PERIYAR UNIVERSITY

NAAC 'A++' Grade - State University - NIRF Rank 56–State Public University Rank 25

SALEM - 636 011, Tamil Nadu, India.

CENTRE FOR DISTANCE AND ONLINE EDUCATION (CDOE)

M.Sc. MATHEMATICS

SEMESTER - I

CORE COURSE: REAL ANALYSIS – I

(Candidates admitted from 2024 onwards)

PERIYAR UNIVERSITY

CENTRE FOR DISTANCE AND ONLINE EDUCATION (CDOE)

M.Sc. Mathematics - 2024 admission onwards

CORE – 2
Real Analysis I

Prepared by:

Centre for Distance and Online Education (CDOE)

PERIYAR UNIVERSITY, Salem - 636011.

REAL ANALYSIS - I

OBJECTIVE: The objective of this course is to work comfortably with functions of bounded variation, Riemann-Stieltjes Integration, convergence of infinite series, infinite product and uniform convergence and its interplay between various limiting operations.

UNIT-I: Functions of bounded variation

Introduction - Properties of monotonic functions - Functions of bounded variation - Total variation - Additive property of total variation - Total variation on $[a, x]$ as a function of x - Functions of bounded variation expressed as the difference of two increasing functions - Continuous functions of bounded variation. Infinite Series Absolute and conditional convergence - Dirichlet's test and Abel's test - Rearrangement of series - Riemann's theorem on conditionally convergent series.

UNIT-II: The Riemann - Stieltjes Integral

Introduction - Notation - The definition of the Riemann - Stieltjes integral - Linear Properties - Integration by parts- Change of variable in a Riemann - Stieltjes integral - Reduction to a Riemann Integral – Euler's summation formula - Monotonically increasing integrators, Upper and lower integrals - Additive and linearity properties of upper, lower integrals - Riemann's condition - Comparison theorems.

UNIT-III: The Riemann-Stieltjes Integral

Integrators of bounded variation-Sufficient conditions for the existence of Riemann-Stieltjes integrals - Necessary conditions for the existence of RS integrals- Mean value theorems -integrals as a function of the interval – Second fundamental theorem of integral calculus-Change of variable -Second Mean Value Theorem for Riemann integral-Riemann-Stieltjes integrals depending on a parameter- Differentiation under integral sign - Interchanging the order of integration

UNIT-IV: Infinite Series and Infinite Products

Double sequences - Double series - Rearrangement theorem for double series - A sufficient condition for equality of iterated series - Multiplication of series – Cesarosummability - Infinite products. Power series - Multiplication of power series - The Taylor's series generated by a function - Bernstein's theorem - Abel's limit theorem - Tauber's theorem.

UNIT-V: Sequences of Functions

Pointwise convergence of sequences of functions - Examples of sequences of real - valued functions - Uniform convergence and continuity - Cauchy condition for uniform convergence - Uniform convergence of infinite series of functions - Riemann - Stieltjes integration – Non-uniform Convergence and term-by-term Integration - Uniform convergence and differentiation - Sufficient condition for uniform convergence of a series - Mean convergence.

TEXT BOOK:

Tom M. Apostol, Mathematical Analysis, 2nd Edition, Addison-Wesley Publishing Company Inc. New York, 1974.

BOOKS FOR SUPPLEMENTARY READING AND REFERENCES:

1. T.M. Apostol, Calculus Vol.2, Multi-Variable Calculus and Linear Algebra with Applications to Differential Equations and Probability, Second Edition - Reprint, John Wiley & Sons, 2016.
2. R.G. Bartle, Real Analysis, John Wiley and Sons Inc., 1976.
3. W. Rudin, Principles of Mathematical Analysis, Third Edition, Mc-Graw Hill Company, New York, 1976.
4. S.C. Malik and S. Arora. Mathematical Analysis, Wiley Eastern Limited, New Delhi, 1991.
5. B.R. Gelbaum and J. Olmsted, Counter Examples in Analysis, Holden day, San Francisco, 1964.
6. A.L. Gupta and N.R. Gupta, Principles of Real Analysis, Pearson Education, (Indian print) 2003.

Contents

Unit 1

Functions of Bounded Variation

Objectives

After the successful completion of this unit, the students are expected to

- recall the basic concepts of monotonic function, bounded functions, convergent series and conditionally convergent series.
- find whether a give series converge (or) diverge.
- understand the fundamental concepts of sequence and series.
- analyse and work with problems related to functions of bounded variation.

1.1 Introduction

Functions of bounded variation is an important class of functions in real analysis. They have important applications across various areas of mathematics and applied disciplines. Functions of bounded variation play a key role in defining and working with the Riemann-Stieltjes integral, a generalization of the Riemann integral. In probability theory, any cumulative distribution function of a probability distribution is a function of bounded variation. In calculus of variations, functions of bounded variation are used to formulate and solve problems that involve optimizing functionals. In image processing, the total variation norm is used to reduce noise in images while preserving edges.

1.2 Properties of Monotonic Functions

Theorem 1.2.1. *Let f be an increasing function defined on $[a, b]$ and let $x_0, x_1, \cdots, x_n$ be n+1 points such that*

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

Then we have the inequality

$$\sum_{k=1}^{n-1} [f(x_k+) - f(x_k-)] \leq f(b) - f(a).$$

Proof. Assume that f is an increasing function on $[a, b]$ and let $x_0, x_1, \cdots, x_n$ be $n + 1$ points such that

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

Let $y_k \in (x_k, x_{k+1})$, for $1 \leq k \leq n - 1$.

Since f is an increasing function on $[a, b]$, $x_k < y_k$ implies $f(x_k+) \leq f(y_k)$. (1)

Also, $x_k > y_{k-1}$ implies $f(x_k-) \geq f(y_{k-1})$.

$\implies -f(x_k-) \leq -f(y_{k-1})$. (2)

Adding (1) and (2), we get

$$f(x_k+) - f(x_k-) \leq f(y_k) - f(y_{k-1}). \tag{3}$$

Putting $k = 1$ in (3), we get

$$f(x_1+) - f(x_1-) \leq f(y_1) - f(y_0).$$

Putting $k = 2$ in (3), we get

$$f(x_2+) - f(x_2-) \leq f(y_2) - f(y_1).$$

and so on. Putting $k = n - 1$ in (3), we get

$$f(x_{n-1}+) - f(x_{n-1}-) \leq f(y_{n-1}) - f(y_{n-2}).$$

Adding all the above inequalities, we get

$$\sum_{k=1}^{n-1} f(x_k+) - f(x_k-) \leq f(y_{n-1}) - f(y_0) \leq f(b) - f(a).$$

Hence the proof. □

Note: The difference $f(x_{k+1}) - f(x_k)$ is the jump of f at x_k. The foregoing theorem tells us that for every finite collection of points x_k in (a, b), the sum of the jumps at these points is always bounded by $f(b) - f(a)$. This result can be used to prove the following theorem.

Theorem 1.2.2. *If f is monotonic on $[a, b]$, then the set of discontinuities of f is countable.*

Proof. Assume that f is increasing on $[a, b]$.

Let S_m be the set of points in (a, b) at which the jump of f exceeds $1/m$, $m > 0$.

If $x_1 < x_2 < \cdots < x_{n-1}$ are in S_m, then, by Theorem 1.2.1, we have

$$\sum_{k=1}^{n-1}[f(x_k+) - f(x_k-)] \leq f(b) - f(a).$$

Since the jump of f exceeds $\frac{1}{m}$, $f(x_k+) - f(x_k-) > 1/m$, for all $k = 1, 2, \cdots, n-1$. Therefore, we have

$$\sum_{k=1}^{n-1}\frac{1}{m} \leq \sum_{k=1}^{n-1}[f(x_k+) - f(x_k-)] \leq f(b) - f(a).$$

which implies that

$$\frac{n-1}{m} \leq f(b) - f(a).$$

Since $f(b), f(a)$ are finite and $\frac{n-1}{m}$ must be finite for any n. The set S_m must be a finite set.

But the set of discontinuities of f in (a, b) is a subset of the union $\bigcup_{m=1}^{\infty} S_m$.

Hence, $\bigcup_{m=1}^{\infty} S_m$ is countable, since the countable union of countable sets is countable.

Hence, the set of discontinuous of f is countable.

Note: If f is decreasing, the argument can be applied to $-f$. □

Let us sum up

- We have discussed the properties of monotonic functions.
- We have shown that set of discontinuities of a monotonic function must be at most countable.

Check your progress

1. Define a monotonic function.

2. Give an example of monotonically increasing function.

1.3 Functions of Bounded Variation

Definition 1.3.1. *If $[a, b]$ is a compact interval, a set of points*

$$P = \{x_0, x_1, x_2, ..., x_n\},$$

satisfying the inequalities

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b,$$

is called a partition of $[a, b]$. The interval $[x_{k-1}, x_k]$ is called the kth subinterval of P and we write $\Delta x_k = x_k - x_{k-1,}$ so that $\sum_{k=1}^{n} \Delta x_k = b - a.$

The collection of all possible partitions of $[a, b]$ will be denoted by $\wp[a, b]$.

Definition 1.3.2. *Let f be defined on $[a, b]$. If $P = \{x_0, x_1, x_2, \cdots, x_n\}$ is a partition of $[a, b]$, write $\Delta f_k = f(x_k) - f(x_{k-1})$, for $k = 1, 2, \cdots, n$. If there exists a positive number M such that*

$$\sum_{k=1}^{n} |\Delta f_k| \leq M,$$

for all partitions of $[a, b]$, then f is said to be bounded variation on$[a, b]$.

Theorem 1.3.3. *If f is monotonic on $[a, b]$, then f is of bounded variation on $[a, b]$.*

Proof. Let $P = \{x_0, x_1, x_2, \cdots, x_n\}$ be a partition of $[a, b]$ such that

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

Since f is an increasing function on $[a, b]$, we have $f(x_{k-1}) \leq f(x_k)$, for all $k = 1, 2, \cdots, n$.

This implies $\Delta f_k = f(x_k) - f(x_{k-1}) \geq 0$ and we have

$$\sum_{k=1}^{n} |\Delta f_k| = \sum_{k=1}^{n} \Delta f_k = \sum_{k=1}^{n} [f(x_k) - f(x_{k-1})]$$

$$\begin{aligned} &= f(x_1) - f(x_0) + f(x_2) - f(x_1) + \\ &\qquad \cdots + f(x_{n-1}) - f(x_{n-2}) + f(x_n) - f(x_{n-1}) \\ &= f(x_n) - f(x_0) = f(b) - f(a). \end{aligned}$$

Hence, f is of bounded variation on $[a, b]$. □

Theorem 1.3.4. *If f is continuous on $[a, b]$ and if f'exists and is bounded in the interior, say $|f'(x)| \leq A$, for all x in (a, b). Then f is of bounded variation on $[a, b]$.*

Proof. Assume that f is continuous on $[a, b]$ and $f(x)$ is differentiable for all x in (a, b) such that $|f'(x)| \leq A$.

Let $P = \{x_0, x_1, x_2, \cdots, x_n\}$ be a partition of $[a, b]$ such that

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

Applying the Mean-Value theorem on (x_{k-1}, x_k), we have

$$f'(t_k) = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}, \quad \text{where } t_k \in (x_{k-1}, x_k),$$

and

$$\Delta f_k = f(x_k) - f(x_{k-1}) = f'(t_k)(x_k - x_{k-1}) = f'(t_k)\Delta x_{k.}$$

Thus, we have

$$\begin{aligned} \sum_{k=1}^{n} |\Delta f_k| &= \sum_{k=1}^{n} |f'(t_k)\Delta x_k| \\ &= \sum_{k=1}^{n} |f'(t_k)||\Delta x_k| \\ &\leq A \sum_{k=1}^{n} |\Delta x_k| \\ &= A(b-a). \end{aligned}$$

Hence, f is of bounded variation on $[a, b]$. □

Theorem 1.3.5. *If f is of bounded variation on $[a, b]$, say $\sum |\Delta f_k| \leq M$, for all partitions of $[a, b]$. Then f is bounded on $[a, b]$. In fact,*

$$|f(x)| \leq |f(a)| + M, \ \ for\ all\ x\ in\ [a, b].$$

Proof. Assume that f is of bounded variation on $[a, b]$. There exists a positive number M such that

$$\sum |\Delta f_k| \leq M \text{ for all partitions of } [a, b]. \tag{4}$$

Let $x \in (a, b)$ be arbitrary and take $P = \{a, x, b\}$ be the special partition such that $a < x < b$.

Now by (4), we have

$$|f(x) - f(a)| + |f(b) - f(x)| \leq M.$$

This implies

$$|f(x) - f(a)| \leq M.$$

Since $|f(x) - f(a)| \geq |f(x)| - |f(a)|$, we have

$$|f(x)| - |f(a)| \leq |f(x) - f(a)| \leq M$$

and hence

$$|f(x)| \leq |f(a)| + M.$$

This inequality remains true when $x = a$ or $x = b$.

Hence, f is bounded on $[a, b]$. □

We provide an example of a continuous function which is not of bounded variation.

Example 1: Consider the function $f : [0, 1] \to \mathbb{R}$ defined by $f(x) = x \cos\left(\frac{\pi}{2x}\right)$, if $x \neq 0$, $f(0) = 0$.

Let $f_1 : (0, 1] \to \mathbb{R}$ be defined by $f_1(x) = x$ and $f_2 : (0, 1] \to \mathbb{R}$ be defined by $f_2(x) = \cos\left(\frac{\pi}{2x}\right)$.

Since f_1 and f_2 are continuous on $(0, 1]$ and we know that product of two continuous functions is continuous, $x\ cos\left(\frac{\pi}{2x}\right)$ is continuous on $(0, 1]$.

Now to prove f is continuous at 0.

Suppose $x_n \to 0$ as $n \to \infty$, we have

$$|f(x_n) - f(0)| = \left|x_n cos\left(\frac{\pi}{2x_n}\right)\right| = |x_n|\left|\cos\left(\frac{\pi}{2x_n}\right)\right| \leq |x_n|.$$

Hence,

$$x_n \to 0 \; as \; n \to \infty, \; f(x_n) \to 0 \; as \; n \to \infty.$$

Thus, f is continuous on $[0,1]$.

Let us choose a P be the partition of $[0,1]$ into $2n$ sub-intervals such that

$$P = \left\{0, \frac{1}{2n}, \frac{1}{2n-1}, \cdots, \frac{1}{3}, \frac{1}{2}, 1\right\}.$$

Then,

$$\begin{aligned}
\sum_{k=1}^{2n} |\Delta f_k| &= \sum_{k=1}^{2n} |f(x_k) - f(x_{k-1})| \\
&= \sum_{k=1}^{2n} \left| x_k cos\left(\frac{\pi}{2x_k}\right) - x_{k-1} cos\left(\frac{\pi}{2x_{k-1}}\right) \right| \\
&= \left| x_1 cos\left(\frac{\pi}{2x_1}\right) - x_0 cos\left(\frac{\pi}{2x_0}\right) \right| + \cdot + \left| x_n cos\left(\frac{\pi}{2x_n}\right) - x_{n-1} cos\left(\frac{\pi}{2x_{n-1}}\right) \right| \\
&= \left| \frac{1}{2n} cos\left(\frac{\pi}{2(\frac{1}{2n})}\right) - 0 \right| + \left| \frac{1}{2n-1} cos\left(\frac{\pi}{2(\frac{1}{2n-1})}\right) - \frac{1}{2n} cos\left(\frac{\pi}{2(\frac{1}{2n})}\right) \right| \\
&\quad + \cdots + \left| cos\left(\frac{\pi}{2}\right) - \frac{1}{2} cos\left(\frac{\pi}{2(\frac{1}{2})}\right) \right| \\
&= \frac{1}{2n} + \frac{1}{2n} + \frac{1}{2n-2} + \frac{1}{2n-2} + \ldots\ldots + \frac{1}{2} + \frac{1}{2} \qquad (1.1) \\
&= 1 + \frac{1}{2} + \ldots.. + \frac{1}{n}. \\
&= \sum_{k=1}^{n} \frac{1}{k} \\
\sum_{k=1}^{n} |\Delta f_k| &= \sum_{k=1}^{n} \frac{1}{k}.
\end{aligned}$$

This is not bounded for all n, since the series $\sum_{k=1}^{\infty} (1/k)$ diverges.

Thus, the function f is not of bounded variation.

Let us sum up

- We have defined a function of bounded variation.
- We have derived sufficient conditions for a function to be of bounded variation.
- We have seen that a function of bounded variation must be bounded.

Check your progress

1. When a function is said to be of bounded variation on $[a, b]$?
2. Give an example of a continuous function which is not of bounded variation.
3. Give an example of a bounded function which is not of bounded variation.

1.4 Total Variation

Definition 1.4.1. *Let f be of bounded variation on $[a, b]$, and let $\sum(P)$denote the sum $\sum_{k=1}^{n} |\Delta f_k|$ corresponding to the partition $P = \{x_0, x_1, ..., x_n\}$ of $[a, b]$. The number*

$$V_f(a, b) = sup\left\{\sum(P) : P \in \wp[a, b]\right\},$$

is called the total variation of f on the interval $[a, b]$.

Note: For simplicity of notation, we will write V_f instead of $V_f(a, b)$.
Since f is of bounded variation on $[a, b]$, the number V_f is finite. Also, $V_f \geq 0$. Since, for each sum $\sum(P) \geq 0$. Moreover, $V_f(a, b) = 0$ if and only if f is constant on $[a, b]$.

Theorem 1.4.2. *Assume that f and g are each of bounded variation on $[a, b]$. Then so are their sum, difference and product. Also, we have*

$$V_{f\pm g} \leq V_f + V_g \;\; and \;\; V_{f.g} \leq AV_f + BV_g,$$

where

$$A = \sup\{|g(x)| : x \in [a, b]\} \text{ and } B = \sup\{|f(x)| : x \in [a, b]\}.$$

Proof. Assume that f and g are each of bounded variation on $[a, b]$.
For any partition of $[a, b]$, there exist a positive number $M_1 > 0$ such that

$$\sum_{k=1}^{n} |\Delta f_k| \leq M_1.$$

Similarly, for any partition of $[a, b]$, there exist a positive number $M_2 > 0$ such that

$$\sum_{k=1}^{n} |\Delta g_k| \leq M_2.$$

Let $M = \max\{M_1, M_2\}$. Therefore, we have

$$\sum_{k=1}^{n} |\Delta f_k| \leq M \quad \text{and} \quad \sum_{k=1}^{n} |\Delta g_k| \leq M.$$

To prove: $f + g$ is of bounded variation on $[a, b]$.

Let $h(x) = f(x) + g(x), x \in [a, b]$ and $P = \{x_0, x_1, x_2, \cdots, x_n\}$ be a partition of $[a, b]$ such that

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

To prove: h is of bounded variation on $[a, b]$, we have

$$\begin{aligned}
\sum_{k=1}^{n} |\Delta h_k| &= \sum_{k=1}^{n} |h(x_k) - h(x_{k-1})| \\
&= \sum_{k=1}^{n} |(f(x_k) + g(x_k)) - (f(x_{k-1}) + g(x_{k-1}))| \\
&= \sum_{k=1}^{n} |f(x_k) - f(x_{k-1}) + g(x_k) - g(x_{k-1})| \\
&= \sum_{k=1}^{n} |\Delta f_k + \Delta g_k| \\
&\leq \sum_{k=1}^{n} |\Delta f_k| + \sum_{k=1}^{n} |\Delta g_k| \\
&\leq M + M = 2M \\
\sum_{k=1}^{n} |\Delta h_k| &\leq 2M.
\end{aligned}$$

Therefore, h is of bounded variation on $[a, b]$.

i.e., $f + g$ is of bounded variation on $[a, b]$.

To prove: $V_{f+g} \leq V_f + V_g$. Consider,

$$\begin{aligned}
&\sum_{k=1}^{n} |\Delta h_k| \leq \sum_{k=1}^{n} |\Delta f_k| + \sum_{k=1}^{n} |\Delta g_k| \\
\implies &\sum_{k=1}^{n} |\Delta h_k| \leq \sup\left\{\sum_{k=1}^{n} |\Delta f_k| : P \in \wp[a, b]\right\} \\
&\qquad\qquad + \sup\left\{\sum_{k=1}^{n} |\Delta g_k| : P \in \wp[a, b]\right\} \\
\implies &\sum_{k=1}^{n} |\Delta h_k| \leq V_f + V_g
\end{aligned}$$

$$\implies \sup\left\{\sum_{k=1}^{n}|\Delta h_k| : P \in \wp[a,b]\right\} \leq V_f + V_g$$
$$\implies V_h(a,b) \leq V_f + V_g$$
$$\implies V_{f+g} \leq V_f + V_g.$$

Now, we prove $f-g$ is of bounded variation on $[a,b]$.

