(G) \leq 1 + \Delta$, where $\Delta = \Delta(G)$.

If G is not $(1 + \Delta)$-edge-colorable, choose a subgraph H of G with a maximum possible number of edges such that H is $(1 + \Delta)$-edge-colorable. We derive a contradiction by showing that there exists a subgraph H_0 of G that is $(1+\Delta)$-edge-colorable and has one edge more than H.

By our assumption, G has an edge $uv_1 \notin E(H)$. Since $d(u) \leq \Delta$, and $1 + \Delta$ colors are being used in H, there is a color c that is not represented at u (i.e., not used for any edge of H incident at u). For the same reason, there is a color c_1 not represented at v_1. (See Fig. 4.4, where the color not represented at a particular vertex is enclosed in a circle and marked near the vertex.)

There must be an edge, say uv_2 of H, colored c_1 otherwise, uv_1 can be assigned the color c_1, and $H \cup (uv_1)$, which has one edge more than H, would have a proper $(1 + \Delta)$-edge coloring. Again, there is a color, say c_2, not represented at v_2. Then as above, there is an edge uv_3 colored c_2 and there is a color, say c_3, not represented at v_3.

In this way, we construct a sequence of edges $\{uv_1, uv_2, \cdots, uv_k\}$ such that color c_i is not represented at vertex $v_i,\ 1 \leq i \leq k$, and the edge uv_{j+1} receives the color $c_j,\ 1 \leq j \leq k - 1$ (see Fig. 4.4).

Suppose at some stage, say the rth stage, where $1 \leq r \leq k$, c (the missing color at u) is not represented at v_r. We then "cascade "(i.e., shift in order) the colors $c_1, \cdots, c_{r-1}$ from $uv_2, uv_3, \cdots, uv_r$ to $uv_1, uv_2, \cdots, uv_{r-1}$. Under this new coloring, c is not represented both at u and at v_r, and therefore we can color uv_r with c. This yields a proper $(1 + \Delta)$-edge coloring to $H \cup (uv_1)$, contradicting the choice of H. Hence, we may assume that c is represented at each of the vertices $v_1, v_2, \cdots, v_k$.

Now we need to know why the sequence of edges $uv_i,\ 1 \leq i \leq k$, had stopped. There are two possible reasons. Either there is no edge incident to u that is colored c_k, or the color $c_k = c_j$ for some $j < k - 1$ and so has already been represented at u. Note that the sequence must stop at some finite stage since $d(u)$ is finite; however, it may as well stop before all the edges incident to u are exhausted.

If c_k is not represented at u in H, then we can cascade as before so that uv_i gets color $c_i,\ 1 \le i \le k-1$, and then color uv_k with color c_k. Once again, we have a contradiction to our assumption on H.

Thus, we must have $c_k = c_j$ for come $j < k - 1$. In this case, cascade the colors $c_1, c_2, \cdots, c_j$ so that uv_i has color $c_i,\ 1 \le i \le j$, and leave uv_{j+1} uncolored (Fig. 4.5). Let $S = (H \cup (uv_i)) - uv_{j+1}$. Then S and H have the same number of edges.

Now consider S_{cc_j}, the subgraph of S defined by the edges of S with colors c and c_j. Clearly, each component of S_{cc_j} is either an even cycle or a path in which the adjacent edges alternate with colors c and c_j.

Now, c is represented at each of the vertices $v_1, v_2, \cdots, v_k$, and in particular at v_{j+1} and v_k. But c_j is not represented at v_{j+1} and v_k, since we have just moved c_j to uv_j, and $c_j = c_k$ is not represented at v_k. Hence in S_{cc_j}, the degrees of v_{j+1} and v_k are both equal to 1. Moreover, c_j is represented at u, but c is not. Therefore, u also has degree 1 in S_{cc_j}. As each component of S_{cc_j} is either a path or an even cycle, not all of $u,\ v_{j+1}$, and v_k can be in the same component of S_{cc_j} (since a nontrivial path has only two vertices of degree 1).

If u and v_{j+1} are in different components of S_{cc_j}, interchange the colors c and $c + j + 1$ in the component containing v_{j+1}. Then c is not represented at both u and v_{j+1}, and so we can color the edge uv_{j+1} with c. This gives a $(1 + \Delta)$-edge coloring to the graph $S \cup (uv_{j+1})$.

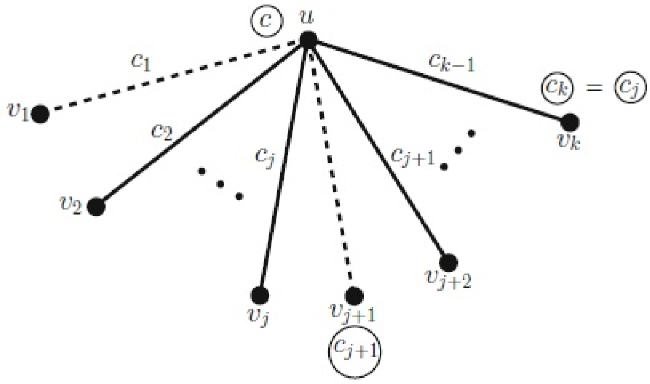


Figure 4.5: Another graph for proof of Theorem 161

Suppose then u and v_{j+1} are in the same components of S_{cc_j}. Then, necessarily, v_k is not in this component. Interchange c and c_j in the component containing v_k. In this case, further cascade the colors so that uv_i has color $c_i,\ 1 \leq i \leq k-1$. Now color uv_k with color c.

Thus, we have extended our edge coloring of S with $1+\Delta$ colors to one more edge of G. This contradiction proves that $H = G$, and thus $\chi'(G) = 1+\Delta$. □

Definition 162. *Graph for which $\chi' = \Delta$ are called Class 1 graphs and those for which $\chi' = 1+\Delta$ are called Class 2 graphs.*

Example 163. *Bipartite graphs are of class 1, whereas the Peterson graph and any simple cubic graph with a cut edge are of class 2.*

Lemma 164. *Let M and N be disjoint matchings of a graph G with $|M| > |N|$. Then there are disjoint matchings M' and N' of G with $|M'| = |M|-1$ and $|N'| = |N|+1$ and with $M' \cup N' = M \cup N$.*

Proof. Consider the subgraph $H = G[M \cup N]$. Each component of H is either an even cycle or a path with edges alternating between M and N. Since $|M| > |N|$, some path component P of H must have its initial and terminal edges in M. Let $P = v_0e_1v_1e_2v_2 \cdots e_{2r+1}v_{2r+1}$.

Now set

$$M' = (M \setminus \{e_1, e_2, \cdots, e_{2r+1}\}) \cup \{e_2, e_4, \cdots, e_{2r}\}$$

and

$$N' = (N \setminus \{e_2, e_4, \cdots, e_{2r}\}) \cup \{e_1, e_3, \cdots, e_{2r+1}\}$$

M' and N' are disjoint matchings of G satisfying the conditions the lemma.

□

Theorem 165. *If G is a bipartite graph (with m edges), and if $m \geq t \geq \Delta$, then there exist t disjoint matchings $M_1,\ M_2, \cdots, M_t$ of G such that*

$$E = M_1 \cup M_2 \cup \cdots \cup M_t$$

and for $1 \leq i \leq t$,

$$\lfloor m/t \rfloor \leq |M_i| \leq \lceil m/t \rceil$$

.