Let $h(x) = f(x) - g(x), x \in [a,b]$ and $P = \{x_0, x_1, x_2, \cdots, x_n\}$ be a partition of $[a,b]$ such that

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

To prove: h is of bounded variation on $[a,b]$. Consider,

$$\begin{aligned}
\sum_{k=1}^{n}|\Delta h_k| &= \sum_{k=1}^{n}|h(x_k) - h(x_{k-1})| \\
&= \sum_{k=1}^{n}|(f(x_k) - g(x_k)) - (f(x_{k-1}) - g(x_{k-1}))| \\
&= \sum_{k=1}^{n}|f(x_k) - g(x_k) - f(x_{k-1}) + g(x_{k-1})| \\
&= \sum_{k=1}^{n}|f(x_k) - f(x_{k-1}) + g(x_{k-1}) - g(x_k)| \\
&\leq \sum_{k=1}^{n}(|f(x_k) - f(x_{k-1})| + |g(x_k) - g(x_{k-1})|) \\
&\leq \sum_{k=1}^{n}(|\Delta f_k| + |\Delta g_k|) \\
&\leq \sum_{k=1}^{n}|\Delta f_k| + \sum_{k=1}^{n}|\Delta g_k| \\
&\leq M + M = 2M.
\end{aligned}$$

Therefore,h is of bounded variation on $[a,b]$.

i.e., $f-g$ is of bounded variation on $[a,b]$.

To prove:$V_{f-g} \leq V_f + V_g$. Consider,

$$\sum_{k=1}^{n}|\Delta h_k| \leq \sum_{k=1}^{n}|\Delta f_k| + \sum_{k=1}^{n}|\Delta g_k|$$
$$\implies \sum_{k=1}^{n}|\Delta h_k| \leq \sup\left\{\sum_{k=1}^{n}|\Delta f_k| : P \in \wp[a,b]\right\}$$

$$+\sup\left\{\sum_{k=1}^{n}|\Delta g_k| : P \in \wp[a,b]\right\}$$

$$\implies \sum_{k=1}^{n}|\Delta h_k| \leq V_f + V_g$$

$$\implies \sup\left\{\sum_{k=1}^{n}|\Delta h_k| : P \in \wp[a,b]\right\} \leq V_f + V_g$$

$$\implies V_h(a,b) \leq V_f + V_g$$

$$\implies V_{f-g} \leq V_f + V_g.$$

To prove: fg is of bounded variation on $[a,b]$.

Let $h(x) = f(x)g(x), x \in [a,b]$ and $P = \{x_0, x_1, x_2, \cdots, x_n\}$ be a partition of $[a,b]$ such that

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

To prove: h is bounded variation on $[a,b]$. Consider,

$$\begin{aligned}
\sum_{k=1}^{n}|\Delta h_k| &= \sum_{k=1}^{n}|h(x_k) - h(x_{k-1})| \\
&= \sum_{k=1}^{n}|f(x_k)g(x_k) - f(x_{k-1})g(x_{k-1})| \\
&= \sum_{k=1}^{n}|f(x_k)g(x_k) - f(x_{k-1})g(x_k) + f(x_{k-1})g(x_k) - f(x_{k-1})g(x_{k-1})| \\
&= \sum_{k=1}^{n}|g(x_k)(f(x_k) - f(x_{k-1})) + f(x_{k-1})(g(x_k) - g(x_{k-1}))| \\
&= \sum_{k=1}^{n}|g(x_k)\Delta f_k + f(x_{k-1})\Delta g_k| \\
&\leq \sum_{k=1}^{n}\Big(|g(x_k)\Delta f_k| + |f(x_{k-1})\Delta g_k|\Big) \qquad (1.2) \\
&\leq \sum_{k=1}^{n}|g(x_k)||\Delta f_k| + \sum_{k=1}^{n}|f(x_{k-1})||\Delta g_k| \\
&\leq \sum_{k=1}^{n}A|\Delta f_k| + \sum_{k=1}^{n}B|\Delta g_k| \\
&= A\sum_{k=1}^{n}|\Delta f_k| + B\sum_{k=1}^{n}|\Delta g_k| \\
&\leq AM + BM \\
&= (A+B)M.
\end{aligned}$$

Hence, h is of bounded variation on $[a,b]$.
i.e., $f.g$ is of bounded variation on $[a,b]$.
To prove: $V_{f.g} \leq AV_f + BV_g$, where $A = \sup\{|g(x)| : x \in [a,b]\}$ and $B = \sup\{|f(x)| : x \in [a,b]\}$.
Consider,

$$\sum_{k=1}^{n} |\Delta h_k| \leq A \sum_{k=1}^{n} |\Delta f_k| + B \sum_{k=1}^{n} |\Delta g_k|$$

$$\implies \sum_{k=1}^{n} |\Delta h_k| \leq A \ \sup\left\{\sum_{k=1}^{n} |\Delta f_k| : P \in \wp[a,b]\right\}$$

$$+ B \ \sup\left\{\sum_{k=1}^{n} |\Delta g_k| : P \in \wp[a,b]\right\} \leq A \ V_f + B \ V_g.$$

$$\implies \sup\left\{\sum_{k=1}^{n} |\Delta h_k| : P \in \wp[a,b]\right\} \leq A \ V_f + B \ V_g.$$

Hence,

$$V_{f.g} \leq A \ V_f + B \ V_g.$$

□

Note: The reciprocal of a function of bounded variation need not be of bounded variation. For example, if $f(x) \to 0$ as $x \to x_0$, then $1/f$ will not be bounded on any interval containing $x_0$and hence, by Theorem 1.3.3 $1/f$ cannot be of bounded variation on such an interval. To extend Theorem 1.4.1 to quotients, it suffices to exclude functions whose values become arbitrarily close to zero.

Theorem 1.4.3. *Let f be of bounded variation on$[a,b]$ and assume that f is bounded away from zero; that is, suppose that there exists a positive number m such that $0 < m \leq |f(x)|$ for all x in $[a,b]$. Then $g = 1/f$ is also of bounded variation on $[a,b]$, and $V_g \leq V_f/m^2$.*

Proof. Assume that,f is of bounded variation on $[a,b]$ and let f be bounded away from zero.
i.e., there exists a positive number m such that $0 < m \leq |f(x)|$ for all$x$ in $[a,b]$.
Let $P = \{x_0, x_1, x_2, \cdots, x_n\}$ be a partition of $[a,b]$ such that

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

Since f is of bounded variation on $[a, b]$, there exist a positive number $M > 0$ such that

$$\sum_{k=1}^{n} |\Delta f_k| \leq M.$$

Let $g = 1/f$.

To prove: g is of bounded variation on $[a, b]$.

Consider,

$$\begin{aligned}
\sum_{k=1}^{n} |\Delta g_k| &= \sum_{k=1}^{n} |g(x_k) - g(x_{k-1})| \\
&= \sum_{k=1}^{n} \left| \frac{1}{f(x_k)} - \frac{1}{f(x_{k-1})} \right| \\
&= \sum_{k=1}^{n} \left| \frac{f(x_{k-1}) - f(x_k)}{f(x_k) f(x_{k-1})} \right| \\
&= \sum_{k=1}^{n} \frac{|f(x_{k-1}) - f(x_k)|}{|f(x_k) f(x_{k-1})|} \\
&= \sum_{k=1}^{n} \frac{|\Delta f_k|}{|f(x_k) f(x_{k-1})|} \quad (\because \Delta f_k = f(x_k) - f(x_{k-1})) \\
&= \sum_{k=1}^{n} \frac{|\Delta f_k|}{|f(x_k)||f(x_{k-1})|} \\
&\leq \sum_{k=1}^{n} \frac{|\Delta f_k|}{m.m} \quad (\because \frac{1}{|f(x)|} \leq \frac{1}{m}, \quad \forall\, x \in [a, b]) \\
&= \sum_{k=1}^{n} \frac{|\Delta f_k|}{m^2} \\
&= \frac{1}{m^2} \sum_{k=1}^{n} |\Delta f_k| \\
&\leq \frac{M}{m^2} \quad (\because \sum_{k=1}^{n} |\Delta f_k| \leq M)
\end{aligned}$$

$$\therefore \quad \sum_{k=1}^{n} |\Delta g_k| \leq \frac{M}{m^2}.$$

Hence, $g = 1/f$ is of bounded variation on $[a, b]$.

Consider,

$$\sum_{k=1}^{n} |\Delta g_k| \leq \sum_{k=1}^{n} \frac{|\Delta f_k|}{m^2}$$

$$\implies \sum_{k=1}^{n} |\Delta g_k| \leq \frac{1}{m^2} \sup \left\{ \sum_{k=1}^{n} |\Delta f_k| : P \in \wp[a, b] \right\} \tag{1.3}$$

$$\implies \sup\left\{\sum_{k=1}^{n} |\Delta g_k| : P \in \wp[a,b]\right\} \le \frac{V_f}{m^2}$$
$$\implies V_g \le \frac{V_f}{m^2}.$$

This completes the proof. □

Let us sum up

- We have introduced the concept of total Variation.
- Also we have discussed if two functions are bounded variation, then their sum, difference and product also bounded variation.
- We have discussed function is bounded away from zero, their reciprocal function also bounded variation.

Check your progress

1. Define total variation.
2. If f and g are bounded variation, then show that $f + g$ and $f \cdot g$ are of bounded variation.

1.5 Additive Property of Total Variation

In the last two theorems the interval $[a, b]$ was kept fixed and $V_f(a, b)$ was considered as a function of f. If we keep f fixed and study the total variation as a function of the interval $[a, b]$, we can prove the following additive property.

Theorem 1.5.1. *Let f be of bounded variation on $[a, b]$, and assume that $c \in (a, b)$.Then f is of bounded variation on $[a, c]$ and on $[c, b]$ and we have*

$$V_f(a, b) = V_f(a, c) + V_f(c, b).$$

Proof. Assume that f is of bounded variation on $[a, b]$ and $c \in (a, b).$
We first prove that f is of bounded variation on $[a, c]$and on $[c, b]$.
Let P_1 be a partition of $[a, c]$, and P_2 be a partition of $[c, b]$.

Then $P_0 = P_1 \cup P_2$ is a partition of [a,b].

$$\implies \sum(P_0) = \sum(P_1) + \sum(P_2), \tag{5}$$

where $\sum(P_0)$ denotes the sum $\sum\limits_{k=1}^{n} |\Delta f_k|$ corresponding to the partition P_0 of $[a, b]$; $\sum(P_1)$ denotes the sum $\sum\limits_{k=1}^{n} |\Delta f_k|$ corresponding to the partition P_1 of $[a, c]$; $\sum(P_2)$ denotes the sum $\sum\limits_{k=1}^{n} |\Delta f_k|$ corresponding to the partition P_2 of $[c, b]$;
Since f is of bounded variation on $[a, b]$, we have

$$\begin{aligned}
\sum(P_0) &= \sum_{k=1}^{n} |\Delta f_k| \\
&\leq \sup\left\{\sum(P_0) : P_0 \in \wp[a, b]\right\} \\
&= V_f(a, b).
\end{aligned}$$

Hence (5) implies

$$\sum(P_1) + \sum(P_2) = \sum(P_0) \leq V_f(a, b) \tag{6}$$

$$\implies \sum(P_1) \leq V_f(a, b) \quad \text{and} \quad \sum(P_2) \leq V_f(a, b).$$

This shows that f is of bounded variation on $[a, c]$ and $[c, b]$.
By using Theorem 1.15, from (6), we have

$$\sup\left\{\sum(P_1) : P_1 \in \wp[a, c]\right\} + \sup\left\{\sum(P_2) : P_2 \in \wp[c, b]\right\} \leq V_f(a, b)$$

Hence,

$$V_f(a, c) + V_f(c, b) \leq V_f(a, b), \tag{7}$$

Next, we prove $V_f(a, c) + V_f(c, b) \geq V_f(a, b)$.
Let $P = \{x_0, x_1, \cdots, x_n\} \in \wp[a, b]$ and $P_0 = P \cup \{x^*\}$ be the partition obtained by adjoining the point x^* to P. If $x^* \in [x_{k-1}, x_k]$, we have

$$|f(x_k) - f(x_{k-1})| \leq |f(x_k) - f(x^*)| + |f(x^*) - f(x_{k-1})|$$

$$\implies \sum(P) \leq \sum(P_0).$$

Now, the points of P_0 in $[a,c]$ determine a partition P_1 of $[a,c]$ and the points of P_0 in $[c,b]$ determine a partition P_2 of $[c,b]$.

Now,

$$\begin{aligned}
&\sum(P) \leq \sum(P_0).\\
\implies &\sum_P |\Delta f_k| \leq \sum_{P_0} |\Delta f_k|\\
\implies &\sum_P |\Delta f_k| \leq \sum_{P_1} |\Delta f_k| + \sum_{P_2} |\Delta f_k|\\
&\leq V_f(a,c) + V_f(c,b)
\end{aligned}$$

and hence

$$\sum_P |\Delta f_k| \leq V_f(a,c) + V_f(c,b)$$

$$\implies \sup\left\{\sum(P) : P \in \wp[a,b]\right\} \leq V_f(a,c) + V_f(c,b)$$

$$\implies V_f(a,b) \leq V_f(a,c) + V_f(c,b) \qquad (8).$$

From (7) and (8), we have

$$V_f(a,b) = V_f(a,c) + V_f(c,b).$$

Hence the proof. □

Let us sum up

- We have discussed the additive property of total variation.

Check your progress

1. State the additive property of total variation.
2. If f is of bounded variation on $[a,b]$, then show that f is of bounded variation on $[a,c]$ and $[c,b]$ where $c \in (a,b)$.

1.6 Total Variation on $[a, x]$ as a Function of$x$

Now we keep the function f and the left endpoint of the interval fixed and study the total variation as a function of the right endpoint. The additive property implies important consequences for this function.

Theorem 1.6.1. *Let f be of bounded variation on $[a, b]$. Let V be defined on $[a, b]$ as follows $V(x) = V_f(a, x)$ if $a < x \leq b$, $V(a) = 0$. Then*
i) $V$is an increasing function on $[a, b]$.
ii) $V - f$ is an increasing function on $[a, b]$.

Proof. To prove that V is an increasing function on $[a, b]$, we need to show that for any $x_1 < x_2$ in $[a, b]$, $V(x_1) \leq V(x_2)$.

Given that f is of bounded variation on $[a, b]$ and $V(x) = V_f(a, x)$ for $a < x < b$, we know that V is the total variation of f on the interval $[a, x]$.

Let's consider $x_1 < x_2$ in $[a, b]$. Then we have

$$V(x_1) = V_f(a, x_1) \quad \text{and} \quad V(x_2) = V_f(a, x_2)$$

Since f is of bounded variation, the total variation of f on any subinterval of $[a, b]$ is non-negative. Therefore, we have

$$V_f(a, x_1) \leq V_f(a, x_2)$$

This implies that $V(x_1) \leq V(x_2)$, which proves that V is an increasing function on $[a, b]$.

To prove that $V - f$ is an increasing function on $[a, b]$, we can consider the function $g(x) = V(x) - f(x)$. To show that g is increasing, we need to show that $g'(x) \geq 0$ for all $x \in [a, b]$.

Since V is increasing on $[a, b]$ and f is of bounded variation, we have that both V and f are continuous functions. Therefore, the difference $g(x) = V(x) - f(x)$ is also continuous on $[a, b]$.

Now, let's compute the derivative of g with respect to x and show that it is non-negative

$$g'(x) = V'(x) - f'(x)$$

Since V is increasing, $V'(x) \geq 0$ for all $x \in [a, b]$. Since f is of bounded variation, $f'(x)$ exists almost everywhere and the total variation of f is non-negative. Therefore, $f'(x)$ can be non-positive, but the term $-f'(x)$ will not exceed the increase of $V(x)$, making $g'(x) \geq 0$ for all $x \in [a, b]$.

Hence, $V - f$ is an increasing function on $[a, b]$. □

Note: For some functions f, the total variation $V_f(a, x)$ can be expressed as an integral.

Let us sum up

- We have studied the total variation as a function.

Check your progress

1. If f is of bounded variation on $[a, b]$ and $V(x) = V_f(a, x)$ if $a < x \leq b,\ V(a) = 0$, then show that V is an increasing function on $[a, b]$.

1.7 Functions of Bounded Variation Expressed as the Difference of Increasing Functions

The following simple and elegant characterization of functions of bounded variation is a consequence of Theorem 1.6.1.

Theorem 1.7.1. *Let f be defined on [a,b].Then f is of bounded variation on [a,b] if and only if f can be expressed as the difference of two increasing functions.*

Proof. Assume that f is of bounded variation on [a,b].
To prove:f can be expressed as the difference of two increasing functions.
Let V be a function defined as follows

$$V(x) = V_f(a, b),\ if\ a < x < b,\ V(a) = 0$$

Now $f = V - V + f = V - (V - f) = V - D$, where $V - f = D$.
Let x and y be any points such that

$$a < x < y \leq b$$

First, we prove V is an increasing function on $[a,b]$.

We can write

$$V_f(a,y) = V_f(a,x) + V_f(x,y)$$
$$\implies V_f(a,y) - V_f(a,x) = V_f(x,y).$$

Since $V_f(x,y) \geq 0$, we have

$$V_f(a,y) - V_f(a,x) \geq 0$$
$$\implies V_f(a,y) \geq V_f(a,x)$$
$$i.e.,\ V_f(a,x) \leq V_f(a,y)$$
$$\implies V(x) \leq V(y).$$

Hence V is an increasing function on $[a,b]$.

Next, we prove D is an increasing function on $[a,b]$.

Let x and y be any two points such that

$$a < x < y \leq b$$

By hypothesis, $V - f$ is a function defined on $[a,b]$.

Let $D = V - f$, we have

$$\begin{aligned} D(y) - D(x) &= (V-f)(y) - (V-f)(x) \\ &= V(y) - f(y) - V(x) + f(x) \\ &= V(y) - V(x) - [f(y) - f(x)] \end{aligned}$$

$$= V_f(a,y) - V_f(a,x) - [f(y) - f(x)] = V_f(x,y) - [f(y) - f(x)]. \tag{10}$$

By the definition of $V_f(x,y)$, it follows that

$$f(y) - f(x) \leq V_f(x,y)$$

$$\implies V_f(x,y) - [f(y) - f(x)] \geq 0.$$

From (10), we have

$$D(y) - D(x) \geq 0.$$

$$\implies D(y) \geq D(x).$$

Hence, both V and $D = V - f$ are increasing functions on $[a, b]$.

Hence, f can be expressed as the difference of two increasing function.

Conversely,

assume that f can be expressed as the difference of two increasing functions.

To prove: f is of bounded variation on $[a, b]$.

Let $f = f_1 - f_2$, where f_1 and f_2 are increasing functions on $[a, b]$.

Hence, by Theorem 1.3.1, the functions f_1 and f_2 are of bounded variation on $[a, b]$.

Hence, by Theorem 1.4.1, $f_1 - f_2$ is also of bounded variation on $[a, b]$.

Hence, f is of bounded variation on $[a, b]$.

Hence the proof. □

Note: The representation of a function of bounded variation as a difference of two increasing functions is by no means unique. If $f = f_1 - f_2$,where f_1 and f_2 are increasing,we also have $f = (f_1 + g) - (f_2 + g)$,where g is an arbitrary increasing function, and we get a new representation of f. If g is strictly increasing, the same will be true of $f_1 + g$ and $f_2 + g$. Therefore, Theorem 1.7.1 also holds if increasing is replaced by strictly increasing.

Let us sum up

- We have shown that a function of bounded variation can be expressed as a difference of two monotonic functions.