Proof. By Theorem 159, $\chi' = \Delta$. Hence $E(G)$ can be partitioned into Δ matchings $M'_1, M'_2, \cdots, M'_\Delta$. So for $t \geq \Delta$, there exist disjoint matchings $M'_1, M'_2, \cdots, M'_t$, where $M'_i = \phi$ for $\Delta + 1 \leq i \leq t$, and

$$E = M'_1 \cup M'_2 \cup \cdots \cup M'_t$$

. Now repeatedly apply Lemma 164 to pairs of matching that differ by more than one in size. This would eventually result in matchings $M_1, M_2, \cdots, M_t$ of G satisfying the condition stated in the theorem. □

Let us Sum Up:

In this section, we have studied definition of edge chromatic number, critical graphs and important theorems for Vizing. Further, Vizing's theorem classifies graphs into Class 1 and Class 2.
Note that bipartite graph are Class 1.

Check your Progress:

Edge chromatic number of Herschel graph is _____

a) 2 b) 3 c) 4 d) 5

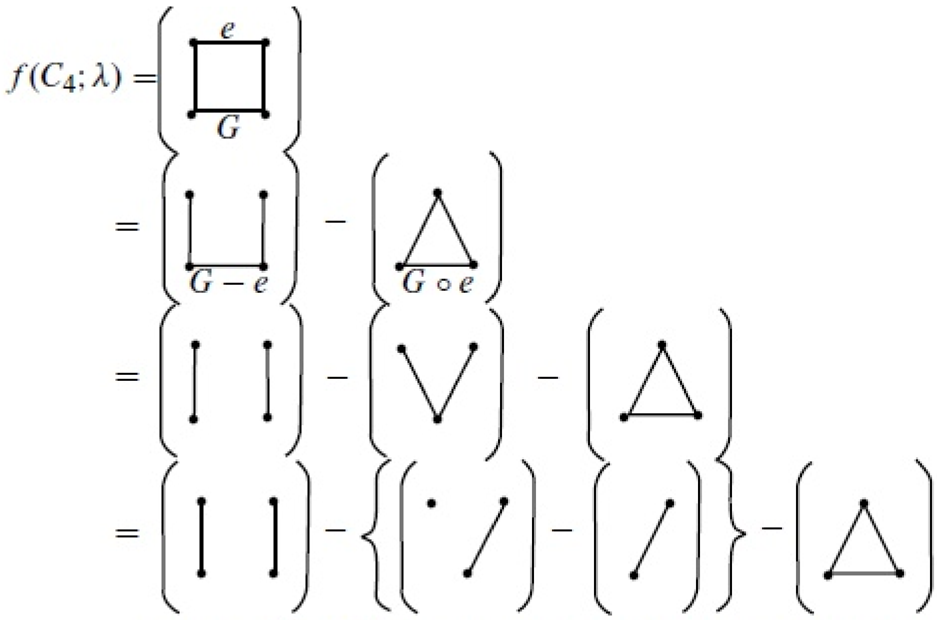
Answer: (c).

4.5 Chromatic Polynomials

For a graph G and a given set of λ colors, the function $f(G;\lambda)$ is defined to be the number of ways of (vertex) coloring G properly using the λ colors. Hence, $f(G;\lambda)$ when G has no proper λ-coloring. Clearly, the minimum λ for which $f(G;\lambda)$ is the chromatic number $\chi(G)$ of G. It is easy to see that $f(K_n;\lambda) = \lambda(\lambda-1)\cdots(\lambda-n+1)$ for $\lambda \geq n$. This is because any vertex of K_n can be colored by any one of the given λ colors. After coloring a vertex of K_n, a second vertex of K_n can be colored by any one of the remaining $(\lambda-1)$ colors, and so on. In particular, $f(K_3;\lambda) = \lambda(\lambda-1)(\lambda-2)$. Also, $f(K_n^c : \lambda) = \lambda''$.

Theorem 166. *Let G be any graph. Then* $f(G;\lambda) = f(G-e;\lambda) - f(G \circ e;\lambda)$ *for any edge* e *of G.*

Proof. $f(G-e;\lambda)$ denotes the number of proper colorings of $G-e$ using λ colors. Hence, it is the sum of the number of proper colorings of $G-e$ in which u and v receive the same color and the number of proper colorings of $G - e$ in which u and v receive distinct colors. The former number is $f(G \circ e; \lambda)$, and the latter number is $f(G; \lambda)$. □



$$= (\lambda(\lambda - 1)^2 - \{\lambda^2(\lambda - 1) - \lambda(\lambda - 1)\} - \lambda(\lambda - 1)(\lambda - 2)$$
$$= \lambda^4 - 4\lambda^3 + 6\lambda^2 - 3\lambda.$$

the function $f(G; \lambda)$ is called the chromatic polynomial of the graph G.

Theorem 167. *For a simple graph G of order n and size m, $f(G; \lambda)$ is a monic polynomial of degree n in λ with integer coefficients and constant term zero. In addition, its coefficients alternate in sign and the coefficient of λ^{n-1} is $-m$:*

Proof. The proof is by induction on m. If $m = 0$, G is K_n^c and $f(K_n^c; \lambda) = \lambda^n$, and if $m = 1$, G is K_2 and $f(K_2; \lambda) = \lambda^2 - \lambda$. and the statement of the theorem is trivially true in these cases. Suppose now that the theorem holds for all graphs with fewer than m edges, where $m \geq 2$. Let G be any simple graph of order n and size m, and let e be any edge of G. Both $G - e$ and $G \circ e$ (after removal of multiple edges, if necessary) are simple graphs with at most $m - 1$ edges, and hence, by the induction hypothesis,

$$f(G - e; \lambda) = \lambda^n - a_0\lambda^{n-1} + a_1\lambda^{n-2} - \cdots + (-1)^{n-1}a_{n-2}\lambda$$

, and

$$f(G \circ e; \lambda) = \lambda^{n-1} - b_1\lambda^{n-2} + \cdots + (-1)^{n-2}b_{n-2}\lambda,$$

where $a_0, \cdots, a_{n-2};\ b_1, \cdots, b_{n-2}$ are nonnegative integers (so that the coefficients alternate in sign), and a_0 is the number of edges in $G - e$, which is $m - 1$. By Theorem 166, $f(G; \lambda) = f(G - e; \lambda) - f(G \circ e; \lambda)$, and hence

$$f(G; \lambda) = \lambda^n - (a_0 + 1)\lambda^{n-1} + (a_1 + b_1)\lambda^{n-2} - \cdots + (-1)^{n-1}(a_{n-2} + b_{n-2})\lambda$$

. Since $a_0 + 1 = m$, $f(G; \lambda)$ has all the stated properties. □

Theorem 168. *A simple graph G on n verices is a tree if and only if $f(G; \lambda) = \lambda(\lambda - 1)^{n-1}$.*

Proof. Let G be a tree. We prove that $f(G; \lambda) = \lambda(\lambda - 1)^{n-1}$ by induction on n. If $n = 1$, the result is trivial. So assume the result for trees with at most $n = 1$ vertices, $n \geq 2$. Let G be a tree with n vertices, and e be a pendent edge of G. By Theorem 166, $f(G; \lambda) = f(G - e; \lambda) - f(G \circ e; \lambda)$. Now,$G - e$ is a forest with two component trees of orders $n - 1$ and 1, and hence $f(G - e; \lambda) = (\lambda(\lambda - 1)^{n-2})\lambda$. Since $G \circ e$ is a tree with $n - 1$ vertices, $f(G \circ e; \lambda) = \lambda(\lambda - 1)^{n-2}$. Thus, $f(G; \lambda) = (\lambda(\lambda - 1)^{n-2})\lambda - \lambda(\lambda - 1)^{n-2} = \lambda(\lambda - 1)^{n-1}$.

Conversely, assume that G is a simple graph with $f(G; \lambda) = \lambda(\lambda -$

$1)^{n-1} = \lambda^n - (n-1)\lambda^{n-1} + \cdots + (-1)^{n-1}\lambda$. Hence, by Theorem 167, G has n vertices and $n-1$ edges. Further, the last term, $(-1)^{n-1}\lambda$, ensures that G is connected. Hence, G is a tree. □

Let us Sum Up

1. The chromatic number $\chi(G)$ of a graph G is the minimum number of independent subset that partition the vertex set of G.

2. A graph G is called critical if for every proper subgraph H of G, $\psi(H) < \psi(a)$.

3. if G is k-critical, then $\delta(G) \geq k - 1$

4. in a critical graph G, no vertex cut is a clique.

5. If a connected graph G is neither an odd cycle nor a complete graph, then $\psi(G) \leq \Delta(G)$.

6. An edge coloring of a loopless graph G is a function $\pi : E(G) \to S$, where S a set if distinct colors; it is proper if no two adjacent edges receive the same color

7. The edge chromatic number is denoted by ψ'.

8. If G is a bipartite graph, then $\psi'(G) = \Delta(G)$.

9. $\chi' = \begin{cases} n-1 & \text{if } n \text{ is even} \\ n & \text{if is } n \text{ is odd} \end{cases}$

10. For any simple G, $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.

11. Graphs for critical $\chi' = \Delta$ are called class 1 graphs.

12. Graphs for which $\chi' = \Delta + 1$ are called class 2 graphs.

13. The chromatic polynomial of the complete graph K_n is $f(K_n; \lambda) = \lambda(\lambda - 1) \cdots (\lambda - n + 1),\ \lambda \geq n$.

14. For any graph G, $f(G; \lambda) = f(G - e; \lambda) - f(G \circ -e; \lambda)$.

15. A simple graph G on n vertices is a tree if and only if $f(G; \lambda) = \lambda(\lambda - 1^{n-1})$.

Check Your Progress

1. A graph G is k-colorable, if
   (a) $\chi(G) = k$ (b) $\chi(G) \leq k$ (c) $\chi < k$ (d) $\chi(G) \geq k$.
2. Chromatic number of the complete graph K_n is
   (a) 1 (b) $n - 1$ (c) n (d) $n + 1$.
3. $\chi(G) = 2$ if and only if G is
   (a) cycle (b) star (c) tree (d) bipartite
4. For any tree T with at least one edge, $\chi(T) =$
   (a) 1 (b) 2 (c) $n - 1$ (d) n.
5. If n is odd, $\chi(C_n) =$
   (a) 1 (b) 2 (c) 3 (d) n
6. If $\chi = 1$, then G is complete
   (a) complete (b) connected (c) disconnected
   (d) totally disconnected
7. A graph G is 1-critical if and only if G is
   (a) C_n (b) K_1 (c) K_2 (d) K_n.
8. A graph G is 2-critical if and only if G is
   (a) C_n (b) K_1 (c) K_2 (d) K_n
9. A graph G is critical if for every proper subgraph H of G
   (a) $\chi(H) < \chi(G)$ (b) $\chi(H) \leq \chi(G)$ (c) $\chi(H) = \chi(G)$
   (d) $\chi(H) > \chi(G)$

10. In a critical graph G, no vertex cut is a _____

    (a) cycle (b) clique (c) block (d) tree

11. if a connected graph G is neither an odd cycle nor a complete graph, then

    (a) $\chi(G) = \Delta(G)$ (b) $\chi(G) < \Delta(G)$ (c) $\chi(G) > \Delta(G)$

    (d) $\chi(G) \leq \Delta(G)$

12. If G is a bipartite graph, then

    (a) $\chi'(G) = \Delta(G)$ (b) $\chi'(G) \leq \Delta(G)$ (c) $\chi'(G) \geq \Delta(G)$

    (d) $\chi'(G) = 2$.

13. If n is odd, then $\chi'(K_n) =$

    (a) 3 (b) n (c) $n - 1$ (d) 1

14. Graphs for which $\chi' = \Delta$ are called

    (a) class 1 graphs (b) class 2 graphs

    (c) bipartite (d) complete

15. The chromatic polynomial of K_3 is

    (a) λ (b) $\lambda(\lambda - 1)$ (c) $\lambda(\lambda - 1)(\lambda - 2)$

    (d) $\lambda(\lambda - 1)(\lambda - 2)(\lambda - 3)$

Answers

1. (a) 2. (c) 3. (d) 4. (b) 5.(c) 6.(d) 7. (b) 8. (c) 9.(a) 10.(b)
11. (d) 12. (a) 13. (b) 14.(a) 15.(c)

Exercises

1. Show that Herschel graph is bipartite.
2. Show that $K_{m,n}, m \neq n$ has no spanning cycle.

References

1. R. Balakrishnan and K. Ranganathan, A Text Book of Graph Theory, second ed., Springer, New York, 2012.
2. J.A. Bondy and U.S.R. Murty, Graph Theory with Application.

Suggested Readings

1. S. Arumugam Issac, introduction to Graph Theory.

Unit 5

Planar Graphs

Objectives

1. To discuss the planar and nonplanar graphs.
2. To learn the significance of Euler's formula for planar graphs.
3. To apply Kuratowski graphs in identifying graphs.
4. To gain knowledge about the famous Four color theorem.
5. To provide a foundation for Tait coloring.

5.1 Introduction

The study of planar and nonplanar graphs and, in particular, the several attempts to solve the four-color conjecture have contributed a great deal to the growth of graph theory. Actually, these efforts have been instrumental to the development of algebraic, topological, and computational techniques in graph theory. In this chapter, we present some of the basic results on planar graphs.

5.2 Planar and Nonplanar Graphs

Definition 169. *A graph G is planar if there exists a drawing of G in the plane in which no two edges intersect in a point other than a vertex of G, where each edge is a Jordan arc (that is, a simple arc). Such a drawing of a planar graph G is called a plane representation of G. In this case, we also say that G has been embedded in the plane. A plane graph is a planar graph that has already been embedded in the plane.*

Example 170. *There exist planar as well as nonplanar graphs. In Fig. 5.1, a planar graph and two of its plane representations are shown. Note that all trees are planar as also are cycles and wheels. The Petersen graph is nonplanar.*

Before proceeding further, let us recall here the celebrated Jordan curve theorem. If J is any closed Jordan curve in the plane, the complement of J (with respect to the plane) is partitioned into two disjoint open connected subsets of the plane, one of which is bounded and the other unbounded. The bounded subset is called the interior of J and is denoted by int J. The unbounded subset is called the exterior of J and is denoted by ext J. The Jordan curve theorem (of topology) states that if J is any closed Jordan curve in the plane, any arc joining a point of int J and a point of ext J must intersect J at some point (see Fig. 5.2). Let G be a plane graph. Then the union of the edges (as Jordan arcs) of a cycle C of G form a closed Jordan curve, which we also denote by C. A plane graph G divides the rest of the plane (i.e., plane minus the edges and vertices of G), say π, into one or more faces.