Check your progress

1. Give an example of a function of bounded variation as a difference of two increasing functions.

2. Show that the difference of two increasing function is a function of bounded variation.

1.8 Continuous Functions of Bounded Variation

Theorem 1.8.1. *Let f be of bounded variation on $[a, b]$. If $x \in (a, b]$, let $V(x) = V_f(a, x)$ and put $V(a) = 0$. Then every point of continuity of f is also a point of continuity of V. The converse is also true.*

Proof. Given, f be of bounded variation on $[a, b]$. Let $x \in (a, b]$ be arbitrary and Let $V(x) = V_f(a, x)$ and put$V(a) = 0$, by Theorem 1.6.1, $V(x)$ is an increasing function. Hence, the right- and left- hand limits $V(x+)$ and $V(x-)$ exist for each point x in (a, b). By Theorem 1.7.1, the same is true for $f(x+)$ and $f(x-)$.
Let $a < x < y \leq b$, by the definition of $V_f(x, y)$,

$$0 \leq |f(y) - f(x)| \leq V_f(a, y) - V_f(a, x) = V(y) - V(x).$$

Letting $y \to x$, we have

$$0 \leq |f(x+) - f(x)| \leq |V(x+) - V(x)|.$$

This implies that a point of continuous of V is a point of an an function of f. Similarly,

$$0 \leq |f(x) - f(x-)| \leq |V(x) - V(x-)|.$$

Conversely, assume that f is continuous at the point c in (a, b).
To prove: V is continuous function.
Since f is continuous at$c$, given $\epsilon > 0$, there exists a $\delta > 0$ such that

$$0 < |x - c| < \delta \implies |f(x) - f(c)| < \epsilon/2.$$

For this same ϵ, there also exist a partition P of $[c, b]$, say

$$P = \{x_0, x_1, \cdots, x_n\}, \quad x_0 = c,\ x_n = b,$$

such that

$$V_f(c, b) - \frac{\epsilon}{2} < \sum_{k=1}^{n} |\Delta f_k|. \quad \left(\because \sum_{k=1}^{n} |\Delta f_k| \leq V_f(c, b) \right) \tag{11}$$

Adding more points to P can only increase the sum $\sum |\Delta f_k|$ and hence
we can assume that $0 < x_1 - x_0 < \delta$, we have

$$|\Delta f_1| = |f(x_1) - f(c)| < \frac{\epsilon}{2}. \tag{12}$$

From (11), we have

$$\begin{aligned}
V_f(c,b) - \frac{\epsilon}{2} &< \sum_{k=1}^{n} |\Delta f_k| \\
&= |\Delta f_1| + \sum_{k=2}^{n} |\Delta f_k| \\
V_f(c,b) - \frac{\epsilon}{2} &< \frac{\epsilon}{2} + \sum_{k=2}^{n} |\Delta f_k| \\
&\leq \frac{\epsilon}{2} + V_f(x_1,b) \;\; (Since,\; \{x_1, x_2, \cdots, x_n\} \text{ is a partition of } [x_1,b])
\end{aligned}$$

$$\implies V_f(c,b) - V_f(x_1,b) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Therefore, we have

$$V_f(c,b) - V_f(x_1,b) < \epsilon.$$

But

$$\begin{aligned}
0 \leq V(x_1) - V(c) = V_f(a,x_1) - V_f(a,c) &= V_f(c,x_1) \\
&= V_f(c,b) - V_f(x_1,b) < \epsilon.
\end{aligned}$$

Hence we have shown that

$$0 < x_1 - c < \delta \quad \text{implies} \quad 0 \leq V(x_1) - V(c) < \epsilon.$$

Hence,

$$V(c+) = V(c)$$

Similarly, we can prove that

$$V(c-) = V(c)$$

Hence, V is continuous at c. □

Theorem 1.8.2. *Let f be continuous on [a,b]. Then f is of bounded variation on [a,b] if and only if, f can be expressed as the difference of two increasing continuous functions.*

Proof. Let f be continuous on [a,b] and assume that f is of bounded variation on [a,b].

To prove: f can be expressed as the difference of two increasing functions.

Let V be a function defined as follows

$$V(x) = V_f(a, b) \quad if\ a < x < b,\ V(a) = 0.$$

Now, $f = V - V + f = V - (V - f) = V - D$, where $V - f = D$.

Let x and y be any points such that

$$a < x < y \leq b.$$

First, we prove V is an increasing function on $[a, b]$.

We can write

$$V_f(a, y) = V_f(a, x) + V_f(x, y).$$

$$\implies V_f(a, y) - V_f(a, x) = V_f(x, y).$$

Since $V_f(x, y) \geq 0$, we have

$$V_f(a, y) - V_f(a, x) \geq 0$$

$$\implies V_f(a, y) \geq V_f(a, x)$$

$$i.e.,\ V_f(a, x) \leq V_f(a, y)$$

$$\implies V(x) \leq V(y).$$

Hence V is an increasing function on $[a, b]$.

Next, we prove D is an increasing function on $[a, b]$.

Let x and y be any two points such that

$$a < x < y \leq b$$

By hypothesis, $V - f$ is a function defined on $[a, b]$.

Let $D = V - f$, we have

$$D(y) - D(x) = (V - f)(y) - (V - f)(x)$$

$$= V(y) - f(y) - V(x) + f(x)$$
$$= V(y) - V(x) - [f(y) - f(x)]$$

$$= V_f(a, y) - V_f(a, x) - [f(y) - f(x)] = V_f(x, y) - [f(y) - f(x)]. \qquad (*)$$

By the definition of $V_f(x, y)$, it follows that

$$f(y) - f(x) \leq V_f(x, y)$$

$$\implies V_f(x, y) - [f(y) - f(x)] \geq 0.$$

From (*), we have

$$D(y) - D(x) \geq 0.$$

$$\implies D(y) \geq D(x).$$

Hence, both V and $D = V - f$ are increasing functions on $[a, b]$.

Hence, f can be expressed as the difference of two increasing function.

Conversely, assume that f can be expressed as the difference of two increasing functions.

To prove : f is of bounded variation on $[a, b]$.

Let $f = f_1 - f_2$, where f_1 and f_2 are increasing functions on $[a, b]$.

By our assumption f is continuous function on $[a, b]$, f_1 and f_2 are both continuous function on $[a, b]$. Hence, by Theorem 1.3.1, functions f_1 and f_2 are of bounded variation on $[a, b]$. Hence, by Theorem 1.4.1, $f_1 - f_2$ is also of bounded variation on $[a, b]$. Hence, f is of bounded variation on $[a, b]$. □

Note: The above theorem also holds if the increasing nature of the function is replaced by strictly increasing.

Let us sum up

- We have discussed if f is of bounded variation on $[a, b]$ and $V(x) = V_f(a, x),\ V(a) = 0,$ then continuity of f implies continuity of V.
- We have studied if function is continuous and bounded variation, then it can be expressed as the difference of two increasing continuous functions.

Check your progress

1. Give an example of continuous function which is of bounded variation.

2. If f can be expressed as the difference of two increasing continuous functions, then show that f is of bounded variation.

1.9 Absolute and Conditional Convergence

Definition 1.9.1. *A series $\sum a_k$ is said to converge absolutely if $\sum |a_k|$ converges. A series $\sum a_k$ is said to converge conditionally if $\sum a_k$ converges but $\sum |a_k|$ diverges.*

Theorem 1.9.2. *Absolute convergence of $\sum a_k$ implies convergence.*

Proof. Let $\sum a_k$ converges absolutely, i.e., the series $\sum |a_k|$ converges.

Therefore, by Cauchy criterion, given $\epsilon > 0$, there exists a positive integer N such that

$$\sum_{k=n}^{m} |a_k| < \epsilon, \quad \text{for } m \geq n \geq N.$$

Now,

$$\left|\sum_{k=n}^{m} a_k\right| \leq \sum_{k=n}^{m} |a_k| < \epsilon, \quad \text{for } m \geq n \geq N.$$

Hence, for every $\epsilon > 0$, there is a positive integer N such that

$$\left|\sum_{k=n}^{m} a_k\right| \leq \epsilon, \quad \text{for } m \geq n \geq N.$$

Hence $\sum a_k$ converges. □

Theorem 1.9.3. *Let $\sum a_n$ be a given series with real-valued terms and define*

$$p_n = \frac{|a_n| + a_n}{2} \quad \textit{and} \quad q_n = \frac{|a_n| - a_n}{2} \qquad (n = 1, 2, \cdots).$$

Then

i) If $\sum a_n$ is conditionally convergent, both $\sum p_n$ and $\sum q_n$ diverge.

ii) If $\sum |a_n|$ converges, both $\sum p_n$ and $\sum q_n$ converge and we have

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} p_n - \sum_{n=1}^{\infty} q_n.$$

Note : *$p_n = a_n$ and $q_n = 0$, if $a_n \geq 0$,whereas $q_n = -a_n$ and $p_n = 0$, if $a_n \leq 0$.*

Proof. Note that if $a_n \geq 0$,

$$p_n = \frac{|a_n| + a_n}{2} = \frac{a_n + a_n}{2} = a_n$$

$$q_n = \frac{|a_n| - a_n}{2} = \frac{a_n - a_n}{2} = 0$$

and similarly if $a_n \leq 0$, we have $q_n = -a_n$ and $p_n = 0$.

$$p_n + q_n = \frac{|a_n| + a_n}{2} + \frac{|a_n| - a_n}{2} = \frac{|a_n| + a_n + |a_n| - a_n}{2} = \frac{2|a_n|}{2} = |a_n| \tag{13}$$

$$p_n - q_n = \frac{|a_n| + a_n}{2} - \frac{[|a_n| - a_n]}{2} = \frac{|a_n| + a_n - |a_n| + a_n}{2} = \frac{2a_n}{2} = a_n. \tag{14}$$

Assume that $\sum a_n$ is conditionally convergent, i.e., $\sum a_n$ converges but $\sum |a_n|$ diverges.

To prove: $\sum p_n$ and $\sum q_n$ diverge.

From (14), we see that $p_n = a_n + q_n$.

Hence, if $\sum q_n$ converges, then $\sum p_n$ also converges.

Similarly, if $\sum p_n$ converges then $\sum q_n$ also converges.

Hence, if either $\sum p_n$ or $\sum q_n$ converges, both must converge.

Hence $\sum |a_n|$ converges, since $|a_n| = p_n + q_n$.

This is a contradiction since $\sum |a_n|$ diverges. Hence both $\sum p_n$ and $\sum q_n$ diverge.

ii) Assume that $\sum |a_n|$ converges.

To prove: $\sum p_n$ and $\sum q_n$ converges and $\sum a_n = \sum p_n - \sum q_n$.

Since $\sum a_n$ converges absolutely, $\sum a_n$ also converges.

Therefore, $\sum p_n$ converges, since $p_n = \dfrac{|a_n| + a_n}{2}$.

Similarly, $\sum q_n$ converges, since $q_n = \dfrac{|a_n| - a_n}{2}$.

Hence, $\sum (p_n - q_n)$ converges.

From (14), $p_n - q_n = a_n$.

Therefore,

$$\sum a_n = \sum (p_n - q_n) = \sum p_n - \sum q_n.$$

Hence the proof. □

Let us sum up

- We have provided some imporant results

Check your progress

1. Given that $\sum a_n$ converge absolutely. Show that $\sum a_n^2$ converges absolutely.

1.10 Dirichlet's Test and Abel's Test

Theorem 1.10.1. *If $\{a_n\}$ and $\{b_n\}$ are two sequences of complex numbers, define*

$$A_n = a_1 + \cdots + a_n.$$

Then we have the identity

$$\sum_{k=1}^{n} a_k b_k = A_n b_{n+1} - \sum_{k=1}^{n} A_k (b_{k+1} - b_k).$$

Therefore, $\sum\limits_{k=1}^{n} a_k b_k$ converges if both the series $\sum\limits_{k=1}^{\infty} A_k(b_{k+1} - b_k)$ and the sequence $\{A_n b_{n+1}\}$ converge.

Proof. Assume $A_0 = 0$.

Now,

$$\begin{aligned}
\sum_{k=1}^{n} a_k b_k &= \sum_{k=1}^{n} (A_k - A_{k-1}) b_k \\
&= \sum_{k=1}^{n} A_k b_k - \sum_{k=1}^{n} A_{k-1} b_k - A_n b_{n+1} + A_n b_{n+1} \\
&= \sum_{k=1}^{n} A_k b_k - \sum_{k=1}^{n} A_k b_{k+1} + A_n b_{n+1} \\
&= A_n b_{n+1} - \sum_{k=1}^{n} A_k (b_{k+1} - b_k) \\
\sum_{k=1}^{n} a_k b_k &= A_n b_{n+1} - \sum_{k=1}^{n} A_k (b_{k+1} - b_k).
\end{aligned}$$

□

Theorem 1.10.2. ***(Dirichlet's test).*** *Let $\sum a_n$ be a series of complex terms whose partial sums form a bounded sequence. Let $\{b_n\}$ be a decreasing sequence which converges to 0. Then $\sum a_n b_n$ converges.*

Proof. Let $A_n = a_1 + a_2 + \cdots + a_n$ be the n-th partial sum of the series $\sum a_n$.

Since the sequence of partial sums of the series $\sum a_n$ is bounded, there exists a positive number M such that

$$|A_n| \leq M, \quad \text{for all } n.$$

Since $\{b_n\}$ is a decreasing sequence which converges to zero, we have

$$\lim_{n\to\infty} A_n b_{n+1} \leq M \lim_{n\to\infty} b_{n+1} = 0$$

Now, by Theorem 1.9.3, we have

$$\sum_{k=1}^{n} a_k b_k = A_n b_{n+1} - \sum_{k=1}^{n} A_k(b_{k+1} - b_k).$$

To prove the convergence of $\sum a_n b_n$, we need only to show that$\sum A_k(b_{k+1} - b_k)$ is convergent. Consider,

$$\begin{aligned} |A_k(b_{k+1} - b_k)| &= |A_k||b_{k+1} - b_k| \\ &\leq M|b_{k+1} - b_k| \ \ (|A_k| \leq M\ ) \\ &= M(b_{k+1} - b_k) \ \ (\{b_n\} \text{ is increasing}) \end{aligned}$$

By the Telescoping series Theorem, " Let $\{a_n\}$ and $\{b_n\}$ be two sequences such that $a_n = b_{n+1} - b_n$ for $n = 1, 2, ...$ Then $\sum a_n$ converges if and only if $\lim\limits_{n\to\infty} b_n$ exists, in which case we have $\sum\limits_{n=1}^{\infty} = \lim\limits_{n\to\infty} b_n - b_1$.J

$$\sum(b_{k+1} - b_k) \text{ is convergent.}$$

By the comparison test, $\sum A_k(b_{k+1} - b_k)$ is absolutely convergent.
Hence, $\sum a_n b_n$ converges. □

Theorem 1.10.3. ***(Abel's test).*** *The series $\sum a_n b_n$ converges if $\sum a_n$ converges and if $\{b_n\}$ is a monotonic convergent sequence.*

Proof. Let $\sum a_n$ be a convergent series, $\{b_n\}$ be a monotonic convergent sequence, $A_n = \sum\limits_{k=1}^{n} a_k$ and let $b = \lim_{n\to\infty} b_n$.
Since $\{a_n\}$ converges, $\{A_n\}$ converges.
Hence, $\{A_n\}$ is bounded.

Case (i):

Suppose $\{b_n\}$ is an increasing sequence

$$i.e., \quad b_1 \leq b_2 \leq \cdots \leq b_n \leq b_{n+1} \leq \cdots$$

Let $c_n = b - b_n$. Then $c_n \geq 0$ and

$$\begin{aligned}
\lim_{n\to\infty} c_n &= \lim_{n\to\infty} (b - b_n) \\
&= b - \lim_{n\to\infty} b_n \\
&= b - b = 0.
\end{aligned}$$

Now,

$$\begin{aligned}
c_n - c_{n+1} &= b - b_n - (b - b_{n+1}) \\
&= b - b_n - b + b_{n+1} \\
&= b_{n+1} - b_n \geq 0 \;\; (\because \; \{b_n\} \text{ is increasing}) \\
\implies \; c_n - c_{n+1} &\geq 0 \\
\implies \; c_n &\geq c_{n+1}.
\end{aligned}$$

Therefore, $\{c_n\}$ is a decreasing sequence, and $\lim_{n\to\infty} c_n = 0$.

By Dirichlet's test, $\sum_{n=1}^{\infty} a_n c_n$ converges.

Now, $c_n = b - b_n \implies b_n = b - c_n$.

$$\begin{aligned}
\implies \sum_{n=1}^{\infty} a_n b_n &= \sum_{n=1}^{\infty} a_n (b - c_n) \\
&= \sum_{n=1}^{\infty} a_n b - \sum_{n=1}^{\infty} a_n c_n \\
&= \sum_{n=1}^{\infty} a_n b - \sum_{n=1}^{\infty} a_n c_n
\end{aligned}$$

Since $\sum a_n$ converges, $\sum b a_n$ also converges.

Hence, $\sum b_n a_n$ converges.

Case (ii):

Suppose $\{b_n\}$ is a decreasing sequence

$$i.e., \quad b_1 \geq b_2 \geq \cdots \geq b_n \geq b_{n+1} \geq \cdots$$

Let $c_n = b_n - b$, where $b = \lim_{n\to\infty} b_n$.

$$\therefore \lim_{n\to\infty} c_n = 0$$

Now,

$$\begin{aligned} c_n - c_{n+1} &= b_n - b - b_{n+1} + b \\ &= b_n - b_{n+1} \geq 0 \end{aligned}$$

$$c_n - c_{n+1} \geq 0 \implies c_n \geq c_{n+1}.$$

Therefore, $\{c_n\}$ is a decreasing sequence and $\lim_{n\to\infty} c_n = 0$.

Hence, by Dirichlet's test, $\sum a_n c_n$ converges.

$c_n = b_n - b \implies b_n = c_n + b$.

$$\implies \sum a_n b_n = \sum a_n c_n + \sum a_n b.$$

Since $\sum a_n$ converges, $\sum b a_n$ also converges.

Hence, $\sum a_n b_n$ converges. $\square$

1.11 Rearrangement of Series

We recall that Z^+ denotes the set of positive integers,$Z^+ = \{1, 2, 3, \cdots\}$.

Definition 1.11.1. *Let f be a function whose domain is Z^+ and whose range is Z^+,and assume that f is one-to-one on Z^+. Let $\sum a_n$ and $\sum b_n$ be two series such that*

$$b_n = a_{f(n)}, \quad for\ n = 1, 2, ... \tag{15}$$

Then $\sum b_n$ is said to be a rearrangement of $\sum a_n$.

Note : Equation (15) implies $a_n = b_{f^{-1}(n)}$ and hence $\sum a_n$ is also a rearrangement of $\sum b_n$.

Theorem 1.11.2. *Let $\sum a_n$ be an absolutely convergent series having sum s. Then every rearrangement of $\sum a_n$ also converges absolutely and has sum s.*

Proof. Let $\{b_n\}$ be defined by $b_n = a_{f(n)}$. Then, we have

$$|b_1| + |b_2| + \cdots + |b_n| = |a_{f(1)}| + \cdots + |a_{f(n)}| \leq \sum_{k=1}^{\infty} |a_k|,$$

Hence, the partial sums of the series $\sum |b_n|$ is bounded.

Hence $\sum b_n$ converges absolutely.

To show that $\sum b_n = s$, let $t_n = b_1 + b_2 + \cdots + b_n,\ s_n = a_1 + a_2 + \cdots + a_n$.

Since, $\{s_n\}$ converges, given $\epsilon > 0$, there exists N so that $|s_N - s| < \epsilon/2$ and $\sum_{k=1}^{\infty} |a_{N+k}| \leq \epsilon/2$ by Cauchy criterion. Now,

$$|t_n - s| = |t_n - s_N + s_N - s| \leq |t_n - s_N| + |s_N - s| < |t_n - s_N| + \frac{\epsilon}{2}.$$

Choose M so that $\{1, 2, ..., N\} \subseteq \{f(1), f(2), ..., f(M)\}$.

Then $n > M$ implies $f(n) > N$. For such n,

$$\begin{aligned} |t_n - s_N| &= |b_1 + ... + b_n - (a_1 + ... + a_N) \\ &= |a_{f(1)} + ... + a_{f(n)} - (a_1 + ... + a_N)| \\ &\leq |a_{N+1}| + |a_{N+2}| + \cdots \leq \frac{\epsilon}{2}, \end{aligned}$$

since all the terms $a_1, a_2, ..., a_N$ cancel out in the subtraction. Hence, $n > M$ implies $|t_n - s| < \epsilon$.