Definition 171. *We say that for points A and B of π, $A \sim B$ if and only if there exists a Jordan arc from A to B in π. Clearly, π is an equivalence*

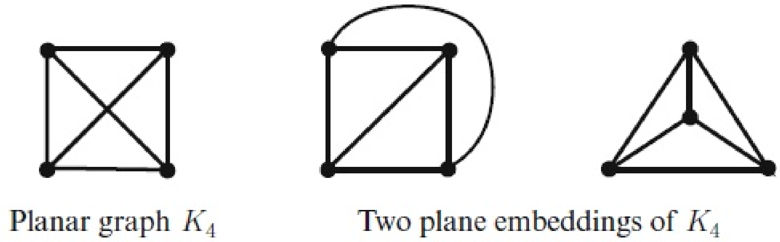


Figure 5.1: A planar graph with two plane embeddings

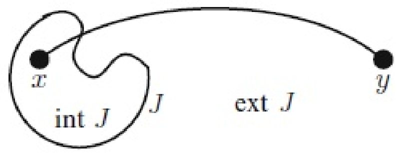


Figure 5.2: Arc connecting point x in int J with point y in int J

relation on π. The equivalence classes of the above equivalence relation are called the faces of G.

Remark 172. 1. *We claim that a connected graph is a tree if and only if it has only one face. Indeed, since there are no cycles in a tree T, the complement of a plane embedding of T in the plane is connected (in the above sense), and hence a tree has only one face. Conversely, it is clear that if a connected plane graph has only one face, then it must be a tree.*

2. *Any plane graph has exactly one unbounded face. The unbounded face is also referred to as the exterior face of the plane graph. All other faces, if any, are bounded. Figure 5.3 represents a plane graph with seven faces.*

Definition 173. *graph is embeddable on a sphere S if it can be drawn on the surface of S so that its edges intersect only at its vertices. Such a drawing, if it exists, is called an embedding of G on S. Embeddings on a sphere are called spherical embeddings.*

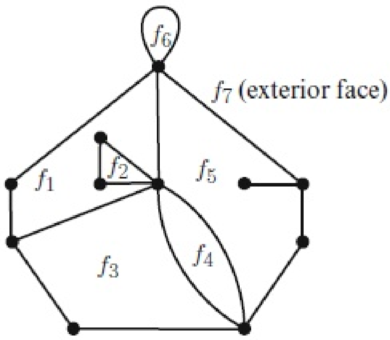


Figure 5.3: A plane graph with seven faces

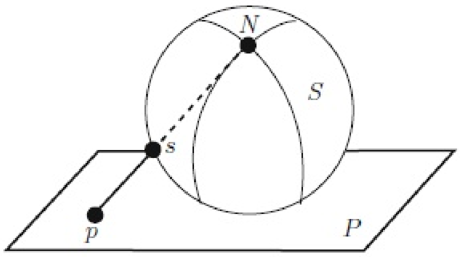


Figure 5.4: Stereographic projection of the sphere S from N

Definition 174. *Let S be a sphere resting on a plane P so that P is a tangent plane to S. Let N be the "north pole,"the point on the sphere diametrically opposite the point of contact of S and P. Let the straight line joining N and a point s of $S \setminus \{N\}$ meet P at p. Then the mapping $\eta : S \setminus \{N\} \to P$ defined by $\eta(s) = p$ is called the stereographic projection of S from N (see Fig. 5.4).*

Theorem 175. *A graph is planar if and only if it is embeddable on a sphere.*

Proof. Let a graph G be embeddable on a sphere and let G' be a spherical embedding of G. The image of G' under the stereographic projection η of the sphere from a point N of the sphere not on G' is a plane representation of G on P. Conversely, if G'' is a plane embedding of G on a plane P, then the inverse of the stereographic projection of G'' on a sphere touching the plane P gives a spherical embedding of G. □

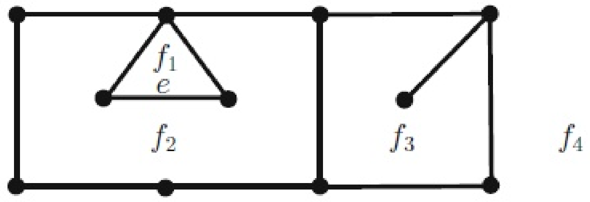


Figure 5.5: Plane graph with four faces

Theorem 176. *(a) Let G be a plane graph and f be a face of G. Then there exists a plane embedding of G in which f is the exterior face.*

(b) Let G be a planar graph. Then G can be embedded in the plane in such a way that any specified vertex (or edge) belongs to the unbounded face of the resulting plane graph.

Proof.

(a) Let n be a point of int f. Let $G' = \sigma(G)$ be a spherical embedding of G and let $N = \sigma(n)$. Let η be the stereographic projection of the sphere with N as the north pole. Then the map $\eta\sigma$ (σ followed by η) gives a plane embedding of G that maps f onto the exterior face of the plane representation $(\eta\sigma)(G)$ of G.

(b) Let f be a face containing the specified vertex (respectively, edge) in a plane representation of G. Now, by part (a) of the theorem, there exists a plane embedding of G in which f becomes the exterior face. The specified vertex (respectively, edge) then becomes a vertex (respectively, edge) of the new unbounded face. □

Remark 177. *1. Let G be a connected plane graph. Each edge of G belongs to one or two faces of G. A cut edge of G belongs to exactly one face, and conversely, if an edge belongs to exactly one face of*

G, it must be a cut edge of G. An edge of G that is not a cut edge belongs to exactly two faces and conversely.

2. *The union of the vertices and edges of G incident with a face f of G is called the boundary of f and is denoted by $b(f)$. The vertices and edges of a plane graph G belonging to the boundary of a face of G are said to be incident with that face. If G is connected, the boundary of each face is a closed walk in which each cut edge of G is traversed twice. When there are no cut edges, the boundary of each face of G is a closed trail in G. (See, for instance, face f_1 of Fig. 5.3.) However, if G is a disconnected plane graph, then the edges and the vertices incident with the exterior face will not define a trail.*

3. *The number of edges incident with a face f is defined as the degree of f. In counting the degree of a face, a cut edge is counted twice. Thus, each edge of a plane graph G contributes two to the sum of the degrees of the faces. It follows that if F denotes the set of faces of a plane graph G, then $\sum_{f \in \mathcal{F}} d(f) = 2m(G)$, where $d(f)$ denotes the degree of the face f.*

   In Fig. 5.5, $d(f_1) = 3,\ d(f_2) = 9,\ d(f_3) = 6$, and $d(f_4) = 8$.

Theorem 178. *A graph G is planar if and only if each of its blocks is planar.*

Proof. If G is planar, then each of its blocks is planar, since a subgraph of a planar graph is planar. Conversely, suppose that each block of G is planar. We now use induction on the number of blocks of G to prove the result. Without loss of generality, we assume that G is connected. If G has only one block, then G is planar.

Now suppose that G has k planar blocks and that the result is true for all connected graphs having $(k-1)$ planar blocks. Choose any end block B_0 of G and delete from G all the vertices of B_0 except the unique cut vertex, say v_0, of G in B_0. The resulting connected subgraph G' of G contains $(k-1)$ planar blocks. Hence, by the induction hypothesis, G' is planar. Let $\bar{G}'$ be a plane embedding of G' such that v_0 belongs to the boundary of the unbounded face, say f' (refer to Theorem 176). Let $\bar{B}_0$ be a plane embedding of B_0 in f' so that v_0 is in the boundary of the exterior face of $\bar{B}_0$. Then (by the identification of v_0 in the two embeddings), $\bar{G}' \cup \bar{B}_0$ is a plane embedding of G. □

Let us Sum Up:

We have studied definitions and some interesting properties of planar graphs. Note that largest complete graph which is planar is K_4. Note that through Petersen graph is 3-regular which is not planar.

5.3 Euler Formula and Its Consequences

Theorem 179. *(Euler formula). For a connected plane graph G, $n-m+\ell = 2$, where n, m,and ℓ denote the number of vertices, edges, and faces of G, respectively.*

Proof. We apply induction on ℓ.