Hence, $\{t_n\}$ converges to s.

i.e., the series $\sum b_n$ converges. □

1.12 Riemann's Theorem on Conditionally Convergent Series

Theorem 1.12.1. *Let $\sum a_n$ be a conditionally convergent series with real valued terms. Let x and y be given numbers in the closed interval $[-\infty, +\infty]$, with $x \leq y$. Then there exists a rearrangement $\sum b_n$ of $\sum a_n$ such that*

$$\lim_{n\to\infty} inf\ t_n = x \quad and \quad \lim_{n\to\infty} \sup\ t_n = y,$$

where $t_n = b_1 + b_2 + ... + b_n$.

Proof. Let $\sum a_n$ be a conditionally convergent series with real valued terms.

Let x and y be given numbers in the closed interval $[-\infty, +\infty]$, with $x \leq y$.

Discarding those terms of a series which are zero does not affect its convergence or divergence.

Hence, we can assume that no terms of $\sum a_n$ are zero.

Let p_n denote the n^{th} positive term of $\sum a_n$ and $-q_n$ denote the n^{th} negative term of $\sum a_n$.

Therefore, $p_n = \frac{|a_n|+a_n}{2}$ and $q_n = \frac{|a_n|-a_n}{2} \implies p_n - q_n = a_n$ and $p_n + q_n = |a_n|$.

Now the series $\sum p_n$ and $\sum q_n$ must both diverge.

Suppose both the series $\sum p_n$ and $\sum q_n$ converge. Then,

$$\sum_{n=1}^{\infty}(p_n + q_n) = \sum_{n=1}^{\infty} |a_n| \quad converges.$$

which is contradiction since $\sum a_n$ is conditionally convergent.

Suppose $\sum p_n$ diverges and $\sum q_n$ converges.

Now,

$$\sum_{n=1}^{N} a_n = \sum_{n=1}^{N}(p_n - q_n) = \sum_{n=1}^{N} p_n - \sum_{n=1}^{N} q_n.$$

Hence, $\sum a_n$ diverges.

which is contradiction since $\sum a_n$ is conditionally convergent.

Similarly, if $\sum p_n$ converges and $\sum q_n$ diverges, then $\sum a_n$ diverges.

which is contradiction since $\sum a_n$ is conditionally convergent.

Hence, the series $\sum p_n$ and $\sum q_n$ must both diverge.

Next, construct two sequences of real numbers, say $\{x_n\}$ and $\{y_n\}$, such that

$$\lim_{n\to\infty} x_n = x, \quad \lim_{n\to\infty} y_n = y$$

with $x_n < y_n, \quad y_1 > 0.$

We take k_1(say) positive terms such that

$$p_1 + \cdots + p_{k_1} > y_1.$$

Similarly, we take r_1(say) negative terms such that

$$p_1 + \cdots + p_{k_1} - q_1 - .... - q_{r_1} < x_1.$$

Next, we take upto k_2 positive terms such that

$$p_1 + \cdots + p_{k_1} - q_1 - \cdots - q_{r_1} + p_{k_1+1} + \cdots + p_{k_2} > y_2.$$

Similarly, we take upto r_2 negative terms such that

$$p_1 + \cdots + p_{k_1} - q_1 - \cdots - q_{r_1} + p_{k_1+1} + \cdots + p_{k_2} - q_{r_1+1} - \cdots - q_{r_2} < x_2.$$

These steps are possible, since $\sum p_n$ and $\sum q_n$ are both divergent series of positive terms.

Continuing this process, we obviously obtain a rearrangement $\sum b_n$ of $\sum a_n$ given by

$$p_1 + \cdots + p_{k_1} - q_1 - \cdots - q_{r_1} + p_{k_1+1} + \cdots + p_{k_2} - q_{r_1+1} - \cdots - q_{r_2} + \cdots \qquad (16)$$

Let α_n andβ_n denote the partial sums of $\sum b_n$ whose last terms are p_{k_n} and q_{r_n} respectively.

Since $p_1 + ... + p_{k_1} - q_1 - .... - q_{r_1} + p_{k_{n-1}} + p_{k_n} > y_n$.

$$\implies p_1 + \cdots + p_{k_1} - q_1 - \cdots - q_{r_1} + p_{k_{n-1}} \le y_n$$

$$\implies p_1 + \cdots + p_{k_1} - q_1 - \cdots - q_{r_1} + p_{k_{n-1}} + p_{k_n} \le y_n + p_{k_n}$$

$$\implies \alpha_n \le y_n + p_{k_n}$$

$$\implies \alpha_n - y_n \le p_{k_n}$$

$$|\alpha_n - y_n| \le p_{k_n} \quad as \quad \alpha_n > y_n \; and \;\; p_{k_n} > 0.$$

Similarly, we have

$$|\beta_n - x_n| \le q_{r_n}.$$

Since $\sum a_n$ is convergent, $a_n \to 0$ as $n \to \infty$.

$\implies p_n \to 0$ and $q_n \to 0$ as $n \to \infty \implies p_{k_n} \to 0$ and $q_{r_n} \to 0$ as $n \to \infty$.

Since $p_{k_n} \to 0$ as $n \to \infty$, for given $\epsilon > 0$, there exist a positive integer N_1 such that

$$p_{k_n} < \epsilon/2, \quad for \;\; n \ge N_1.$$

Again, since $y_n \to y$ as $n \to \infty$, there exist a positive integer N_2 such that

$$|y_n - y| < \epsilon/2, \quad for \;\; n \geq N_2.$$

Choose $N = Max\{N_1, N_2\}$. Then for $n \geq N$, we have

$$\begin{aligned} |\alpha_n - y| &= |\alpha_n - y_n + y_n - y| \\ &\leq |\alpha_n - y_n| + |y_n - y| \\ &\leq p_{k_n} + |y_n - y| < \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

$\implies \alpha_n \to y$ as $n \to \infty$.

Similarly, we can obtain $\beta_n \to x$ as $n \to \infty$.

Finally, it is clear that no number less than x or no number greater then y can be sub-sequential limit of the partial sums of (16).

$$\therefore \quad \lim_{n\to\infty} inf\ t_n = x \quad and \quad \lim_{n\to\infty} \sup\ t_n = y,$$

where $t_n = b_1 + b_2 + ... + b_n$. □

Let us sum up

- We have discussed Absolute convergence.
- Also discussed Conditional Convergence.
- We have studied Dirichlet's Test and Abel's Test.
- We have discussed Rearrangement of Series.
- We have discussed Riemann's Theorem on Conditionally Convergent Series.

Check your progress

1. Define absolute and conditional convergence of a series.
2. Give an example of a series that converges absolutely and a series that converges conditionally.
3. State Dirichlet's Test.

4. State Abel's Test.

5. Define Rearrangement of Series.

6. State Riemann's Theorem on Conditionally Convergent Series.

Check your progress

1. Which of the following statements is/are true?

   a) If $f : [a, b] \to \mathbb{R}$ is monotonic, then it is of bounded variation.

   b) If $f \in C^1[a, b]$, then it is bounded variation.

   c) If $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are of bounded variation, then $f + g$ is also of bounded variation.

   d) If $f : [a, b] \to \mathbb{R}$ is monotonic, then it is not of bounded variation.

2. $f(x) = \begin{cases} x^2 \cos(\frac{1}{x}), \ x \neq 0 \\ 0, \ x = 0 \end{cases}$ then

   a) f is of bounded variation on $[-1, 1]$

   b) f' is of bounded variation on $[-1, 1]$

   c) $|f'| \leq 1$

   d) $|f'| \leq 4$

3. For non-negative integers $k \geq 1$ define

$$f_k(x) = \frac{x^k}{(1+x)^2} \ \forall \ x \geq 0.$$

   Which of the following statements are true?

   a) For each k, f_k is a function of bounded variation on compact intervals

   b) For every k, $\displaystyle\int_0^\infty f_k(x)dx < \infty$

   c) $\displaystyle\lim_{k\to\infty} \int_0^1 f_k(x)dx$ exists

   d) The sequence of functions f_k converge uniformly on $[0, 1]$ as $k \to \infty$

4. Which of the following is not function of bounded variation.

   a) $x^2 + x + 1$ for $x \in (-1, 1)$

   b) $\tan \frac{\pi x}{2}$ for $x \in (-1, 1)$

   c) $\sin \frac{x}{2}$ for $x \in (-\pi, \pi)$

   d) $\sqrt{1 - x^2}$ for $x \in (-1, 1)$

5. $f(x) = \begin{cases} x^{\alpha}\cos(\frac{1}{x^{\beta}}), \ x \neq 0 \\ 0, \ x = 0. \end{cases}$ Then

a) $f(x)$ is bounded variation on $[0,1]$ if $\alpha < \beta$.

b) $f(x)$ is bounded variation on $[0,1]$ if $\alpha > \beta$.

c) $f(x)$ is bounded variation on $[0,1]$ if $\alpha \geq \beta$.

d) All of the above.

6. $f(x) = \begin{cases} x\sin(\frac{1}{x}), \ x \in (0,1] \\ 0, \ x = 0. \end{cases}$ and $g(x) = xf(x)$ for $0 \leq x \leq 1$. Then which of the following are true?

a) f is of bounded variation

b) f is not of bounded variation

c) g is of bounded variation

d) g is not of bounded variation

7. Let $[a,b] \subset \mathbb{R}$ be a finite interval. Let $f : [a,b] \to \mathbb{R}$ be a bounded and Riemann integrable function. Define, for $x \in [a,b]$,

$$F(x) = \int_a^x f(t)dt$$

which of the following statements is/are true?

a) The function F is uniformly continuous.

b) The function F is of bounded variation.

c) The function F is differentiable on (a,b).

d) The function F is not uniformly continuous.

8. The $\lim\limits_{n\to\infty} \dfrac{1^2 + 2^2 + \cdots + n^2}{n^3}$ equals

a) 1 b) $\frac{1}{2}$ c) $\frac{1}{3}$ d) None of the above

Summary

- Discussed the properties of Monotonic Functions.
- Introduced the concept of function of bounded variation and total variation.
- Proved some important properties of function of bounded variation.
- Derived sufficient conditions for a function to be of bounded variation.

- Introduced the concept of absolute and conditional Convergence of a series.
- Proved some important test for absolute and conditional Convergence of a series.
- Discussed rearrangement of series.
- Discussed Riemann's Theorem for conditionally convergent series.

Glossary

- **Increasing function:** A real-valued function f defined on a subset S of $\mathbb{R}$ is said to be increasing (or decreasing) on S if

$$x < y \implies f(x) \leq f(y)$$

  for every $x, y \in S$

- **Monotonic function:** A function f is said to be monotonic on S if it is increasing or decreasing on S.

- **Additive property of Total variation :** Let f be of bounded variation on $[a, b]$, and assume that $c \in (a, b)$.Then f is of bounded variation on $[a, c]$ and on $[c, b]$ and we have

$$V_f(a, b) = V_f(a, c) + V_f(c, b).$$

- **Telescoping series Theorem:** Let $\{a_n\}$ and $\{b_n\}$ be two sequences such that $a_n = b_{n+1} - b_n$, for $n = 1, 2, ...$ Then $\sum a_n$ converges if and only if $\lim_{n\to\infty} b_n$ exists, in which case we have $\sum_{n=1}^{\infty} = \lim_{n\to\infty} b_n - b_1$.

- **Cauchy Condition:** The series $\sum a_n$ converges if and only if, for every $\epsilon > 0$ there exists an integer N such that

$$n > N \implies |a_{n+1} + \cdots + a_{n+p}| < \epsilon, \text{for each } p = 1, 2, \ldots$$

(or)

$$m \geq n > N \implies |\sum_{k=n}^{m} a_k| < \epsilon$$

Self-Assessment Questions

Short-Answer Questions

1. State and prove Dirichlet's test.
2. State and prove Abel's test.
3. Absolute convergence of series implies convergence.
4. If f is of bounded variation on $[a, b]$, say $\sum |\Delta f_k| \leq M$ for all partitions of $[a, b]$, then f is bounded on $[a, b]$. In fact,

$$|f(x)| \leq |f(a)| + M, \ for \ all \ x \ in \ [a, b].$$

5. If f is monotonic on $[a, b]$, then f is of bounded variation on $[a, b]$.

Long-Answer Questions

1. State and prove Riemann's Theorem on Conditionally Convergent Series.
2. Let f be defined on [a,b].Then f is of bounded variation on [a,b] if and only if f can be expressed as the difference of two increasing functions.
3. Assume that f and g are each of bounded variation on $[a, b]$. Then so are their sum, difference and product. Also, we have

$$V_{f \pm g} \leq V_f + V_g \ \ and \ \ V_{f.g} \leq AV_f + BV_g,$$

where $A = \sup\{|g(x)| : x \in [a, b]\}$ and $B = \sup\{|f(x)| : x \in [a, b]\}$.

Exercises

1. Determine which of the following functions are of bounded variation on $[0, 1]$.

   a) $f(x) = x^2 \sin(\frac{1}{x})$ if $x \neq 0, \ f(0) = 0.$

   b) $f(x) = \sqrt{x} \sin(\frac{1}{x})$ if $x \neq 0, \ f(0) = 0.$

2. Show that a polynomial f is of bounded variation on every compact interval $[a, b]$. Describe a method for finding the total variation of f on $[a, b]$ if the zeros of the derivative f' are known.

3. Let f be a real-valued function defined on $[0,1]$ such that $f(0) > 0, f(x) \neq x$ for all x, and $f(x) \leq f(y)$ whenever $x \leq y$. Let $A = x : f(x) > x$. Prove that $\sup A \in A$ and that $f(1) > 1$.

4. Given that $\sum a_n$ converges absolutely. Show that each of the following series also converges absolutely:

   a) $\sum a_n^2$, b) $\sum \frac{a_n}{1+a_n}$ (if no $a_n = -1$), c) $\sum \frac{a_n^2}{1+a_n^2}$.

5. Praove the following statements:

   a) $\sum a_n b_n$ converges if $\sum a_n$ converges and if $\sum(b_n - b_{n+1})$ converges absolutely.

   b) $\sum a_n b_n$ converges if $\sum a_n$ has bounded partial sums and if $\sum(b_n - b_{n+1})$ converges absolutely, provided that $b_n \to 0$ as $n \to \infty$.

References

Tom M. Apostol, Mathematical Analysis, Second Edition, Addison-Wesley Publishing Company Inc., New York, 1974.

Suggested Readings

1. T.M. Apostol, Calculus Vol.2, Multi-Variable Calculus and Linear Algebra with Applications to Differential Equations and Probability, Second Edition - Reprint, John Wiley & Sons, 2016.

2. R.G. Bartle, Real Analysis, John Wiley and Sons Inc., 1976.

3. W. Rudin, Principles of Mathematical Analysis, Third Edition, Mc-Graw Hill Company, New York, 1976.

4. S.C. Malik and S. Arora. Mathematical Analysis, Wiley Eastern Limited, New Delhi, 1991.

5. B.R. Gelbaum and J. Olmsted, Counter Examples in Analysis, Holden day, San Francisco, 1964.

6. A.L. Gupta and N.R. Gupta, Principles of Real Analysis, Pearson Education, (Indian print) 2003.

Unit 2

THE RIEMANN-STIELTJES INTEGRAL

Objectives

After the successful completion of this unit; the students are expected

- To recall the basic concepts of bounded real-valued function, partition, upper and lower Riemann integral.
- To analyze the properties of Riemann integral.
- To understand the fundamental concepts of Riemann-Stieltjes integral.
- To analyse and work with problems related to Riemann-Stieltjes integral function.

2.1 Introduction

First let us define the notion of Riemann integral of a function.

Let f be a bounded real-valued function defined on $[a, b]$. There exists a real number $M > 0$ such that

$$|f(x)| \leq M \ (\ a \leq x \leq b \ )$$

Let $P = \{x_0, x_1, \cdots, x_n\}$ be a partition of $[a, b]$ such that

$$a = x_0 < x_1 < \cdots < x_n = b.$$

Let us define the following:

$$\Delta x_k = x_k - x_{k-1}.$$

$$\begin{aligned} M_k &= \sup\{f(x) : x \in [x_{k-1}, x_k]\} = \sup_{x\in[x_{k-1},x_k]} f(x). \\ m_k &= \inf\{f(x) : x \in [x_{k-1}, x_k]\} = \inf_{x\in[x_{k-1},x_k]} f(x). \\ U(P,f) &= \sum_{k=1}^{n} M_k \Delta x_k \quad and \quad L(P,f) = \sum_{k=1}^{n} m_k \Delta x_k. \end{aligned}$$

Now let us define

$$\int_{\bar{a}}^{b} f = \sup_P \ L(P,f) \quad and \quad \int_{a}^{\bar{b}} f = \inf_P \ U(P,f),$$

where the sup and inf are taken over all possible partitions $P \in \wp[a,b]$.

If $\int_{\bar{a}}^{b} f = \int_{a}^{\bar{b}} f$, then the common value is denoted by $\int_{a}^{b} f$ or $\int_a^b f(x)dx$ and the function f is said to be Riemann integrable on $[a,b]$.

Let $\mathcal{R}[a,b]$ denotes the set of all Riemann integrable functions on $[a,b]$.

Now, we are ready to define Riemann-Stieltjes integral:

Let $f : [a,b] \to \mathbb{R}$ and α be a monotonically increasing function on $[a,b]$.

Let $f : [a,b] \to \mathbb{R}$ be a bounded function and let $P = \{x_0, x_1, ..., x_n\}$ be any partition of $[a,b]$ such that

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

Define $\Delta\alpha_k = \alpha(x_k) - \alpha(x_{k-1})$.

Let $M_k = \sup\limits_{\alpha\in[x_{k-1},x_k]} f(x)$ and $m_k = \inf\limits_{\alpha\in[x_{k-1},x_k]} f(x)$.

Let $U(P,f) = \sum\limits_{k=1}^{n} M_k\Delta\alpha_k$ and $L(P,f) = \sum\limits_{k=1}^{n} m_k\Delta\alpha_k$.

Define $\int_{a}^{\bar{b}} f d\alpha = \inf U(P,f,\alpha)$ and $\int_{\bar{a}}^{b} f d\alpha = \sup L(P,f,\alpha)$

If $\int_{\bar{a}}^{b} f d\alpha = \int_{a}^{\bar{b}} f d\alpha = \int_{a}^{b} f d\alpha$(say), then f is said to be Riemann integrable on $[a,b]$.

i.e.,$f \in \mathcal{R}(\alpha)$, where $\mathcal{R}(\alpha)$ denotes the set of all Riemann integrable on $[a,b]$.

Let us sum up

- We have discussed definition of Riemann integrable.
- Also discussed bounded function.
- We have defined Upper and Lower Riemann integrable.

2.2 Notation

1. Let f be a bounded function defined on $[a, b]$. Let α be also a bounded function defined on $[a, b]$.

2. A partition $P = \{x_0, x_1, \cdots, x_n\}$ of $[a, b]$ is a finite set of points such that

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

3. A partition P^* of $[a, b]$ is said to be refinement of P if $P \subseteq P^*$.

4. $\Delta\alpha_k = \alpha(x_k) - \alpha(x_{k-1})$

$$\begin{aligned}\sum_{k=1}^{n} \Delta\alpha_k &= [\alpha(x_1) - \alpha(x_0)] + \cdots + [\alpha(x_n) - \alpha(x_{n-1})] \\ &= \alpha(x_n) - \alpha(x_0) \\ &= \alpha(b) - \alpha(a).\end{aligned}$$

5. The set of all possible partitions of $[a, b]$ is denoted by $\wp[a, b]$.

6. The norm of a partition P is the length of the largest subinterval of P is denoted by $\|P\|$. Note that

$$P \subseteq P' \implies \|P'\| \leq \|P\|.$$

7. If we need to subdivide $[a, b]$ into n intervals, then the length of each sub-interval is given by $\dfrac{b-a}{n}$.

Let us sum up

- We have discussed basic notations for Riemann integable.

2.3 The Definition of the Riemann-Stieltjes Integral

Definition 2.3.1. *Let $P = \{x_0, x_1, \cdots, x_n\}$ be a partition of $[a, b]$ and let t_k be a point in the sub-interval $[x_{k-1}, x_k]$. A sum of the form*

$$S(P, f, \alpha) = \sum_{k=1}^{n} f(t_k)\Delta\alpha_k$$

is called a Riemann-Stieltjes sum of f with respect to α. We say f is Riemann-integrable with respect to α on $[a,b]$, and we write $f \in \mathcal{R}(\alpha)$ on $[a,b]$, if there exists a number A having the following property: For every $\epsilon > 0$, there exists a partition P_ϵ of $[a,b]$ such that for every partition P finer than P_ϵ and for every choice of the points t_k in $[x_{k-1}, x_k]$, we have $|S(P,f,\alpha) - A| < \epsilon$.