If $\ell = 1$, then G is a tree and $m = n-1$. Hence, $n-m+\ell = 2$.

Now assume that the result is true for all plane graphs with $\ell - 1$ faces, $\ell \geq 2$, and suppose that G has ℓ faces. Since $\ell \geq 2$, G is not a tree, and hence contains a cycle C. Let e be an edge of C. Then e belongs to exactly two faces, say f_1 and f_2, of G and the deletion of e from G

results in the formation of a single face from f_1 and f_2 (see Fig. 5.5). Also, since e is not a cut edge of G, $G - e$ is connected. Further, the number of faces of $G - e$ is $\ell - 1$. So applying induction to $G - e$, we get $n - (m - 1) + (\ell - 1) = 2$, and this implies that $n - m + \ell = 2$. This completes the proof of the theorem. □

Corollary 180. *If G is a simple planar graph with at least three vertices, then* $m \leq 3n \leq 6$.

Proof. Without loss of generality, we can assume that G is a simple connected plane graph. Since G is simple and $n \geq 3$, each face of G has degree at least 3. Hence, if $\mathcal{F}$ denotes the set of faces of G, $\sum_{f \in \mathcal{F}} d(f) \geq 3\ell$. But $\sum_{f \in \mathcal{F}} d(f) = 2m$. Consequently, $2m \geq 3\ell$, so that $\ell \leq \frac{2m}{3}$.

By the Euler formula, $m = n + \ell - 2$. Now $\ell \leq \frac{2m}{3}$ implies that $m \leq n + \left(\frac{2m}{3}\right) - 2$. This gives $m \leq 3n - 6$. □

Example 181. *Show that the complement of a simple planar graph with 11 vertices is nonplanar.*

Proof. Let G be a simple planar graph with $n(G) = 11$. Since G is planar, $m(G) = 3n - 6 = 27$. If G^c were also planar, then $m(G^c) \leq 3n - 6 = 27$. On the one hand, $m(G) + m(G^c) \leq 27 + 27 = 54$, whereas, on the other hand, $m(G) + m(G^c) = m(K_{11}) = \binom{11}{2} = 55$. Hence, we arrive at a contradiction. This contradiction proves that G^c is nonplanar. □

Corollary 182. *For any simple planar graph G,* $\delta(G) \leq 5$.

Proof. If $n \leq 6$, then $\Delta(G) \leq 5$. Hence $\delta(G) \leq \Delta(G) \leq 5$, proving the result for such graphs. So assume that $n \geq 7$. By Corollary 180, $m \leq 3n \leq 6$. Now, $\delta_n \leq \sum_{v \in V(G)} d_G(v) = 2m \leq 2(3n - 6) = 6n - 12$. Hence $n(\delta - 6) \leq -12$. Consequently, $\delta - 6$ is negative, implying that $\delta \leq 5$. □

Theorem 183. *If the girth k of a connected plane graph G is at least 3, then $m \leq \frac{k(n-2)}{(k-2)}$.*

Proof. Let $\mathcal{F}$ denote the set of faces and ℓ, as before, denote the number of faces of G. If $f \in \mathcal{F}$, then $d(f) \geq k$. Since $2m = \sum_{f \in \mathcal{F}} d(f)$, we get $2m \geq k\ell$. By Theorem 183, $\ell = 2 - n + m$. Hence, $2m \geq k(2 - n + m)$, implying that $m(k-2) \leq k(n-2)$. Thus, $m \leq \frac{k(n-2)}{(k-2)}$. □

Corollary 184. *The Petersen graph P is nonplanar.*

Proof. The girth of the Petersen graph P is 5, $n(P) = 10$, and $m(p) = 15$. Hence, if P were planar, $15 \leq \frac{5(10-2)}{5-2}$, which is not true. Hence, P is nonplanar. □

Definition 185. *A graph G is maximal planar if G is planar, but for any pair of nonadjacent vertices u and v of G, $G + uv$ is nonplanar.*

Definition 186. *A plane triangulation is a plane graph in which each of its faces is bounded by a triangle. A plane triangulation of a plane graph G is a plane triangulation H such that G is a spanning subgraph of H.*

To any simple plane graph G that is not already a plane triangulation, we can add a set of new edges to obtain a plane triangulation. The set of new edges thus added need not be unique.

Figure 5.6a is a simple plane graph G and Fig. 5.6b is a plane triangulation of G, Fig. 5.6c is a plane triangulation of G isomorphic to the graph of Fig. 5.6b having only straight-line edges.

Let us Sum Up:

We have studied Euler's formula, an important necessary condition for a graph to be planar. It provides many interesting properties on planar

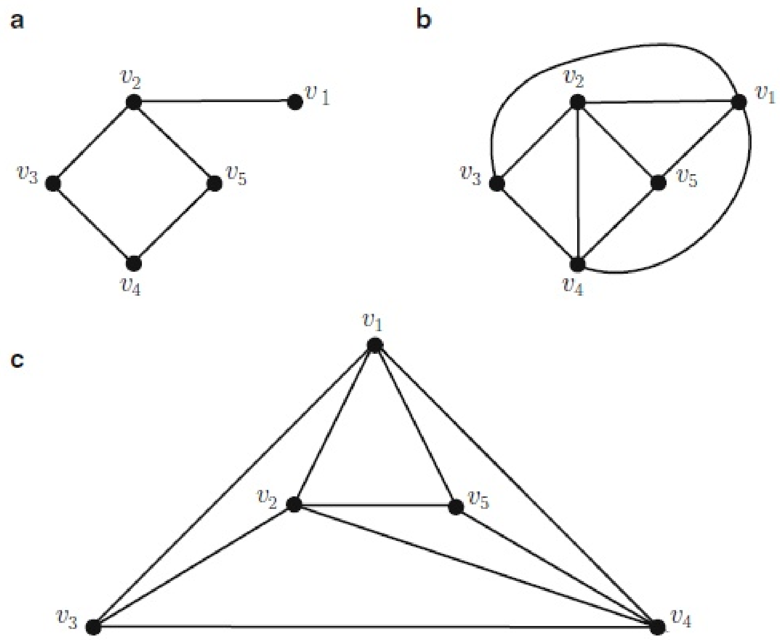


Figure 5.6: (a) Graph G and (b), (c) are plane triangulations of G

graphs. Note that there exists no regular planar graph of degree greater than 5.

5.4 K_5 and $K_{3,3}$ are Nonplanar Graphs

In this section we prove that K_5 and $K_{3,3}$ are nonplanar. These two graphs are basic in Kuratowski's characterization of planar graphs. For this reason, they are often referred to as the two Kuratowski graphs.