When such a number A exists, it is uniquely determined and it is denoted by $\int_a^b f d\alpha$ or $\int_a^b f(x) d\alpha(x)$. We also say that the Riemann-Stieltjes integral $\int_a^b f d\alpha$ exists. The functions f and α are referred to as the integrand and the integrator, respectively. In the special case when $\alpha(x) = x$, we write $S(P,f)$ instead of $S(P,f,\alpha)$, and $f \in \mathcal{R}$ instead of $f \in \mathcal{R}(\alpha)$. The integral is then called a Riemann integral and is denoted by $\int_a^b f dx$ or $\int_a^b f(x) dx$.

2.4 Linear Properties

Theorem 2.4.1. *If $f \in \mathcal{R}(\alpha)$ and $g \in \mathcal{R}(\alpha)$ on $[a,b]$, then $c_1 f + c_2 g \in \mathcal{R}(\alpha)$ on $[a,b]$, where c_1 and c_2 are constants and we have*

$$\int_a^b (c_1 f + c_2 g)\, d\alpha = c_1 \int_a^b f\, d\alpha + c_2 \int_a^b g\, d\alpha.$$

Proof. Let $h = c_1 f + c_2 g$ and P be a partition of $[a,b]$, we have

$$\begin{aligned} S(P,h,\alpha) &= \sum_{k=1}^{n} h(t_k)\Delta\alpha_k \\ &= c_1 \sum_{k=1}^{n} f(t_k)\Delta\alpha_k + c_2 \sum_{k=1}^{n} g(t_k)\Delta\alpha_k \\ &= c_1 S(P,f,\alpha) + c_2 S(P,g,\alpha). \end{aligned}$$

Let $\epsilon > 0$ and $f, g \in \mathcal{R}(\alpha)$, we can choose partitions P'_ϵ and P''_ϵ such that for every partition P with $P'_\epsilon \subseteq P$ and $P''_\epsilon \subseteq P$, we have

$$\left| S(P,f,\alpha) - \int_a^b f d\alpha \right| < \frac{\epsilon}{2|c_1|}$$

and

$$\left| S(P,g,\alpha) - \int_a^b g d\alpha \right| < \frac{\epsilon}{2|c_2|}.$$

Take $P_\epsilon = P'_\epsilon \cup P''_\epsilon$, then for every partition P finer than P_ϵ we have

$$\left| S(P,h,\alpha) - c_1 \int_a^b f d\alpha - c_2 \int_a^b g d\alpha \right| = \left| c_1 S(P,f,\alpha) + c_2 S(P,g,\alpha) - c_1 \int_a^b f d\alpha - c_2 \int_a^b g d\alpha \right|$$

$$
\begin{aligned}
&\leq |c_1| \left| S(P, f, \alpha) - \int_a^b f d\alpha \right| + |c_2| \left| S(P, g, \alpha) - \int_a^b g d\alpha \right| \\
&< |c_1| \frac{\epsilon}{2|c_1|} + |c_2| \frac{\epsilon}{|c_2|} \\
&= \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}
$$

Since ϵ was arbitrary, we have

$$
\int_a^b (c_1 f + c_2 g)\, d\alpha = c_1 \int_a^b f\, d\alpha + c_2 \int_a^b g\, d\alpha.
$$

□

Theorem 2.4.2. *If $f \in \mathcal{R}(\alpha)$ and $f \in \mathcal{R}(\beta)$ on $[a, b]$, then $f \in \mathcal{R}(c_1\alpha + c_2\beta)$ on $[a, b]$, where c_1 and c_2 are constants and we have*

$$
\int_a^b f d(c_1\alpha + c_2\beta) = c_1 \int_a^b f\, d\alpha + c_2 \int_a^b f\, d\beta.
$$

Proof. Let $\eta = c_1\alpha + c_2\beta$, and P be a partition of $[a, b]$, we have

$$
\begin{aligned}
S(P, f, \eta) &= \sum_{k=1}^{n} f(t_k)\Delta\eta_k \\
&= \sum_{k=1}^{n} f(t_k)[\eta_k - \eta_{k-1}] \\
&= \sum_{k=1}^{n} f(t_k)[c_1\alpha_k + c_2\beta_k - c_1\alpha_{k-1} - c_2\beta_{k-1}] \\
&= \sum_{k=1}^{n} f(t_k)c_1\Delta\alpha_k + \sum_{k=1}^{n} f(t_k)c_2\Delta\beta_k \\
&= c_1 \sum_{k=1}^{n} f(t_k)\Delta\alpha_k + c_2 \sum_{k=1}^{n} f(t_k)\Delta\beta_k \\
&= c_1 S(P, f, \alpha) + c_2 S(P, f, \beta).
\end{aligned}
$$

Let $\epsilon > 0$, and $f \in \mathcal{R}(\alpha)$ and $f \in \mathcal{R}(\beta)$ we can choose partitions P'_ϵ and P''_ϵ such that for every partition $P'_\epsilon \subseteq P$ and $P''_\epsilon \subseteq P$, we have

$$
\left| S(P, f, \alpha) - \int_a^b f d\alpha \right| < \frac{\epsilon}{2|c_1|}
$$

and

$$
\left| S(P, f, \beta) - \int_a^b f d\beta \right| < \frac{\epsilon}{2|c_2|}.
$$

Take $P_\epsilon = P'_\epsilon \cup P''_\epsilon$, then for every partition P finer than P_ϵ, we have

$$\begin{aligned}\left|S(P,f,\eta) - c_1\int_a^b f d\alpha - c_2\int_a^b f d\beta\right| &= \left|c_1 S(P,f,\alpha) + c_2 S(P,g,\alpha) - c_1\int_a^b f d\alpha - c_2\int_a^b f d\beta\right| \\ &\le |c_1|\left|S(P,f,\alpha) - \int_a^b f d\alpha\right| + |c_2|\left|S(P,f,\beta) - \int_a^b f d\beta\right| \\ &\le |c_1|\frac{\epsilon}{2|c_1|} + |c_2|\frac{\epsilon}{|c_2|} \\ &= \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

Since ϵ was arbitrary, we have

$$\int_a^b f\ d(c_1\alpha + c_2\beta)\ =\ c_1\int_a^b f\ d\alpha\ +\ c_2\int_a^b f\ d\beta.$$

□

Theorem 2.4.3. *Assume that $c \in (a,b)$. If two of the three integrals in (17) exist, then the third also exists and we have*

$$\int_a^c f\ d\alpha + \int_c^b f\ d\alpha = \int_a^b f\ d\alpha. \tag{17}$$

Proof. Let P be a partition of $[a,b]$ such that $c \in P$ and $P' = P \cap [a,c]$ and $P'' = P \cap [c,b]$, denote the corresponding partitions of $[a,c]$ and $[c,b]$, respectively. The Riemann-Stieltjes sums for these partitions satisfy the equation

$$S(P,f,\alpha) = S(P',f,\alpha) + S(P'',f,\alpha).$$

Assume that $\int_a^c f d\alpha$ and $\int_c^b f d\alpha$ exist.
Let $\epsilon > 0$ be given. Then we can choose a partition P'_ϵ of $[a,c]$ such that

$$\left|S(P',f,\alpha) - \int_a^c f d\alpha\right| < \frac{\epsilon}{2},$$

whenever P' is finer than P'_ϵ. Similarly, we can choose a partition P''_ϵ of $[c,b]$ such that

$$\left|S(P'',f,\alpha) - \int_c^b f d\alpha\right| < \frac{\epsilon}{2},$$

whenever P'' is finer than P''_ϵ. Then, $P_\epsilon = P'_\epsilon \cup P''_\epsilon$ is a partition of $[a,b]$ such that P finer than P_ϵ implies $P'_\epsilon \subseteq P'$ and $P''_\epsilon \subseteq P''$.
Hence, if P is finer than P_ϵ, we have

$$\left|S(P,f,\alpha) - \int_a^c f d\alpha - \int_c^b f d\alpha\right| = \left|S(P',f,\alpha) + S(P'',f,\alpha) - \int_a^c f d\alpha - \int_c^b f d\alpha\right|$$

$$\leq \left|S(P', f, \alpha) - \int_a^c f d\alpha\right| + \left|S(P'', f, \alpha) - \int_c^b f d\alpha\right|$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This proves that $\int_a^b f d\alpha$ exists. Hence

$$\int_a^b f \, d\alpha = \int_a^c f \, d\alpha + \int_c^b f \, d\alpha.$$

□

Definition 2.4.4. *If $a < b$, we define $\displaystyle\int_b^a f d\alpha = -\int_a^b f d\alpha$ whenever $\displaystyle\int_a^b f d\alpha$ exists. We also define $\displaystyle\int_a^a f d\alpha = 0$.*

Let us sum up

- We have discussed the linear properties of Riemann-stieltjes integral.

2.5 Integration by Parts

Theorem 2.5.1. *If $f \in \mathcal{R}(\alpha)$ on $[a, b]$, then $\alpha \in \mathcal{R}(f)$ on $[a, b]$ and we have*

$$\int_a^b f(x) d\alpha(x) + \int_a^b \alpha(x) df(x) = f(b)\alpha(b) - f(a)\alpha(a).$$

This equation is known as the formula for integration by parts.

Proof. Let $\epsilon > 0$, and $\int_a^b f dx$ exists, there is a partition P_ϵ of $[a, b]$ such that for every partition P' finer than P_ϵ, we have

$$\left|S(P', f, \alpha) - \int_a^b f d\alpha\right| < \epsilon. \tag{18}$$

Consider an arbitrary Riemann-Stieltjes sum for the integral $\int_a^b \alpha df$, say

$$S(P, \alpha, f) = \sum_{k=1}^{n} \alpha(t_k)\Delta f_k$$
$$= \sum_{k=1}^{n} \alpha(t_k)[f(x_k) - f(x_{k-1})$$
$$= \sum_{k=1}^{n} \alpha(t_k) f(x_k) - \sum_{k=1}^{n} \alpha(t_k) f(x_{k-1}),$$

where P is finer than P_ϵ. Let $A = f(b)\alpha(b) - f(a)\alpha(a)$, then

$$A = \sum_{k=1}^{n} f(x_k)\alpha(x_k) - \sum_{k=1}^{n} f(x_{k-1})\alpha(x_{k-1}).$$

Therefore, we have

$$A - S(P, \alpha, f) = \sum_{k=1}^{n} f(x_k)\alpha(x_k) - \sum_{k=1}^{n} f(x_{k-1})\alpha(x_{k-1}) - \sum_{k=1}^{n} \alpha(t_k)f(x_k) + \sum_{k=1}^{n} \alpha(t_k)f(x_{k-1})$$

$$= \sum_{k=1}^{n} f(x_k)[\alpha(x_k) - \alpha(t_k)] + \sum_{k=1}^{n} f(x_{k-1})[\alpha(t_k) - \alpha(x_{k-1})].$$

The two sums on the right can be combined into a single sum of the form $S(P', f, \alpha)$, where P' is that partition of $[a, b]$ obtained by taking the points x_k and t_k together. Then P' is finer than P and P' is also finer than P_ϵ.

Now,

$$A - S(P, \alpha, f) = S(P', \alpha, f).$$

From (18), we have

$$\left| A - S(P, \alpha, f) - \int_a^b f d\alpha \right| < \epsilon,$$

where P is finer than P_ϵ.

This implies that $\alpha \in \mathcal{R}(f)$ and

$$\int_a^b \alpha df = A - \int_a^b f \, d\alpha = f(b)\alpha(b) - f(a)\alpha(a) - \int_a^b f \, d\alpha.$$

Therefore, we get

$$\int_a^b f d\alpha + \int_a^b \alpha df = f(b)\alpha(b) - f(a)\alpha(a).$$

□

Let us sum up

- We have derived the formula for integration by parts.

2.6 Change of Variable in a Riemann-Stieltjes Integral

Theorem 2.6.1. *Let $f \in \mathcal{R}(\alpha)$ on $[a, b]$ and g be a strictly monotonic continuous function defined on an interval S having endpoints c and d. Assume that $a = g(c), b = g(d)$. Let h and β be the composite functions defined as follows:*

$$h(x) = f[g(x)], \quad \beta(x) = \alpha[g(x)], \quad if\ x \in S.$$

Then $h \in \mathcal{R}(\beta)$ on S and we have $\int_a^b f\, d\alpha = \int_c^d h\, d\beta$. That is,

$$\int_{g(c)}^{g(d)} f(t)\, d\alpha(t) = \int_c^d f[g(x)]\, d\left\{\alpha[g(x)]\right\}.$$

Proof. Assume that g is strictly increasing function on S. Also g is continuous.
Hence, g is one to one and onto function from $[c, d]$ to $[a, b]$.
Then g^{-1} exists, which is also strictly increasing function on $[a, b]$.
Therefore, for every partition $P = \{y_0, y_1, \cdots, y_n\}$ of $[c, d]$, there corresponds one and only one partition $P' = \{x_0, \cdots, x_n\}$ of $[a, b]$ such that

$$x_k = g(y_k).$$

We can write

$$P' = g(P) \quad and \quad P = g^{-1}(P').$$

Let $\epsilon > 0$, and $f \in \mathcal{R}(\alpha)$ on $[a, b]$, there is a partition P'_ϵ of $[a, b]$ with $P'_\epsilon \subseteq P'$ such that

$$\left| S(P', f, \alpha) - \int_a^b f\, d\alpha \right| < \epsilon. \tag{19}$$

Let $P_\epsilon = g^{-1}(P'_\epsilon)$ be the corresponding partition of $[c, d]$, and let $P = \{y_0, \cdots, y_n\}$ be a partition of $[c, d]$ finer than P_ϵ. Form a Riemann-Stieltjes sum

$$S(P, h, \beta) = \sum_{k=1}^{n} h(u_k) \Delta\beta_k,$$

where $u_k \in [y_{k-1}, y_k]$ and $\Delta\beta_k = \beta(y_k) - \beta(y_{k-1})$.
Put $t_k = g(u_k)$ and $x_k = g(y_k)$. Then $P' = \{x_0, \cdots, x_n\}$ is a partition of $[a, b]$ finer than P'_ϵ. Now,

$$S(P, h, \beta) = \sum_{k=1}^{n} f[g(u_k)] \left\{\Delta\alpha[g(y_k)]\right\}$$

$$= \sum_{k=1}^{n} f[g(u_k)] \{\alpha[g(y_k)] - \alpha[g(y_{k-1})]\}$$
$$= \sum_{k=1}^{n} f(t_k) \{\alpha(x_k) - \alpha(x_{k-1})\}$$
$$= \sum_{k=1}^{n} f(t_k)\Delta\alpha_k$$
$$= S(P', f, \alpha) \ \ (\because \ t_k \in [x_{k-1}, x_k]).$$

From (19), we have

$$\left| S(P, h, \beta) - \int_a^b f \ d\alpha \right| < \epsilon$$

$$\implies \left| S(P, h, \beta) - \int_c^d h \ d\beta \right| < \epsilon$$

Therefore, $h \in \mathcal{R}(\beta)$ on $[c, d]$. Also

$$\int_a^b f \ d\alpha = \int_c^d h \ d\beta$$
$$\int_{g(c)}^{g(d)} f(t) \ d\alpha(t) = \int_c^d f[g(x)] \ d\{\alpha[g(x)]\}.$$

Hence the proof. □

Let us sum up

- We have discussed the change of variable in a Riemann-Stieltjes integral.

2.7 Reduction to a Riemann Integral

The next theorem tells us that we are permitted to replace the symbol $d\alpha(x)$ by $\alpha'(x)dx$ in the integral $\int_a^b f(x)d\alpha(x)$ whenever α has a continuous derivative α'.

Theorem 2.7.1. *Assume $f \in \mathcal{R}(\alpha)$ on $[a, b]$ and assume that α has a continuous derivative α' on $[a, b]$. Then the Riemann integral $\int_a^b f(x)\alpha'(x)dx$ exists and we have*

$$\int_a^b f(x) \ d\alpha(x) \ = \ \int_a^b f(x) \ \alpha'(x) \ dx.$$

Proof. Assume $f \in \mathcal{R}(\alpha)$ on $[a,b]$ and assume that α has a continuous derivative α' on $[a,b]$. Let $g(x) = f(x)\alpha'(x)$ and $P = \{x_0, x_1, \cdots, x_n\}$ be the partition of $[a,b]$ and $t_k \in (x_{k-1}, x_k)$.

Consider a Riemann sum

$$\begin{aligned} S(P,g) &= \sum_{k=1}^{n} g(t_k)\Delta x_k \\ &= \sum_{k=1}^{n} f(t_k)\alpha'(t_k)\Delta x_k \\ &= \sum_{k=1}^{n} f(t_k)\alpha'(t_k)[x_k - x_{k-1}]. \end{aligned}$$

For the same partition P and the same choice of the t_k, consider the Riemann-Stieltjes sum

$$\begin{aligned} S(P,f,\alpha) &= \sum_{k=1}^{n} f(t_k)\Delta\alpha_k \\ &= \sum_{k=1}^{n} f(t_k)[\alpha(x_k) - \alpha(x_{k-1})]. \end{aligned}$$

Since, α is continuous and differentiable on $[a,b]$, by mean value theorem, we have

$$\alpha(x_k) - \alpha(x_{k-1}) = \alpha'(v_k)[x_k - x_{k-1}], \quad where \;\; v_k \in (x_{k-1}, x_k).$$

Therefore, we get

$$\begin{aligned} S(P,f,\alpha) &= \sum_{k=1}^{n} f(t_k)\alpha'(v_k)[x_k - x_{k-1}] \\ S(P,f,\alpha) - S(P,g) &= \sum_{k=1}^{n} f(t_k)\alpha'(v_k)\Delta x_k - \sum_{k=1}^{n} f(t_k)\alpha'(t_k)\Delta x_k \\ &= \sum_{k=1}^{n} f(t_k)[\alpha'(v_k) - \alpha'(t_k)]\Delta x_k. \end{aligned}$$

Since f is bounded, there exist $M > 0$ such that $|f(x)| \le M$, for all x in $[a,b]$.

Since α' is continuous on $[a,b]$, and $[a,b]$ is closed and bounded, α' is uniformly continuous on $[a,b]$.

i.e., Given $\epsilon > 0$, there exist a $\delta > 0$ such that

$$0 \le |x - y| < \delta \quad \implies \quad |\alpha'(x) - \alpha'(y)| < \frac{\epsilon}{2M(b-a)}.$$

If we take a partition P'_ϵ with norm $\|P'_\epsilon\| < \delta$, then for any finer partition P we have

$$|\alpha'(v_k) - \alpha'(t_k)| < \frac{\epsilon}{2M(b-a)}.$$

For such partition P, we have

$$\begin{aligned}
|S(P,f,\alpha) - S(P,g)| &= \left|\sum_{k=1}^{n} f(t_k)[\alpha'(v_k) - \alpha'(t_k)]\Delta x_k\right| \\
&\leq \sum_{k=1}^{n} |f(t_k)||\alpha'(v_k) - \alpha'(t_k)|\Delta x_k \\
&\leq M.\frac{\epsilon}{2M(b-a)}(b-a) = \frac{\epsilon}{2}.
\end{aligned}$$

Since $f \in \mathcal{R}(\alpha)$ on $[a,b]$, there exists a partition P''_ϵ such that P finer than P''_ϵ implies

$$\left|S(P,f,\alpha) - \int_a^b f\ d\alpha\right| < \frac{\epsilon}{2}$$

Let $P_\epsilon = P'_\epsilon \cup P''_\epsilon$ with $P_\epsilon \subseteq P$. Consider,

$$\begin{aligned}
\left|S(P,g) - \int_a^b f\ d\alpha\right| &= \left|S(P,g) - S(P,f,\alpha) + S(P,f,\alpha) - \int_a^b f\ d\alpha\right| \\
&\leq \left|S(P,g) - S(P,f,\alpha)\right| + \left|S(P,f,\alpha) - \int_a^b f\ d\alpha\right| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Therefore,g is Riemann integrable.
i.e., $\displaystyle\int_a^b f(x)\alpha' dx$ exists and

$$\int_a^b f(x)\ d\alpha(x) \ = \ \int_a^b f(x)\ \alpha'(x)\ dx.$$

□

Let us sum up

- We have permitted to replace the symbol $d\alpha(x)$ by $\alpha'(x)dx$ in the integral $\int_a^b f(x)d\alpha(x)$ whenever α has a continuous derivative α'.

2.8 Euler's Summation Formula

Theorem 2.8.1. ***(Euler's summation formula).*** *If f has a continuous derivative f' on $[a,b]$, then we have*

$$\sum_{a<n\leq b} f(n) = \int_a^b f(x)\ dx + \int_a^b f'(x)((x))dx + f(a)((a)) - f(b)((b)),$$

where $((x)) = x - [x]$. *When* a *and* b *are integers, this becomes*

$$\sum_{n=a}^{b} f(n) = \int_a^b f(x)\, dx + \int_a^b f'(x)\left(x - [x] + \frac{1}{2}\right) dx \;+\; \frac{f(a)+f(b)}{2}.$$

Note. $\sum\limits_{a<n\le b}$ *means the sum from* $n = [a]+1$ *to* $n = [b]$.

Proof. Using integration by parts rule, we have

$$\int_a^b f(x) d\alpha(x) + \int_a^b \alpha(x) df(x) = f(b)\alpha(b) - f(a)\alpha(a).$$

Put $\alpha(x) = [x - [x]]$, $\alpha(a) = [a - [a]]$, $\alpha(b) = [b - [b]]$, we have

$$\int_a^b f(x) d[x - [x]] + \int_a^b [x - [x]] df(x) = f(b)[b - [b]] - f(a)[a - [a]]$$

$$\int_a^b f(x)dx - \int_a^b f(x)d[x] + \int_a^b ((x))df(x) = f(b)((b)) - f(a)((a))$$

$$\int_a^b f(x)dx + \int_a^b ((x))df(x) - f(b)((b)) + f(a)((a)) = \int_a^b f(x)d[x].$$

Since, $\sum\limits_{a<n\le b} f(n) = \int_a^b f(x)d[x]$, we get

$$\sum_{a<n\le b} f(n) = \int_a^b f(x)dx + \int_a^b ((x))df(x) - f(b)((b)) + f(a)((a)). \tag{20}$$

For every a and b are integers,

$$((a)) = a - [a] = a - a = 0 \text{ and } ((b)) = b - [b] = b - b = 0.$$

Substituting these values in (20), we get

$$\begin{aligned}\sum_{a<n\le b} f(n) &= \int_a^b f(x)dx + \int_a^b ((x))df(x) \\ &= \int_a^b f(x)dx + \int_a^b (x - [x])f'(x)\, dx \\ &= \int_a^b f(x)dx + \int_a^b f'(x)(x - [x])\, dx.\end{aligned}$$

Adding and subtracting $\frac{1}{2}\int_a^b f'(x)dx$ on RHS, we get

$$\sum_{a<n\le b} f(n) = \int_a^b f(x)dx + \int_a^b f'(x)(x - [x])\, dx - \frac{1}{2}\int_a^b f'(x)dx + \frac{1}{2}\int_a^b f'(x)dx$$

$$= \int_a^b f(x)dx + \int_a^b f'(x)\left(x - [x] - \frac{1}{2}\right)\, dx + \frac{1}{2}\int_a^b f'(x)dx$$
$$= \int_a^b f(x)dx + \int_a^b f'(x)\left[x - [x] - \frac{1}{2}\right]\, dx + \frac{1}{2}[f(x)]_a^b$$
$$= \int_a^b f(x)dx + \int_a^b f'(x)\left[x - [x] - \frac{1}{2}\right]\, dx + \frac{1}{2}[f(b) - f(a)].$$

Adding $f(a)$ on both sides, we have

$$\sum_{a<n\leq b} f(n) + f(a) = \int_a^b f(x)dx + \int_a^b f'(x)\left[x - [x] - \frac{1}{2}\right]\, dx + \frac{1}{2}f(b) - \frac{1}{2}f(a) + f(a)$$
$$\sum_{n=a}^{b} f(n) = \int_a^b f(x)dx + \int_a^b f'(x)\left[x - [x] - \frac{1}{2}\right]\, dx + \frac{1}{2}f(b) + \frac{1}{2}f(a)$$
$$\sum_{n=a}^{b} f(n) = \int_a^b f(x)dx + \int_a^b f'(x)\left[x - [x] - \frac{1}{2}\right]\, dx + \frac{f(b) + f(a)}{2}.$$

Hence the proof. □

Let us sum up

- We have derived Euler's Summation Formula.

2.9 Monotonically Increasing Integrators Upper and Lower Integrals

Definition 2.9.1. *Let P be a partition of $[a, b]$ and let*

$$M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\} \text{ and}$$
$$m_k = inf\{f(x) : x \in [x_{k-1}, x_k]\}.$$

The numbers

$$U(P, f, \alpha) = \sum_{k=1}^{n} M_k(f)\Delta\alpha_k \qquad and \qquad L(P, f, \alpha) = \sum_{k=1}^{n} m_k(f)\Delta\alpha_k,$$

are called, respectively, the upper and lower Stieltjes sums of f with respect to α for the partition P.

Note: We always have $m_k(f) \leq M_k(f)$.

If α increasing on $[a, b]$, then $\Delta\alpha_k \geq 0$. Therefore

$$m_k(f)\Delta\alpha_k \le M_k(f)\Delta\alpha_k.$$

i.e., the lower sums do not exceed the upper sums.

Furthermore, if $t_k \in [x_{k-1}, x_k]$, then

$$m_k(f) \le f(t_k) \le M_k(f).$$

Therefore, when $\alpha \nearrow$, we have the inequalities

$$L(P, f, \alpha) \le S(P, f, \alpha) \le U(P, f, \alpha)$$

relating the upper and lower sums to the Riemann-Stieltjes sums.

Theorem 2.9.2. *Assume that $\alpha \nearrow$ on $[a, b]$. Then*

i) If P' is finer than P, we have

$$U(P', f, \alpha) \le U(P, f, \alpha) \quad and \quad L(P', f, \alpha) \ge L(P, f, \alpha).$$

ii) For any two partitions P_1 and P_2, we have

$$L(P_1, f, \alpha) \le U(P_2, f, \alpha).$$

Proof. (i) First we prove $U(P', f, \alpha) \le U(P, f, \alpha)$.

It suffices to prove this when P' contains exactly one more point than P, say the point x^*.

Let x^* is in the i^{th} sub-interval of P.

i.e., $x^* \in [x_{i-1}, x_i],\ i = 1, 2, \cdots, n$ we can write

$$U(P', f, \alpha) = \sum_{k=1, k\neq i}^{n} M_k(f)\Delta\alpha_k + M'[\alpha(x^*) - \alpha(x_{i-1})] + M''[\alpha(x_i) - \alpha(x^*)],$$

where

$$M' = \sup\{f(x) : x \in [x_{i-1}, x^*]\}$$

and

$$M'' = \sup\{f(x) : x \in [x^*, x_i]\}.$$

Since, $M' \le M_i(f)$ and $M'' \le M_i(f)$, we have

$$U(P', f, \alpha) \le \sum_{k=1, k\neq i}^{n} M_k(f)\Delta\alpha_k + M_i(f)[\alpha(x^*) - \alpha(x_{i-1})] + M_i(f)[\alpha(x_i) - \alpha(x^*)]$$

$$
\begin{aligned}
&= \sum_{k=1,k\neq i}^{n} M_k(f)\Delta\alpha_k + M_i(f)[\alpha(x_i) - \alpha(x_{i-1})] \\
&= \sum_{k=1,k\neq i}^{n} M_k(f)\Delta\alpha_k + M_i(f)\Delta\alpha_i \\
&= U(P, f, \alpha).
\end{aligned}
$$

Next, we prove $L(P', f, \alpha) \geq L(P, f, \alpha)$.

It suffices to prove this when P' contains exactly one more point than P, say the point x^*. Let x^* is in the i^{th} sub-interval of P.

i.e., $x^* \in [x_{i-1}, x_i]$, we can write

$$
L(P', f, \alpha) = \sum_{k=1,k\neq i}^{n} m_k(f)\Delta\alpha_k + m'[\alpha(x^*) - \alpha(x_{i-1})] + m^{''}[\alpha(x_i) - \alpha(x^*)],
$$

where

$$
m' = inf\,\{f(x) : x \in [x_{i-1}, x^*]\}
$$

and

$$
m^{''} = inf\,\{f(x) : x \in [x^*, x_i]\}\,.
$$

Since $m' \geq m_i(f)$ and $m^{''} \geq m_i(f)$, we have

$$
\begin{aligned}
L(P', f, \alpha) &\geq \sum_{k=1,k\neq i}^{n} m_k(f)\Delta\alpha_k + m_i(f)[\alpha(x^*) - \alpha(x_{i-1})] + m_i(f)[\alpha(x_i) - \alpha(x^*)] \\
&= \sum_{k=1,k\neq i}^{n} m_k(f)\Delta\alpha_k + m_i(f)[\alpha(x_i) - \alpha(x_{i-1})] \\
&= \sum_{k=1,k\neq i}^{n} m_k(f)\Delta\alpha_k + m_i(f)\Delta\alpha_i \\
&= L(P, f, \alpha).
\end{aligned}
$$

To prove (ii). Let $P = P_1 \cup P_2$. Then by (i), we have

$$
L(P_1, f, \alpha) \leq L(P, f, \alpha) \leq U(P, f, \alpha) \leq U(P_2, f, \alpha).
$$

□

Note: If $M = \sup\limits_{a\leq x\leq b} f(x)$ and $m = \inf\limits_{a\leq x\leq b} f(x)$, then

$$
m[\alpha(b) - \alpha(a)] \leq L(P_1, f, \alpha) \leq U(P_2, f, \alpha) \leq M[\alpha(b) - \alpha(a)].
$$

Definition 2.9.3. *Assume that $\alpha \nearrow$ on $[a,b]$. The upper Stieltjes integral of f with respect to α is defined as follows*

$$\int_a^{\bar{b}} f d\alpha = inf \{U(P,f,\alpha) : P \in \wp[a,b]\}.$$

The lower Stieltjes integral is similarly defined

$$\int_{\bar{a}}^{b} f d\alpha = \sup \{L(P,f,\alpha) : P \in \wp[a,b]\}.$$

Note: $\overline{I}(f,\alpha) = \int_a^{\bar{b}} f d\alpha$.

$\underline{I}(f,\alpha) = \int_{\bar{a}}^{b} f d\alpha$.

If $\alpha(x) = x$, then $U(P,f,\alpha) = U(P,f)$ and $L(P,f,\alpha) = L(P,f)$.

Theorem 2.9.4. *Assume that $\alpha \nearrow$ on $[a,b]$. Then $\underline{I}(f,\alpha) \leq \overline{I}(f,\alpha)$.*

Proof. Let $\epsilon > 0$, then $\overline{I}(f,\alpha)+\epsilon$ is not a lower bound to the set $\{U(P,f,\alpha) : P \in \wp[a,b]\}$ and so there is a partition P_1 such that

$$U(P_1,f,\alpha) < \overline{I}(f,\alpha) + \epsilon.$$

By Theorem 2.9.1 (ii), for any partition P of $[a,b]$, we have

$$L(P,f,\alpha) \leq U(P_1,f,\alpha) < \overline{I}(f,\alpha) + \epsilon.$$

i.e., $\overline{I}(f,\alpha) + \epsilon$ is an upper bound to all lower sums $L(P,f,\alpha)$.

Thus,

$$\sup \{L(P,f,\alpha) : P \in \wp[a,b]\} \leq \overline{I}(f,\alpha) + \epsilon$$

$$\implies \quad \underline{I}(f,\alpha) \leq \overline{I}(f,\alpha) + \epsilon.$$

Since ϵ was arbitrary.

$$\underline{I}(f,\alpha) \leq \overline{I}(f,\alpha).$$

□

Example: Let $\alpha(x) = x$ and define a function f on $[0,1]$ as follows:

$$f(x) = 1, \;\; if\ x\ is \text{ rational}, \qquad f(x) = 0, \;\; if\ x \text{ is irrational}.$$

Let P be any partition of $[0,1]$, we have

$$M_k(f) = 1 \text{ and } m_k(f) = 0.$$

Therefore,

$$U(P,f,\alpha) = \sum_{k=1}^{n} M_k(f)\Delta x_k = \sum_{k=1}^{n} 1.\ \Delta x_k = 1(1-0) = 1.$$

and

$$L(P,f,\alpha) = 0.$$

Thus,

$$\underline{I}(f,\alpha) = \sup\{L(P,f,\alpha) : P \in \wp[0,1]\} = 0,$$
$$\overline{I}(f,\alpha) = inf\,\{U(P,f,\alpha) : P \in \wp[0,1]\} = 1.$$

Hence,

$$\underline{I}(f,\alpha) < \overline{I}(f,\alpha).$$

Let us sum up

- We have discussed the definition of upper and lower Stieltjes sums of the function with respect to α for the partition.
- We also discussed the definition of upper and lower Stieltjes integral.

2.10 Additive and Linearity Properties of Upper and Lower Integrals

Additive property:

$$\begin{aligned} \underline{\int_a^b} f\,d\alpha &= \underline{\int_a^c} f\,d\alpha + \underline{\int_c^b} f\,d\alpha, \quad for\ any\ c \in (a,b) \\ \overline{\int_a^b} f\,d\alpha &= \overline{\int_a^c} f\,d\alpha + \overline{\int_c^b} f\,d\alpha, \quad for\ any\ c \in (a,b) \end{aligned}$$

Linearity property:

$$\overline{\int_a^b} (f+g)\,d\alpha = \overline{\int_a^b} f\,d\alpha + \overline{\int_a^b} g\,d\alpha.$$

Let us sum up

- We have discussed the Additive and Linearity Properties of Upper and Lower Integrals.

2.11 Riemann's Condition

Definition 2.11.1. *We say that f satisfies Riemann's condition with respect to α on $[a, b]$, if for every $\epsilon > 0$, there exists a partition P_ϵ such that P finer than P_ϵ implies*

$$0 \leq U(P, f, \alpha) - L(P, f, \alpha) < \epsilon.$$

Theorem 2.11.2. *Assume that $\alpha \nearrow$ on $[a, b]$. Then the following three statements are equivalent:*

i) $f \in \mathcal{R}(\alpha)$ *on* $[a, b]$.

ii) f *satisfies Riemann's condition with respect to* α *on* $[a, b]$.

iii) $\underline{I}(f, \alpha) = \overline{I}(f, \alpha)$.

Proof. We will prove (i) $\implies$ (ii) $\implies$ (iii) $\implies$ (i).

To prove (i) $\implies$ (ii):

Assume that $f \in \mathcal{R}(\alpha)$ *on* $[a, b]$.

If $\alpha(a) = \alpha(b)$, then for given $\epsilon > 0$ and for any partition $P_\epsilon \in \wp[a, b]$, we have

$$\begin{aligned}
U(P_\epsilon, f, \alpha) - L(P_\epsilon, f, \alpha) &= \sum_{k=1}^{n} M_k(f)\Delta\alpha_k - \sum_{k=1}^{n} m_k(f)\Delta\alpha_k \\
&= \sum_{k=1}^{n} [M_k(f) - m_k(f)]\Delta\alpha_k \\
&\leq \sum_{k=1}^{n} (M - m)\Delta\alpha_k \ \ (where\ M = \sup_{x\in[a,b]} f(x)\ and\ m = \inf_{x\in[a,b]} f(x)) \\
&= (M - m)\sum_{k=1}^{n} \Delta\alpha_k \\
&= (M - m)[\alpha(b) - \alpha(a)] \\
&= 0 < \epsilon. (\text{since } \alpha(a) = \alpha(b))
\end{aligned}$$

Thus, for any partition P finer than P_ϵ, we have

$$U(P, f, \alpha) - L(P, f, \alpha) \leq U(P_\epsilon, f, \alpha) - L(P_\epsilon, f, \alpha) < \epsilon.$$

Hence, we can assume that $\alpha(a) \le \alpha(b)$.

Given $\epsilon > 0$, there is a partition P_ϵ on $[a, b]$ such that for any partition P finer than P_ϵ and all choices t_k and $t'_k \in [x_{k-1}, x_k]$, we have

$$\left|\sum_{k=1}^{n} f(t_k)\Delta\alpha_k - \int_a^b f\ d\alpha\right| < \frac{\epsilon}{3} \text{ and}$$
$$\left|\sum_{k=1}^{n} f(t'_k)\Delta\alpha_k - \int_a^b f\ d\alpha\right| < \frac{\epsilon}{3}.$$

Combining the above inequalities, we have

$$\begin{aligned}\left|\sum_{k=1}^{n}[f(t_k) - f(t'_k)]\Delta\alpha_k\right| &= \left|\sum_{k=1}^{n} f(t_k)\Delta\alpha_k - \int_a^b f\ d\alpha + \int_a^b f\ d\alpha - \sum_{k=1}^{n} f(t'_k)\Delta\alpha_k\right| \\ &\le \left|\sum_{k=1}^{n} f(t_k)\Delta\alpha_k - \int_a^b f\ d\alpha\right| + \left|\int_a^b f\ d\alpha - \sum_{k=1}^{n} f(t'_k)\Delta\alpha_k\right| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} = \frac{2\epsilon}{3}.\end{aligned}$$

We know that $M_k(f) - m_k(f) = \sup\{f(x) - f(x') : x,\ x' \in [x_{k-1}, x_k]\}$.

Hence, for any $h > 0$, $M_k(f) - m_k(f) - h$ is not an upper bound of the set $\{f(x) - f(x') : x,\ x' \in [x_{k-1}, x_k]\}$.

Hence, we can choose t_k and $t'_k \in [x_{k-1}, x_k]$ so that

$$M_k(f) - m_k(f) - h < f(t_k) - f(t'_k).$$

Let $h = \dfrac{\epsilon}{3[\alpha(b) - \alpha(a)]}$. Thus, for any partition P finer than P_ϵ, we have

$$\begin{aligned}U(P, f, \alpha) - L(P, f, \alpha) &= \sum_{k=1}^{n}[M_k(f) - m_k(f)]\Delta\alpha_k \\ &< \sum_{k=1}^{n}[f(t_k) - f(t'_k) + h]\Delta\alpha_k \\ &= \sum_{k=1}^{n}[f(t_k) - f(t'_k)]\Delta\alpha_k + h\sum_{k=1}^{n}\Delta\alpha_k \\ &< \frac{2\epsilon}{3} + \frac{\epsilon}{3[\alpha(b) - \alpha(a)]}(\alpha(b) - \alpha(a)) \\ &= \frac{2\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.\end{aligned}$$

Hence,

$$(i) \implies (ii).$$

To prove (ii) $\implies$ (iii):

Assume that f satisfies Riemann's condition with respect to α on $[a, b]$.

Let $\epsilon > 0$, then there is a partition P_ϵ such that for any partition P finer than P_ϵ, we have

$$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$$

$$\implies U(P, f, \alpha) < L(P, f, \alpha) + \epsilon.$$

For such P we have

$$\overline{I}(f, \alpha) \leq U(P, f, \alpha) < L(P, f, \alpha) + \epsilon \leq \underline{I}(f, \alpha) + \epsilon$$

$$i.e., \ \overline{I}(f, \alpha) < \underline{I}(f, \alpha) + \epsilon.$$

Since ϵ was arbitrary,

$$\overline{I}(f, \alpha) \leq \underline{I}(f, \alpha).$$

By Theorem 2.9.2, we have $\underline{I}(f, \alpha) \leq \overline{I}(f, \alpha)$. Hence $\underline{I}(f, \alpha) = \overline{I}(f, \alpha)$.

Hence $(ii) \implies (iii)$.

To prove (iii) $\implies$ (i):

Assume that $\underline{I}(f, \alpha) = \overline{I}(f, \alpha) = A$(say).

We shall prove that $\int_a^b f d\alpha$ exists and is equal to A.

Given $\epsilon > 0$, choose P'_ϵ such that

$$U(P, f, \alpha) < \overline{I}(f, \alpha) + \epsilon, \text{ for all } P \text{ finer than } P'_\epsilon.$$

Also choose P''_ϵ such that

$$L(P, f, \alpha) > \underline{I}(f, \alpha) - \epsilon, \text{ for all } P \text{ finer than } P''_\epsilon.$$

Let $P_\epsilon = P'_\epsilon \cup P''_\epsilon$, we have

$$\underline{I}(f, \alpha) - \epsilon < L(P, f, \alpha) \leq S(P, f, \alpha) \leq U(P, f, \alpha) < \overline{I}(f, \alpha) + \epsilon,$$

for every P finer than P_ϵ.