Theorem 187. *K_5 is nonplanar*

First proof. This proof uses the Jordan curve theorem. Assume the contrary, namely, K_5 is planar. Let v_1, v_2, v_3, v_4, and v_5 be the vertices of K_5 in a plane representation of K_5. The cycle $C = v_1v_2v_3v_4v_1$ (as a closed Jordan curve) divides the plane into two faces, namely, the interior and the exterior of C. The vertex v_5 must belong either to int C or to ext C. Suppose that v_5 belongs to int C (a similar proof holds if v_5 belongs to ext C). Draw the edges $v_5v_1,\ v_5v_2,\ v_5v_3$ and v_5v_4 in int C. Now there remain two more edges v_1v_3 and v_2v_4 to be drawn. None of these can be drawn in int C, since it is assumed that K_5 is planar. Thus, v_1v_3 lies in ext C. Then one of v_2 and v_4 belongs to the interior of the closed Jordan curve $C_1 = v_1v_5v_3v_1$ and the other to its exterior (see Fig. 5.7). Hence, v_2v_4 cannot be drawn without violating planarity. □

Second proof. If K_5 were planar, it follows from Theorem 183 that $10 \leq \frac{3(5-2)}{(3-2)}$, which is not true. Hence K_5 is nonplanar. □

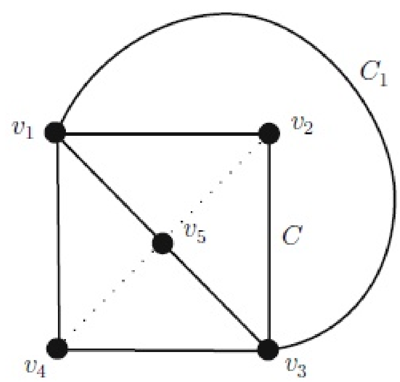


Figure 5.7: Graph for first proof of Theorem 187

Theorem 188. $K_{3,3}$ *is nonplanar.*

First proof. The proof is by the use of the Jordan curve theorem. Suppose that $K_{3,3}$ is planar. Let $U = \{u_1,\ u_2,\ u_3\}$ and $V = \{v_1,\ v_2,\ v_3\}$ be the bipartition of $K_{3,3}$ in a plane representation of the graph. Consider the cycle $C = u_1v_1u_2v_2u_3v_3u_1$. Since the graph is assumed to be planar, the edge u_1v_2 must lie either in the interior of C or in its exterior. For the sake of definiteness, assume that it lies in int C (a similar proof holds if one assumes that the edge u_1v_2 lies in ext C). Two more edges remain to be drawn, namely, u_2v_3 and u_3v_1. None of these can be drawn in int

C without crossing the edge u_1v_2. Hence, both of them are to be drawn in ext C. Now draw u_2v_3 in ext C. Then one of v_1 and u_3 belongs to the interior of the closed Jordan curve $C_1 = u_1v_2u_2v_3u_1$ and the other to the exterior of C_1 (see Fig. 5.8). Hence, the edge v_1u_3 cannot be drawn without violating planarity. This shows that $K_{3,3}$ is nonplanar. □

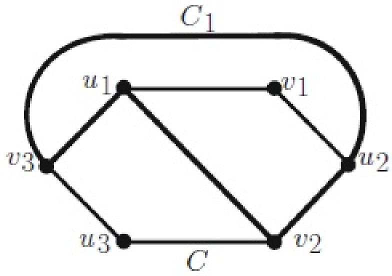


Figure 5.8: Graph for first proof of Theorem 188

Second proof. Suppose $K_{3,3}$ is planar. Let ℓ be the number of faces of $G = K_{3,3}$ in a plane embedding of G and $\mathcal{F}$, the set of faces of G. As the girth of $K_{3,3}$ is 4, we have $m = \frac{1}{2}\sum_{f\in\mathcal{F}} d(f) \geq \frac{4\ell}{2} = 2\ell$. By Theorem 179, $n - m + \ell = 2$. For $K_{3,3}$ have $n = 6$, and $m = 9$. Hence, $\ell = 2 + m - n = 5$. Thus, $9 \geq 2.5 = 10$, a contradiction. □

5.5 Dual of a Plane Graph

Let G be a plane graph. One can form out of G a new graph H in the following way. Corresponding to each face f of G, take a vertex f^* and corresponding to each edge e of G, take an edge e^*. Then edge e^* joins vertices f^* and g^* in H if and only if edge e is common to the boundaries of faces f and g in G. (It is possible that f may be the same as g.) The graph H is then called the dual (or more precisely, the geometric dual) of G (see Fig. 5.9). The definition of the dual implies that $m(G^*) = m(G)$,

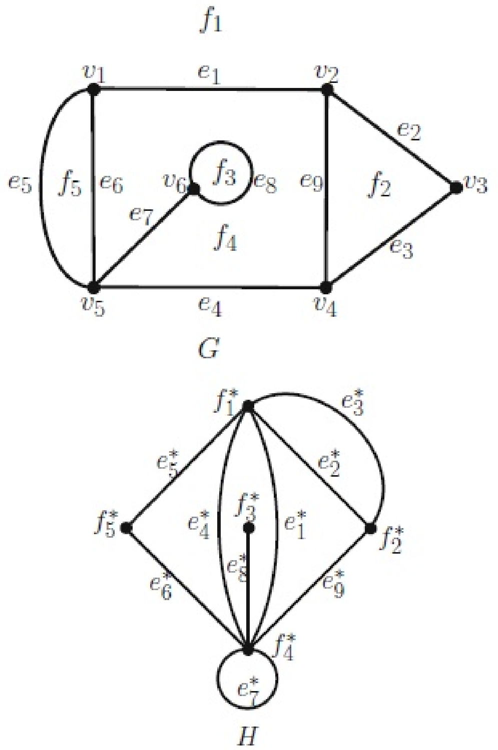


Figure 5.9: A plane graph G and its dual H

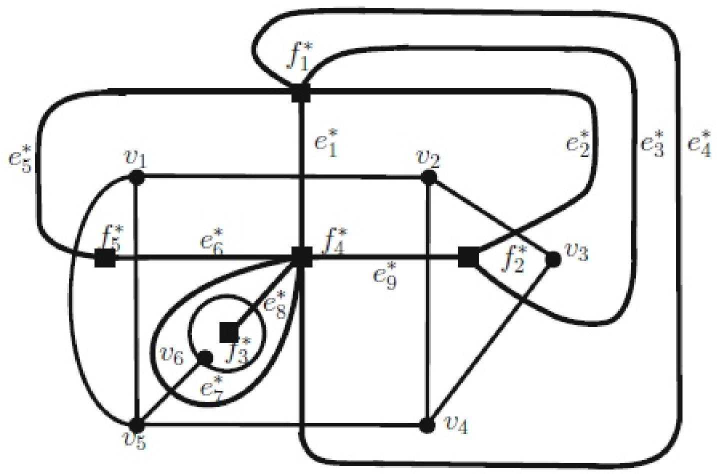


Figure 5.10: Procedure for drawing the dual graph

$n(G^*) = \ell(G)$, and $d_{G^*} = d_G(f)$, where $d_G f$ denotes the degree of the face f of G. From the manner of construction of G^*, it follows that

(i) An edge e of a plane graph G is a cut edge of G if and only if e^* is a loop of G^*, and it is a loop of G if and only if e^* is a cut edge of G^*.

(ii) G^* is connected whether G is connected or not.

5.6 The Four-Color Conjecture and the Heawood Five-Color Theorem

What is the minimum number of colors required to color the world map of countries so that no two countries having a common boundary receive the same color? This simple-looking problem manifested itself into one of the most challenging problems of graph theory, popularly known as the four-color conjecture $(4CC)$.

An assignment of colors to the faces of a plane graph G so that no two faces having a common boundary containing at least one edge receive the same color is a face coloring of G. The face-chromatic number $\chi^*(G)$ of a plane graph G is the minimum k for which G has a face coloring using k colors.