Since, $\underline{I}(f, \alpha) = \overline{I}(f, \alpha) = A$, from the above inequality we have

$$A - \epsilon < S(P, f, \alpha) < A + \epsilon$$

$$\implies -\epsilon < S(P, f, \alpha) - A < \epsilon$$

$$\implies |S(P, f, \alpha) - A| < \alpha,$$

where $A = \int_a^b f d\alpha$.

This proves that $\int_a^b f d\alpha$ exists and is equal to A.

i.e., $f \in \mathcal{R}(\alpha)$ on $[a, b]$.

$$\therefore \ (iii) \implies (i).$$

□

Let us sum up

- We have discussed the equivalent conditions for the existence of Riemann integral.

2.12 Comparison Theorems

Theorem 2.12.1. *Assume that $\alpha \nearrow$ on $[a, b]$. If $f \in \mathcal{R}(\alpha)$ and $g \in \mathcal{R}(\alpha)$ on $[a, b]$ and if $f(x) \leq g(x)$ for all x in $[a, b]$, then we have*

$$\int_a^b f(x)\, d\alpha(x) \leq \int_a^b g(x)\, d\alpha(x).$$

Proof. Given that $f(x) \leq g(x)$ for all $x \in [a, b]$, let P be any partition of $[a, b]$.

Since, $\alpha \nearrow$ on $[a, b]$, we have $\Delta\alpha_k \geq 0$.

Now $f(t_k) \leq g(t_k)$, for every $t_k \in [x_{k-1}, x_k]$, we have

$$\Delta\alpha_k\ f(t_k) \leq \Delta\alpha_k\ g(t_k)$$

$$\implies \sum_{k=1}^{n} f(t_k)\Delta\alpha_k \leq \sum_{k=1}^{n} g(t_k)\Delta\alpha_k.$$

The corresponding Riemann-Stieltjes sums satisfy

$$S(P, f, \alpha) = \sum_{k=1}^{n} f(t_k)\Delta\alpha_k \leq \sum_{k=1}^{n} g(t_k)\Delta\alpha_k = S(P, g, \alpha).$$

Therefore, we get

$$\int_a^b f(x)\,d\alpha(x) \le \int_a^b g(x)\,d\alpha(x).$$

Hence the proof. $\square$

Note: If $g(x) \ge 0$ and $\alpha \nearrow$ on $[a,b]$, then $\int_a^b g(x)\,d\alpha(x) \ge 0$.

Theorem 2.12.2. *Assume that $\alpha \nearrow$ on $[a,b]$. If $f \in \mathcal{R}(\alpha)$ on $[a,b]$, then $|f| \in \mathcal{R}(\alpha)$ on $[a,b]$ and we have the inequality*

$$\left| \int_a^b f(x)\,d\alpha(x) \right| \le \int_a^b |f(x)|\,d\alpha(x).$$

Proof.

$$Let\ \ M_k = \sup\left\{f(x) : x \in [x_{k-1}, x_k]\right\} \ \ and \ \ m_k = inf\left\{f(x) : x \in [x_{k-1}, x_k]\right\}.$$

We can write

$$M_k(f) - m_k(f) = \sup\left\{f(x) - f(y) : x,\ y \in [x_{k-1}, x_k]\right\}.$$

We know that $|f(x)| - |f(y)| \le |f(x) - f(y)|$. Therefore, we have

$$\sup\left\{|f(x)| - |f(y)| : x,\ y \in [x_{k-1}, x_k]\right\} \le \sup\left\{|f(x) - f(y)| : x,\ y \in [x_{k-1}, x_k]\right\}$$

$$\implies \quad M_k(|f|) - m_k(|f|) \le M_k(f) - m_k(f).$$

$$\implies \quad \sum_{k=1}^n M_k(|f|)\Delta\alpha_k - \sum_{k=1}^n m_k(|f|)\Delta\alpha_k \le \sum_{k=1}^n M_k(f)\Delta\alpha_k - \sum_{k=1}^n m_k(f)\Delta\alpha_k$$

Hence, $U(P,|f|,\alpha) - L(P,|f|,\alpha) \le U(P,f,\alpha) - L(P,f,\alpha)$, for any partition P of $[a,b]$. Since $f \in \mathbb{R}(\alpha)$, for given $\epsilon > 0$ we can choose a partition P_ϵ of $[a,b]$ such that

$$U(P,f,\alpha) - L(P,f,\alpha) < \epsilon, \quad \text{for } P \text{ finer than } P_\epsilon.$$

Thus, for all partition P finer than P_ϵ, we have

$$U(P,|f|,\alpha) - L(P,|f|,\alpha) < \epsilon.$$

Hence,

$$|f| \in \mathcal{R}(\alpha).$$

We know that $f(x) \le |f(x)|$, for all $x \in [a, b]$.

Hence, by using Theorem 2.12.1 with $g = |f|$, we have

$$\left| \int_a^b f(x)\, d\alpha(x) \right| \le \int_a^b |f(x)|\, d\alpha(x).$$

□

Theorem 2.12.3. *Assume that* $\alpha \nearrow$ *on* $[a, b]$. *If* $f \in \mathcal{R}(\alpha)$ *on* $[a, b]$, *then* $f^2 \in \mathcal{R}(\alpha)$ *on* $[a, b]$.

Proof. Let $\epsilon > 0$ and $f \in \mathcal{R}(\alpha)$, by Theorem 2.12.2, $|f| \in \mathcal{R}(\alpha)$.

We can choose a partition P_ϵ of $[a, b]$ such that for all P finer than P_ϵ, we have

$$U(P, |f|, \alpha) - L(P, |f|, \alpha) < \tfrac{\epsilon}{2M}. \quad \text{(where } M = \sup_{x\in[a,b]} f(x)\text{)}$$

For this partition P_ϵ, we shall prove that

$$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon \quad \text{for all } P \text{ finer than } P_\epsilon.$$

Now,

$$M_k(f^2) = \sup_{x\in[x_{k-1},x_k]} f^2(x) = \left(\sup_{x\in[x_{k-1},x_k]} |f(x)| \right)^2 = [M_k(|f|)]^2.$$

Similarly, we have

$$m_k(f^2) = [m_k(|f|)]^2.$$

Consider

$$\begin{aligned} M_k(f^2) - m_k(f^2) &= [M_k(|f|)]^2 - [m_k(|f|)]^2 \\ &= [M_k(|f|) + m_k(|f|)][M_k(|f|) - m_k(|f|)] \\ &\le (M + M)[M_k(|f|) - m_k(|f|)] \\ &= 2M[M_k(|f|) - m_k(|f|)], \end{aligned}$$

where $M = \sup_{x\in[a,b]} |f(x)|$ is an upper bound for $|f|$ on $[a, b]$. Therefore,

$$M_k(f^2) - m_k(f^2) \le 2M[M_k(|f|) - m_k(|f|)]$$

$$\implies \quad \sum_{k=1}^n M_k(f^2)\Delta\alpha_k - \sum_{k=1}^n m_k(f^2)\Delta\alpha_k \le 2M\left[\sum_{k=1}^n M_k(|f|)\Delta\alpha_k - \sum_{k=1}^n m_k(|f|)\Delta\alpha_k \right]$$

$$\implies U(P,f^2,\alpha) - L(P,f^2,\alpha) \leq 2M\Big(U(P,|f|,\alpha) - L(P,|f|,\alpha)\Big) < 2M.\,\frac{\epsilon}{2M} = \epsilon$$

$$\implies U(P,f^2,\alpha) - L(P,f^2,\alpha) < \epsilon,$$

Hence,

$$f^2 \in \mathcal{R}(\alpha).$$

□

Theorem 2.12.4. *Assume that* $\alpha \nearrow$ *on* $[a,b]$. *If* $f \in \mathcal{R}(\alpha)$ *and* $g \in \mathcal{R}(\alpha)$ *on* $[a,b]$, *then the product* $f.g \in \mathcal{R}(\alpha)$ *on* $[a,b]$.

Proof. Consider

$$2f(x)g(x) = [f(x)+g(x)]^2 - [f(x)]^2 - [g(x)]^2.$$

By using Linearity property of Riemann-Stieltjes integral, we have $f+g \in \mathcal{R}(\alpha)$.
Since $f+g,\ f,\ g\ \in \mathcal{R}(\alpha)$ on $[a,b]$, by Theorem 2.12.3, we have
$(f+g)^2,\ f^2,\ g^2\ \in \mathcal{R}(\alpha)$ on $[a,b]$. Therefore,

$$fg \in \mathcal{R}(\alpha)\ on\ [a,b].$$

□

Let us sum up

- We have discussed the product of two Riemann functions is also Riemann function.
- We have discussed the modulus function of Riemann function is also Riemann function.

Check your progress

1. Give an example of a bounded function f and an increasing function α defined on $[a,b]$ such that $|f| \in \mathcal{R}(\alpha)$ but for which $\int_a^b f d\alpha$ does not exist.

Summary

- Defined the notion of Riemann-Stieltjes integral of a bounded funcion f with respect to an arbitrary α.
- Derived the formula for integration by parts.
- Discussed the change of variable in a Riemann-Stieltjes integral.
- Permitted to replace the symbol $d\alpha(x)$ by $\alpha'(x)dx$ in the integral $\int_a^b f(x)d\alpha(x)$ whenever α has a continuous derivative α'.
- Derived Euler's Summation Formula.
- Defined the notion of Riemann-Stieltjes integral of a bounded funcion f with respect to an monotonic function α.

Exercises

1. Prove that $\displaystyle\int_a^b d\alpha(x) = \alpha(b) - \alpha(a)$.
2. If $f \in \mathcal{R}(\alpha)$ on $[a, b]$ and if $\int_a^b d\alpha = 0$ for every f whichi is monotonic on $[a, b]$, prove that α must be constant on $[a, b]$.
3. Use Euler's summation formula, or integration by parts in a Stieltjes integral, to derive the following identities:

   a) $\displaystyle\sum_{k=1}^{n} \frac{1}{k^s} = \frac{1}{n^{s-1}} + s\int_1^n \frac{[x]}{x^{s+1}}dx \quad \text{if } s \neq 1.$

   b) $\displaystyle\sum_{k=1}^{n} \frac{1}{k} = \log n + s\int_1^n \frac{x-[x]}{x^{s+1}}dx + 1.$
4. Assume f' is continuous on $[1, 2n]$ and use Euler's summation formula or integration by parts to prove that

$$\sum_{k=1}^{2n} (-1)^k f(k) = \int_1^{2n} f'([x] - 2[x/2])dx.$$

5. Let $\phi_1(x) = x - [x] - \frac{1}{2}$ if $x \neq$ integer, and let $\phi_1(x) = 0$ if $x =$ integer. Also, let $\phi_2(x) = \int_0^x \phi_1(t)dt$. If f'' is continuous on $[1, n]$ prove that Euler's summation formula implies that

$$\sum_{k=1}^{n} f(k) = \int_1^n f(x)dx - \int_1^n \phi_2(x)f''dx + \frac{f(1)+f(n)}{2}.$$

6. If $\alpha \uparrow$ on $[a, b]$, prove that we have

a) $\overline{\int_a^b} f d\alpha = \overline{\int_a^c} f d\alpha + \overline{\int_c^b} f d\alpha, \quad (a < c < b),$

b) $\overline{\int_a^b} (f+g)d\alpha \leq \overline{\int_a^b} f d\alpha + \overline{\int_a^b} g d\alpha$

c) $\underline{\int_a^b} (f+g)d\alpha \geq \underline{\int_a^b} f d\alpha + \underline{\int_a^b} g d\alpha$

References

Tom M. Apostol, Mathematical Analysis, Second Edition, Addison-Wesley Publishing Company Inc., New York, 1974.

Suggested Readings

1. T.M. Apostol, Calculus Vol.2, Multi-Variable Calculus and Linear Algebra with Applications to Differential Equations and Probability, Second Edition - Reprint, John Wiley & Sons, 2016.

2. R.G. Bartle, Real Analysis, John Wiley and Sons Inc., 1976.

3. W. Rudin, Principles of Mathematical Analysis, Third Edition, Mc-Graw Hill Company, New York, 1976.

4. S.C. Malik and S. Arora. Mathematical Analysis, Wiley Eastern Limited, New Delhi, 1991.

5. B.R. Gelbaum and J. Olmsted, Counter Examples in Analysis, Holden day, San Francisco, 1964.

6. A.L. Gupta and N.R. Gupta, Principles of Real Analysis, Pearson Education, (Indian print) 2003.

Unit 3

THE RIEMANN-STIELTJES INTEGRAL

Objectives

After the successful completion of this unit; the students are expected to

- To recall the basic concepts of total variation, integral exist and integrators.
- To analyze the properties of Riemann-Stieltjes integral.
- To understand the fundamental concepts of Riemann-Stieltjes integral.
- To analyse and work with problems related to Riemann-Stieltjes integral function.

We have see that that every function α of bounded variation on $[a, b]$ can be expressed as the difference of two increasing functions. However, the converse is not always true. If $f \in \mathcal{R}(\alpha)$ on $[a, b]$, it is quite possible to choose increasing functions α_1 and α_2 such that $\alpha = \alpha_1 - \alpha_2$, but such that neither integral $\int_a^b f\, d\alpha_1$, $\int_a^b f\, d\alpha_2$ exists. The difficulty, of course, is due to the nonuniqueness of the decomposition $\alpha = \alpha_1 - \alpha_2$. However, we can prove that there is at least one decomposition for which the converse is true, namely, when α_1 is the total variation of α and $\alpha_2 = \alpha_1 - \alpha$.

3.1 Integrators of Bounded Variation

Theorem 3.1.1. *Assume that α is of bounded variation on $[a, b]$. Let $V(x)$ denote the total variation of α on $[a, x]$ if $a < x \leq b$, and let $V(a) = 0$. Let f be a bounded function defined on $[a, b]$. If $f \in \mathcal{R}(\alpha)$ on $[a, b]$, then $f \in \mathcal{R}(V)$ on $[a, b]$.*

Proof. If $V(b) = 0$, then V is constant and the result is trivial.

Assume that $V(b) > 0$ and f is bounded on $[a, b]$, there is a number $M > 0$ such that

$$|f(x)| \leq M, \quad a \leq x \leq b.$$

Since V is increasing, we need only to verify that f satisfies Riemann's condition with respect to V on $[a, b]$.

Since $f \in \mathcal{R}(\alpha)$ on $[a, b]$, given $\epsilon > 0$, and choose P_ϵ so that for any finer P and all choices of points t_k and t'_k in $[x_{k-1}, x_k]$, we have

$$\left| \sum_{k=1}^{n} [f(t_k) - f(t'_k)] \Delta\alpha_k \right| < \frac{\epsilon}{4}.$$

Since, V is increasing on $[a, b]$, we have

$$V(b) < \sum_{k=1}^{n} |\Delta\alpha_k| + \frac{\epsilon}{4M}.$$

For P finer than P_ϵ we will establish the two inequalities

$$\sum_{k=1}^{n} [M_k(f) - m_k(f)](\Delta V_k - |\Delta\alpha_k|) < \frac{\epsilon}{2}$$

and

$$\sum_{k=1}^{n} [M_k(f) - m_k(f)] |\Delta\alpha_k| < \frac{\epsilon}{2}.$$

Adding the above two inequalities, we get

$$\sum_{k=1}^{n} [M_k(f) - m_k(f)] \Delta V_k < \epsilon$$

$$\implies U(P, f, V) - L(P, f, V) < \epsilon.$$

To prove the first inequality:

We know that $\Delta V_k - |\Delta\alpha_k| \geq 0$

$\therefore \sum_{k=1}^{n} [M_k(f) - m_k(f)](\Delta V_k - |\Delta\alpha_k|) \leq 2M \ \sum_{k=1}^{n} (\Delta V_k - |\Delta\alpha_k|)$ (where $M = \sup_{x \in [a,b]} f(x)$)

$$= 2M \left(V(b) - \sum_{k=1}^{n} |\Delta\alpha_k| \right) < 2M \left(\frac{\epsilon}{4M} \right) = \frac{\epsilon}{2}.$$

To prove the second inequality, let

$$A(P) = \{k : \Delta\alpha_k \geq 0\} \;\; and \;\; B(P) = \{k : \Delta\alpha_k < 0\}\,.$$

Let $h = \dfrac{\epsilon}{4V(b)}$. If $k \in A(P)$, choose t_k and t'_k so that

$$f(t_k) - f(t'_k) > M_k(f) - m_k(f) - h$$

and if $k \in B(P)$, choose t_k and t'_k so that

$$f(t'_k) - f(t_k) > M_k(f) - m_k(f) - h.$$

Then

$$\begin{aligned}\sum_{k=1}^{n}[M_k(f) - m_k(f)]|\Delta\alpha_k| &< \sum_{k\in A(P)}[f(t_k) - f(t'_k)]|\Delta\alpha_k| + \sum_{k\in B(P)}[f(t'_k) - f(t_k)]|\Delta\alpha_k| \\ &\quad + h\sum_{k=1}^{n}|\Delta\alpha_k| \\ &= \sum_{k\in A(P)}[f(t_k) - f(t'_k)]\Delta\alpha_k + \sum_{k\in B(P)}[f(t_k) - f(t'_k)](-\Delta\alpha_k) + \\ &\quad + h\sum_{k=1}^{n}|\Delta\alpha_k| \\ &= \sum_{k=1}^{n}[f(t_k) - f(t'_k)]\Delta\alpha_k + h\sum_{k=1}^{n}|\Delta\alpha_k| \\ &< \frac{\epsilon}{4} + h\,.V(b) = \frac{\epsilon}{4} + h\,.\frac{\epsilon}{4h} = \frac{\epsilon}{2}.\end{aligned}$$

It follows that $f \in \mathcal{R}(V)$ on $[a, b]$. □

Theorem 3.1.2. *Let α be of bounded variation on $[a, b]$ and assume that $f \in \mathcal{R}(\alpha)$ on $[a, b]$. Then $f \in \mathcal{R}(\alpha)$ on every subinterval $[c, d]$ of $[a, b]$.*

Proof. Let $V(x)$ denote the total variation of α on $[a, x]$, with $V(a) = 0$.
Then $\alpha = V - (V - \alpha)$, where both V and $V - \alpha$ are increasing on $[a, b]$ (by Theorem 1.7.1).
By Theorem 3.1.1, $f \in \mathcal{R}(\alpha)$, and hence $f \in \mathcal{R}(V - \alpha)$ on $[a, b]$.
Therefore, if the theorem is true for increasing integrators, it follows that $f \in \mathcal{R}(V)$ on $[c, d]$ and $f \in \mathcal{R}(V - \alpha)$ on $[c, d]$. Therefore, $f \in \mathcal{R}(\alpha)$ on $[c, d]$.
It suffices to prove the theorem when $\alpha \nearrow$ on $[a, b]$.