Theorem 189. *Every planar graph is 6-vertex-colorable.*

Proof. The proof is by induction on n, the number of vertices of the graph. The result is trivial for planar graphs with at most six vertices. Assume the result for planar graphs with $n-1$, $n \geq 7$, vertices. Let G be a planar graph with n vertices. By Corollary 182, $\delta(G) \leq 5$, and hence G has a vertex v of degree at most 5. By hypothesis, $G - v$ is 6-vertex-colorable. In any proper 6-vertex coloring of $G - v$, the neighbors of v in G would have used only at most five colors, and hence v can be colored by an unused color. In other words, G is 6-vertex colorable. □

Theorem 190. *(Heawood's five-color theorem). Every planar graph is 5-vertex colorable.*

Proof. The proof is by induction on $n(G) = n$. Without loss of generality, we assume that G is a connected plane graph. If $n \leq 5$, the result is clearly true. Hence, assume that $n \geq 6$ and that any planar graph with fewer than n vertices is 5-vertex-colorable. G being planar, $\delta(G) \leq 5$ by Corollary 182, and so G contains a vertex v_0 of degree not exceeding 5. By the induction hypothesis, $G - v_0$ is 5-vertex-colorable.

If $d(v_0) \leq 4$, at most four colors would have been used in coloring the neighbors of v_0 in G in a 5-vertex coloring of $G - v_0$. Hence, an unused color can then be assigned to v_0 to yield a proper 5-vertex coloring of G.

If $d(v_0) = 5$, but only four or fewer colors are used to color the neighbors of v_0 in a proper 5-vertex coloring of $G - v_0$, then also an unused color can be assigned to v_0 to yield a proper 5-vertex coloring of G.

Hence assume that the degree of v_0 is 5 and that in every 5-coloring of $G - v_0$, the neighbors of v_0 in G receive five distinct colors. Let $v_1,\ v_2,\ v_3,\ v_4$, and v_5 be the neighbors of v_0 in a cyclic order in a plane embedding of G. Choose some proper 5-coloring of $G - v_0$ with colors, say, $c_1,\ c_2, \cdots, c_5$. Let $\{V_1,\ V_2, \cdots, V_5\}$ be the color partition of $G - v_0$, where the vertices in V_i are colored $c_i,\ 1 \leq i \leq 5$. Assume further that $v_i \in V_i,\ 1 \leq i \leq 5$.

Let G_{ij} be the subgraph of $G - v_0$ induced by $V_i \cup V_j$. Suppose v_i and $v_j,\ 1 \leq i,\ j \leq 5$, belong to distinct components of G_{ij}. Then the interchange of the colors c_i and c_j in the component of G_{ij} containing v_i would give a recoloring of $G - v_0$ in which only four colors are assigned to the neighbors of v_0. But this is against our assumption. Hence, v_i and v_j must belong to the same component of G_{ij}. Let $P_{i,j}$ be a $v_i - v_j$ path in G_{ij}. Let C denote the cycle $v_0 v_1 P_{13} v_3 v_0$ in G (Fig. 5.11). Then C separates v_2 and v_4; that is, one of v_2 and v_4 must lie in int C and the other in ext C. In Fig. 5.11, $v_2 \in int\ C$ and $v_4 \in\ extC$. Then P_{24} must cross C at a vertex of C. But this is clearly impossible since no vertex of C receives either of the colors c_2 and c_4. Hence this possibility cannot arise, and G is 5-vertex-colorable. □

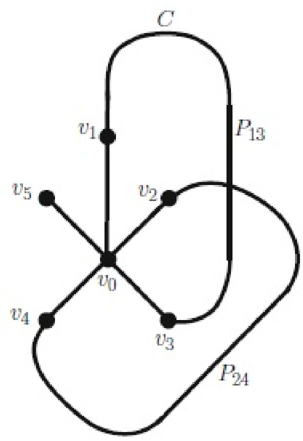


Figure 5.11: Graph for proof of Theorem 190

Let us Sum Up:

We have studied very interesting result on coloring planar graph namely 5-color theorem, the best theoretical proof known. Note that Herschel graph is 2- colorable.

5.7 Hamiltonian Plane Graphs

An elegant necessary condition for a plane graph to be Hamiltonian was given by Grinberg [**?**].

Theorem 191. *Let G be a loopless plane graph having a Hamilton cycle C. Then $\sum_{i=2}^{n}(i-2)(\phi' - \phi'')$ are the numbers of faces of G of degree i contained in int C and ext C, respectively.*

Proof. Let E' and E'' denote the sets of edges of G contained in int C and ext C, respectively, and let $|E'| = m'$ and $|E''| = m''$. Then int C contains exactly $m' + 1$ faces (see Fig. 5.12), and so

$$\sum_{i=2}^{n} \phi'_i = m' + 1. \tag{5.1}$$

(Since G is loopless, $\phi' = \phi'' = 0$). Moreover, each edge in int C is on the boundary of exactly two faces in int C, and each edge of C is on the boundary of exactly one face in int C. Hence, counting the edges of all the faces in int C, we get

$$\sum_{i=2}^{n} i\phi' \;=\; 2m' + n \tag{5.2}$$

Eliminating m' from (5.1) and (5.2), we get

$$\sum_{i=2}^{n} (i-2)\phi'_i = n - 2. \tag{5.3}$$

Similarly,

$$\sum_{i=2}^{n} (i-2)\phi'' = n - 2. \tag{5.4}$$

Equqtions (5.3) and (5.4) give the required result. □

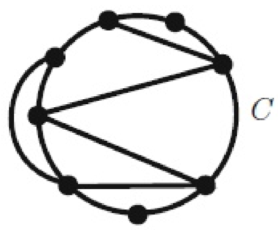


Figure 5.12: Graph for proof of Theorem 191

5.8 Tait Coloring

A 3-edge coloring of a cubic planar graph is often called a Tait coloring.

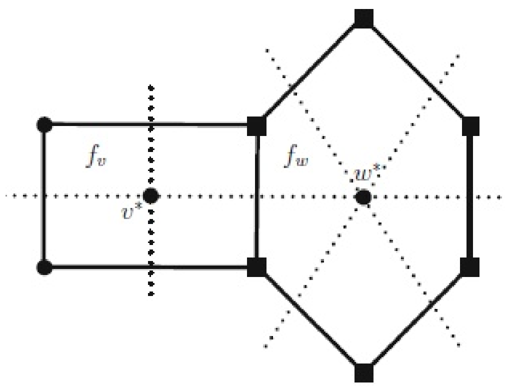


Figure 5.13: Graph for proof of (i) ⇒ (ii) in Theorem 192

Theorem 192. *The following statements are equivalent:*

(i) All plane graphs are 4-vertex-colorable.

(ii) All plane graphs are 4-face-colorable.

(iii) All simple 2-edge-connected cubic planar graphs are 3-edge-colorable (i.e., Tait colorable).

Proof. (i) ⇒ (ii)

Let G be a plane graph. Let G^* be the dual of G. Then, since G^* is a plane graph, it is 4-vertex-colorable. If v^* is a vertex of G^*, and f_v is the face of G corresponding to v^*, assign to f_v the color of v^* in a 4-vertex coloring of *G. Then, by the definition of G^*, it is clear that adjacent faces of G will receive distinct colors. (See Fig. 8.30, in which f_v and f_w receive the colors of v^* and w^*,respectively.) Thus, G is 4-face-colorable.

(ii) ⇒ (iii)

Let G be a plane embedding of a 2-edge-connected cubic planar graph. By assumption, G is 4-face-colorable. Denote the four colors by $(0,0),\ (1,0),\ (0,1),$ and $(1,1)$ the elements of the ring $\mathbb{Z}_2 \times \mathbb{Z}_2$. If e is an edge of G that separates the faces, say f_1 and f_2, color e with the color given by the sum (in $\mathbb{Z}_2 \times \mathbb{Z}_2$) of the colors of f_1 and f_2. Since G has no cut edge, each edge is the common boundary of exactly two faces of G. This gives a 3-edge coloring of G using the colors $(1,0),(0,1)$, and $(1,1)$ since the sum of any two distinct elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$ is not $(0,0)$ (see Fig. 5.14).

(iii) $\Rightarrow$ (i)

Let G be a planar graph.We want to show that G is 4-vertex-colorable. We may assume without loss of generality that G is simple. Let $\tilde{G}$ be a plane embedding of G. Then $\tilde{G}$ is a spanning subgraph of a plane triangulation T, and hence it suffices to prove that T is 4-vertex-colorable.

Let T^* be the dual of T. Then T^* is a 2-edge-connected cubic plane graph. By our assumption, T^* is 3-edge-colorable using, for example, the colors $c_1,\ c_2$, and c_3. Since T^* is cubic, each of the above three colors is represented at each vertex of T^*. Let T^*_{ij} be the edge subgraph of T^* induced by the edges of T^* which have been colored using the colors c_i and c_j. Then T^*_{ij} is a disjoint union of even cycles, and thus it is 2-face-colorable. But each face of T^* is the intersection of a face of T^*_{12} and a face of T^*_{23} (see Fig. 5.15). Now the 2-face colorings of T^*_{12} and T^*_{23} induce a 4-face coloring of T^* if we assign to each face of T^* the (unordered) pair of colors assigned to the faces whose intersection is f. Since $T^* = T^*_{12} \cup T^*_{23}$, this defines a proper 4-face coloring of T^*. Thus, $\chi(G) = \chi(\tilde{G}) \leq \chi(T) = \chi^*(T^*) \leq 4$, and G is 4-vertex-colorable. (Recall that $\chi^*(T^*)$ is the face-chromatic number of T^*.) □

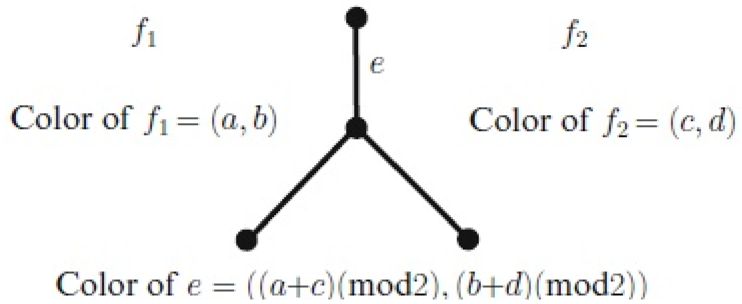


Figure 5.14: Graph for proof of (ii) ⇒ (iii) in Theorem 192

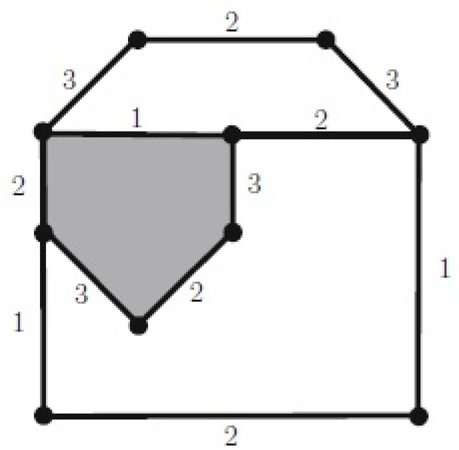


Figure 5.15: Graph for proof of (iii) ⇒ (i) in Theorem 192

Let Us Sum Up

1. A graph G is planar if there is a drawing of G in the plane in which no two edges intersect in a point other than a vertex of G.
2. All trees, cycles and wheels are planar
3. The Petersen graph is nonplanar
4. The Jordan curve theorem states that if J is any closed Jordan curve in the plane, any arc joining a point of int J and a point of ext J must intersect J at some point
5. A plane graph G divides the rest of the plane into one or more faces.
6. A connected graph is a tree if and only if it has only one face.
7. Any plane graph has exactly one unbounded face.
8. Embeddings on a sphere are called spherical embeddings.
9. A graph is planar if and only if it is embeddable on a sphere.
10. A cut edge of G belongs to exactly one face.
11. The number of edges incident with a face f is defined as the degree of f.
12. For a connected plane graph G, $n - m + f = 2$, where n, m and f denote the number of vertices, edges and faces of G, respectively.
13. For any simple planar graph G, $\delta(G) \leq 5$.
14. A graph G is maximal planar if G is planar, but for any pair of non-adjacent vertices u and v of G, $G + uv$ is nonplanar.
15. The graphs K_5 and $K_{3,3}$ are refered to as the two Kuratowski graphs.

Check Your Progress

1. Which one of the following is a nonplanar graph?

   (a) Tress (b) Cycles (c)Wheels (d) petersen graph

2. Which one of the following is a planar graph?

   (a) K_4 (b) K_5 (c) K_6 (d) K_7

3. If $\mathcal{F}$ denotes the set of faces of a plane graph G, then $\sum_{f \in \mathcal{F}} d(f) =$ ___

   (a) $2n(G)$ (b) $2m(G)$ (c) $3n(G)$ (d) $3m(G)$

4. For any connected planar graph G, $n - m + f$=___

   (a) 0 (b) 1 (c) 2 (d) 3

5. If G is a simple planar graph then

   (a) $m = 2n - b$ (b) $m = 3n - b$ (c) $m \leq 3n - b$ (d) $m \leq 2n - 5$

6. For any simple planar graph G,

   (a) $\delta(G) = 5$ (b) $\delta(G) \leq 5$ (c) $\delta(G) \leq 6$ (d) $\delta(G) = 6$

7. If G^* is the dual of a plane graph G, then $M(G^*)$=

   (a) $m(G)$ (b) $n(G)$ (c) $f(G)$ (d) $2m(G)$

8. If G^* is the dual of a plane graph G, then $n(G^*)$=

   (a) $m(G)$ (b) $n(G)$ (c) $f(G)$ (d) $2n(G)$

9. An edge e of a plane graph G is a cut edge of G if and only if e^* is a

   (a) cut edge of G^* (b) pendant edge of G^*

   (c) parallel edge of G^* (d) loop of G^*

10. A 3-edge coloring of a cubic planar graph is called a

    (a) face coloring (b) Tait coloring

    (c) chromatics index (d) b-coloring

11. All simple 2-edge-connected cubic planar graphs are
    (a) Tait colorable (b) b-colorable
    (c) 3-vertex colorable (d) complete-3- colorable

12. Every planar graph is
    (a) 4-vertex colorable (b) 5-vertex colorable
    (c) Tait colorable (d) complete-3-colorable

13. The face-chromatic number of a planes graph G is denoted by
    (a) $\chi(G)$ (b) $\chi'(G)$ (c) $\chi^*(G)$ (d) $\chi'(G^*)$

14. the graph G^{**} is isomorphic to G if and only if G is
    (a) complete (b) bipartite (c) planar (d) connected

15. A graph is planar if and only if each of its blocks is
    (a) complete (b) bipartite (c) planar (d) connected

Answers

1. (d) 2. (a) 3. (b) 4. (c) 5. (c) 6. (b) 7. (a) 8. (c) 9. (d) 10. (b)
11.(a) 12. (b) 13. (c) 14.(d) 15. (c)

Exercises

1. Show that Herschel graph is bipartite.
2. Show that $K_{m,n}, m \neq n$ has no spanning cycle.

References

1. R. Balakrishnan and K. Ranganathan, A Text Book of Graph Theory, second ed., Springer, New York, 2012.

2. J.A. Bondy and U.S.R. Murty, Graph Theory with Application.

Suggested Readings

1. S. Arumugam Issac, introduction to Graph Theory.