By Theorem 2.4.3, it suffices to prove that each integral $\int_a^c f\, d\alpha$ and $\int_a^d f\, d\alpha$ exists. Assume that $a < c < b$, and let P be a partition of $[a, x]$ and $\Delta(P, x)$ denote the difference of the upper and lower sums associated with the interval $[a, x]$.

$$i.e., \quad \Delta(P, x) = U(P, f, \alpha) - L(P, f, \alpha).$$

Since $f \in \mathcal{R}(\alpha)$ on $[a, b]$, f satisfies Riemann's condition.
Let $\epsilon > 0$, then there is a partition P_ϵ of $[a, b]$ such that for all P finer than P_ϵ,

$$\Delta(P, b) = U(P, f, \alpha) - L(P, f, \alpha) < \epsilon.$$

We can assume that $c \in P_\epsilon$. The points of P_ϵ in $[a, c]$ form a partition P'_ϵ of $[a, c]$. Let P' be a partition of $[a, c]$ finer than P'_ϵ, then $P = P' \cup P_\epsilon$ is a partition of $[a, b]$ composed of the points of P' along with those points of P_ϵ in $[c, b]$.
Now the sum $\Delta(P', c)$ contains only few terms in the sum $\Delta(P, b)$.
Since each term is nonnegative and P is finer than P_ϵ, we have

$$\Delta(P', c) \le \Delta(P, b) < \epsilon.$$

i.e., P' finer than P'_ϵ implies $\Delta(P', c) < \epsilon$.
Hence, f satisfies Riemann's condition on $[a, c]$ and $\int_a^c f\, d\alpha$ exists.
In a similar way, we can prove that $\int_a^d f\, d\alpha$ exists.
We know that

$$\int_a^d f\, d\alpha = \int_a^c f\, d\alpha + \int_c^d f\, d\alpha.$$

Hence, by Theorem 2.4.3, $\int_c^d f\, d\alpha$ exists. $\square$

Theorem 3.1.3. *Assume $f \in \mathcal{R}(\alpha)$ and $g \in \mathcal{R}(\alpha)$ on $[a, b]$, where $\alpha \nearrow$ on $[a, b]$. Define*

$$F(x) = \int_a^x f(t)\, d\alpha(t)$$

and

$$G(x) = \int_a^x g(t)\, d\alpha(t), \quad if\ x \in [a, b].$$

Then $f \in \mathcal{R}(G)$, $g \in \mathcal{R}(F)$, and the product $f.g \in \mathcal{R}(\alpha)$ on $[a, b]$, and we have

$$\begin{aligned} \int_a^b f(x)g(x)\, d\alpha(x) &= \int_a^b f(x)\, dG(x) \\ &= \int_a^b g(x)\, dF(x). \end{aligned}$$

Proof. By Theorem 2.12.4, the integral $\int_a^b f.g\ d\alpha$ exists and let P be a partition of $[a,b]$, we have

$$\begin{aligned} S(P,f,G) &= \sum_{k=1}^{n} f(t_k)\Delta G_k \\ &= \sum_{k=1}^{n} f(t_k)[G(x_k) - G(x_{k-1})] \\ &= \sum_{k=1}^{n} f(t_k)\int_{x_{k-1}}^{x_k} g(t)\ d\alpha(t) \\ &= \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} f(t_k).g(t)\ d\alpha(t) \end{aligned}$$

and

$$\int_a^b f(x)g(x)\ d\alpha(x) = \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} f(t)g(t)\ d\alpha(t).$$

Let $M_g = \sup\{|g(x)| : x \in [a,b]\}$. Now,

$$\left|S(P,f,G) - \int_a^b f.g\ d\alpha\right| = \left|\sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} \{f(t_k) - f(t)\}\, g(t)\ d\alpha(t)\right|$$

$$\leq \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} |f(t_k) - f(t)||g(t)|\ d\alpha(t)$$

$$\leq M_g \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} |f(t_k) - f(t)|\ d\alpha(t)$$

$$\leq M_g \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} [M_k(f) - m_k(f)] d\alpha(t) \quad (\because |f(t_k) - f(t)| \leq M_k(f) - m_k(f))$$

$$= M_g[U(P,f,\alpha) - L(P,f,\alpha)].$$

Since $f \in \mathcal{R}(\alpha)$, for every $\epsilon > 0$ there is a partition P_ϵ such that P finer than P_ϵ implies

$$U(P,f,\alpha) - L(P,f,\alpha) < \frac{\epsilon}{M_g}$$

$$\therefore \quad \left|S(P,f,G) - \int_a^b f.g\ d\alpha\right| < M_g.\frac{\epsilon}{M_g} = \epsilon.$$

This proves that $f \in \mathcal{R}(G)$ on $[a,b]$ and $\int_a^b f(t)g(t)\ d\alpha(t) = \int_a^b f(t)\ dG(t)$.

To prove : $g \in \mathcal{R}(F)$ on $[a,b]$

Let P be a partition of $[a,b]$, we have

$$\begin{aligned}
S(P,g,F) &= \sum_{k=1}^{n} g(t_k)\Delta F_k \\
&= \sum_{k=1}^{n} g(t_k)[F(x_k) - F(x_{k-1})] \\
&= \sum_{k=1}^{n} g(t_k) \int_{x_{k-1}}^{x_k} f(t)\ d\alpha(t) \\
&= \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} g(t_k).f(t)\ d\alpha(t)
\end{aligned}$$

and

$$\int_a^b f(x)g(x)\ d\alpha(x) = \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} f(t)g(t)\ d\alpha(t).$$

Let $M_f = \sup\{|f(x)| : x \in [a,b]\}$. Now,

$$\begin{aligned}
\left|S(P,g,F) - \int_a^b f.g\ d\alpha\right| &= \left|\sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} \{g(t_k) - g(t)\} f(t)\ d\alpha(t)\right| \\
&\leq \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} |g(t_k) - g(t)||f(t)|\ d\alpha(t) \\
&\leq M_f \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} |g(t_k) - g(t)|\ d\alpha(t) \\
&\leq M_f \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} [M_k(g) - m_k(g)] d\alpha(t) \quad (\because |g(t_k) - g(t)| \leq M_k(g) - m_k(g)) \\
&= M_f[U(P,g,\alpha) - L(P,g,\alpha)].
\end{aligned}$$

Since $g \in \mathcal{R}(\alpha)$, for every $\epsilon > 0$ there is a partition P_ϵ such that P finer than P_ϵ implies

$$U(P,g,\alpha) - L(P,g,\alpha) < \frac{\epsilon}{M_f}$$

$$\therefore \quad \left|S(P,g,F) - \int_a^b f.g\ d\alpha\right| < M_f.\frac{\epsilon}{M_f} = \epsilon.$$

This proves that $g \in \mathcal{R}(F)$ on $[a,b]$ and $\int_a^b f(t)g(t)\ d\alpha(t) = \int_a^b g(t)\ dF(t)$.

$\square$

Note: Theorem 3.1.3 is also valid if α is of bounded variation on $[a,b]$.

Let us sum up

- We have discussed α is of bounded variation on $[a, b]$ and assume that $f \in \mathcal{R}(\alpha)$ on $[a, b]$. Show that $f \in \mathcal{R}(\alpha)$ on every subinterval $[c, d]$ of $[a, b]$.
- We have discussed the integrators.

3.2 Sufficient Conditions for Existence of Riemann-Stieltjes Integrals

Theorem 3.2.1. *If f is continuous on $[a, b]$ and if α is of bounded variation on $[a, b]$, then $f \in \mathcal{R}(\alpha)$ on $[a, b]$.*

Proof. It suffices to prove the theorem when $\alpha \nearrow$ on $[a, b]$ with $\alpha(a) < \alpha(b)$.
To prove: $f \in \mathcal{R}(\alpha)$ on $[a, b]$.
Since f is continuous on $[a, b]$ and $[a, b]$ is closed and bounded, f is uniformly continuous on $[a, b]$.
i.e., if $\epsilon > 0$ is given, there exist $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \frac{\epsilon}{A}$, where $A = 2[\alpha(b) - \alpha(a)]$.
Let P_ϵ be a partition of $[a, b]$ such that

$$\|P_\epsilon\| < \delta.$$

We know that $M_k(f) - m_k(f) = \sup\{f(x) - f(y) : x, y \in [x_{k-1}, x_k]\}$.
Hence, for any partition P finer than P_ϵ, we have

$$M_k(f) - m_k(f) \le \frac{\epsilon}{A}$$
$$\implies \sum_{k=1}^{n} M_k(f)\Delta\alpha_k - \sum_{k=1}^{n} m_k(f)\Delta\alpha_k \le \frac{\epsilon}{A}\sum_{k=1}^{n}\Delta\alpha_k$$
$$\implies U(P, f, \alpha) - L(P, f, \alpha) < \frac{\epsilon}{2[\alpha(b) - \alpha(a)]}(\alpha(b) - \alpha(a)) = \epsilon.$$

Hence, $f \in \mathcal{R}(\alpha)$ on $[a, b]$. □

Theorem 3.2.2. *Each of the following conditions is sufficient for the existence of the Riemann integral $\int_a^b f(x)\,dx$*

a) f is continuous on $[a, b]$.

b) f is of bounded variation on $[a, b]$.

Proof. Assume f is continuous on $[a, b]$.

It suffices to prove the theorem when $\alpha \nearrow$ on $[a, b]$ with $\alpha(a) < \alpha(b)$.

To prove: $f \in \mathcal{R}(\alpha)$ on $[a, b]$.

Since f is continuous on $[a, b]$ and $[a, b]$ is closed and bounded, f is uniformly continuous on $[a, b]$.

i.e., if $\epsilon > 0$ is given, there exist $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \frac{\epsilon}{A}$, where $A = 2[\alpha(b) - \alpha(a)]$.

Let P_ϵ be a partition of $[a, b]$ such that

$$\|P_\epsilon\| < \delta.$$

We know that $M_k(f) - m_k(f) = \sup \{f(x) - f(y) : x, y \in [x_{k-1}, x_k]\}$.

Hence, for any partition P finer than P_ϵ, we have

$$\begin{aligned} M_k(f) - m_k(f) &\leq \frac{\epsilon}{A} \\ \implies \sum_{k=1}^{n} M_k(f)\Delta\alpha_k - \sum_{k=1}^{n} m_k(f)\Delta\alpha_k &\leq \frac{\epsilon}{A} \sum_{k=1}^{n} \Delta\alpha_k \\ \implies U(P, f, \alpha) - L(P, f, \alpha) &< \frac{\epsilon}{2[\alpha(b) - \alpha(a)]}(\alpha(b) - \alpha(a)) = \epsilon. \end{aligned}$$

Hence, $f \in \mathcal{R}(\alpha)$ on $[a, b]$.

Assume f is of bounded variation on $[a, b]$. □

Let us sum up

- We have discussed sufficient conditions for existence of Riemann-Stieltjes integrals.

3.3 Necessary Conditions for Existence of Riemann-Stieltjes Integrals

Theorem 3.3.1. *Assume that $\alpha \nearrow$ on $[a, b]$ and let $a < c < b$. Assume further that both α and f are discontinuous from the right at $x = c$ that is, assume that there exists an $\epsilon > 0$ such that for every $\delta > 0$ there are values x and y in the interval $(c, c + \delta)$ for which*

$$|f(x) - f(c)| \geq \epsilon \quad \textit{and} \quad |\alpha(y) - \alpha(c)| \geq \epsilon.$$

Then the integral $\int_a^b f(x)\, d\alpha(x)$ cannot exist. The integral also fails to exist if α and f are discontinuous from the left at c.

Proof. Let P be a partition of $[a,b]$ containing c as a point of subdivision.
Consider

$$U(P,f,\alpha) - L(P,f,\alpha) = \sum_{k=1}^{n}[M_k(f) - m_k(f)]\Delta\alpha_k.$$

If the i^{th} subinterval has c as its left endpoint (i.e.,$c < x_i$), then

$$\sum_{k=1}^{n}[M_k(f) - m_k(f)]\Delta\alpha_k \geq [M_i(f) - m_i(f)][\alpha(x_i) - \alpha(c)], \quad (\because \alpha(x_i) > \alpha(c)) \tag{21}$$

If c is a common point of discontinuity from the right, we can assume that the point x_i is chosen so that $\alpha(x_i) - \alpha(c) \geq \epsilon$.
Since $M_i(f) - m_i(f) \geq f(x_i) - f(c)$, by our assumption we have $M_i(f) - m_i(f) \geq \epsilon$.
Hence, from (21) we have

$$\therefore \quad U(P,f,\alpha) - L(P,f,\alpha) \geq \epsilon^2$$

Riemann's condition cannot be satisfied.
Hence the integral $\int_a^b f(x)\, d\alpha(x)$ cannot exist. □

Note: If c is a common discontinuity from the left, the argument is similar.

Let us sum up

- We derived the necessary condition for the existence of Riemann-Stieltjes integrals.

3.4 Mean-Value Theorems for Riemann-Stieltjes Integrals

Theorem 3.4.1. ***(First Mean-Value Theorem for Riemann-Stieltjes integrals).*** *Assume that $\alpha \nearrow$ and let $f \in \mathcal{R}(\alpha)$ on $[a,b]$. Let M and m denote, respectively, the sup and inf of the set $\{f(x) : x \in [a,b]\}$. Then there exists a real number c satisfying $m \leq c \leq M$ such that*

$$\int_a^b f(x)\, d\alpha(x) = c\int_a^b d\alpha(x) = c[\alpha(b) - \alpha(a)].$$

In particular, if f is continuous on $[a,b]$, then $c = f(x_0)$ for some x_0 in $[a,b]$.

Proof. If $\alpha(a) = \alpha(b)$, then $\int_a^b f(x)\, d\alpha(x) = 0$ and the theorem holds trivially.
Assume that $\alpha(a) < \alpha(b)$ and let P be a partition of $[a, b]$.
Given that $M = \sup\{f(x) : x \in [a, b]\}$ and $m = inf\{f(x) : x \in [a, b]\}$. Therefore, we have

$$L(P, f, \alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i \geq m \sum_{i=1}^{n} \Delta\alpha_i = m(\alpha(b) - \alpha(a)),$$

similarly

$$U(P, f, \alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i \leq M \sum_{i=1}^{n} \Delta\alpha_i = M(\alpha(b) - \alpha(a)).$$

All upper and lower sums satisfy

$$m[\alpha(b) - \alpha(a)] \leq L(P, f, \alpha) \leq U(P, f, \alpha) \leq M[\alpha(b) - \alpha(a)].$$

Since $f \in \mathcal{R}(\alpha)$, the integral $\int_a^b f\, d\alpha$ exists and

$$m[\alpha(b) - \alpha(a)] \leq L(P, f, \alpha) \leq \int_a^b f\, d\alpha \leq U(P, f, \alpha) \leq M[\alpha(b) - \alpha(a)]$$

$$\implies m[\alpha(b) - \alpha(a)] \leq \int_a^b f\, d\alpha \leq M[\alpha(b) - \alpha(a)]$$

$$\implies m \leq \frac{\int_a^b f\, d\alpha}{\int_a^b\, d\alpha} \leq M.$$

Therefore, the quotient $c = \frac{\int_a^b f\, d\alpha}{\int_a^b\, d\alpha}$ satisfies $m \leq c \leq M$ and

$$\int_a^b f\, d\alpha = c \int_a^b\, d\alpha = c[\alpha(b) - \alpha(a)].$$

If f is continuous on $[a, b]$, then by intermediate value theorem there is a point $x_0 \in [a, b]$ such that $c = f(x_0)$. □

Theorem 3.4.2. ***(Second Mean-Value Theorem for Riemann-Stieltjes integrals).*** *Assume that α is continuous and that $f \nearrow$ on $[a, b]$. Then there exists a point x_0 in $[a, b]$ such that*

$$\int_a^b f(x)\, d\alpha(x) = f(a) \int_a^{x_0} d\alpha(x) + f(b) \int_{x_0}^b d\alpha(x).$$

Proof. By Theorem 2.5.1, we have

$$\int_a^b f(x)d\alpha(x) = f(b)\alpha(b) - f(a)\alpha(a) - \int_a^b \alpha(x)df(x).$$

By Theorem 3.4.1, for some $x_0 \in [a,b]$, we have

$$\int_a^b \alpha(x)df(x) = \alpha(x_0)[f(b) - f(a)].$$

Therefore,

$$\begin{aligned}\int_a^b f(x)d\alpha(x) &= f(b)\alpha(b) - f(a)\alpha(a) - \alpha(x_0)[f(b) - f(a)]\\ &= f(a)[\alpha(x_0) - \alpha(a)] + f(b)[\alpha(b) - \alpha(x_0)]\\ &= f(a)\int_a^{x_0} d\alpha(x) + f(b)\int_{x_0}^b d\alpha(x)\\ \therefore \quad \int_a^b f(x)\ d\alpha(x) &= f(a)\int_a^{x_0} d\alpha(x) + f(b)\int_{x_0}^b d\alpha(x).\end{aligned}$$

□

Let us sum up

- We have proved the first and second Mean-Value Theorems of Riemann-Stieltjes integrals.

3.5 The Integral as a Function of The Interval

Theorem 3.5.1. *Let α be of bounded variation on $[a,b]$ and assume that $f \in \mathcal{R}(\alpha)$ on $[a,b]$. Define F by the equation*

$$F(x) = \int_a^x f\ d\alpha, \qquad if\ x \in [a,b].$$

Then we have te following

i) F is of bounded variation on $[a,b]$.

ii) Every point of continuity of α is also a point of continuity of F.

iii) If $\alpha \nearrow$ on $[a,b]$, the derivative $F'(x)$ exists at each point x in (a,b) where $\alpha'(x)$ exists and where f is continuous. For such x, we have

$$F^prime(x) = f(x)\alpha'(x).$$

Proof. Without loss of generality, we may assume that $\alpha \nearrow$ on $[a, b]$.

To prove (i):

By Theorem 3.4.1, for $a \leq x < y \leq b$ we have

$$\int_x^y f(t)\, d\alpha(t) = c[\alpha(y) - \alpha(x)],$$

where $m \leq c \leq M$, $m = \inf_{x \in [a,b]} f(x)$ and $M = \sup_{x \in [a,b]} f(x)$.

Now,

$$F(y) - F(x) = \int_x^y F'(t)\, dt$$

$$= \int_x^y f(t)\, d\alpha(t).$$

Therefore, we have

$$F(y) - F(x) = c[\alpha(y) - \alpha(x)].$$

For any subdivision of $[a, b]$, we have

$$\sum_{k=1}^{n} |\Delta F_k| = \sum_{k=1}^{n} |F(x_k) - F(x_{k-1})|$$

$$= |c| \sum_{k=1}^{n} |\alpha(x_k) - \alpha(x_{k-1})|.$$

Since α is of bounded variation on $[a, b]$, there is a number $M_1 > 0$ such that

$$\sum_{k=1}^{n} |\Delta \alpha_k| \leq M_1$$

$$\text{Hence, } \sum_{k=1}^{n} |\Delta F_k| \leq |c| M_1 = M \text{(say)}.$$

Hence F is of bounded variation on $[a, b]$.

To prove (ii):

Let α be continuous at $x_0 \in [a, b]$.

Claim: F is continuous at x_0.

Let $\epsilon > 0$ be given. Then there is a $\delta > 0$ such that

$$|x - x_0| < \delta \implies |\alpha(x) - \alpha(x_0)| < \frac{\epsilon}{|c|} \quad \forall\, x \in [a, b]$$

Hence, if $|x - x_0| < \delta$, then by (i), we have

$$|F(x) - F(x_0)| = |c||\alpha(x) - \alpha(x_0)| < |c|.\frac{\epsilon}{|c|} = \epsilon.$$

Hence, F is continuous at x_0.

To prove (iii):

Consider

$$F(y) - F(x) = c[\alpha(y) - \alpha(x)].$$

Dividing $y - x$ on both sides, we get

$$\frac{F(y) - F(x)}{y - x} = c\frac{[\alpha(y) - \alpha(x)]}{y - x} \tag{22}$$

Since f is continuous on $[a, b]$, then $c = f(x_0)$ for some $x_0 \in [a, b]$.

$$\lim_{x \to x_0} f(x) = f(x_0) \implies c = f(x_0).$$

Letting $y \to x$ in (22), we have

$$\lim_{y \to x} \frac{F(y) - F(x)}{y - x} = c \lim_{y \to x} \frac{[\alpha(y) - \alpha(x)]}{y - x}$$

$$\implies F'(x) = f(x_0)\alpha'(x).$$

□

Theorem 3.5.2. *If $f \in \mathcal{R}$ and $g \in \mathcal{R}$ on $[a, b]$, let*

$$F(x) = \int_a^x f(t)\,dt, \quad G(x) = \int_a^x g(t)\,dt, \; if \; x \in [a, b].$$

Then F and G are continuous functions of bounded variation on $[a, b]$. Also, $f \in \mathcal{R}(G)$ and $g \in \mathcal{R}(F)$ on $[a, b]$, and we have

$$\int_a^b f(x)g(x)\,dx = \int_a^b f(x)\,dG(x) = \int_a^b g(x)\,dF(x).$$

Proof. Assume that $\alpha \nearrow$ on $[a, b]$.

Let $f \in \mathcal{R}$ and $g \in \mathcal{R}$ on $[a, b]$.

By parts (i) and (ii) of Theorem 3.5.1, F and G are continuous functions of bounded variation on $[a, b]$.

Hence, by Theorem 3.1.3, we have $f \in \mathcal{R}(G)$ and $g \in \mathcal{R}(F)$ on $[a, b]$.

Also

$$\int_a^b f(x)g(x)\ d\alpha(x) = \int_a^b f(x)\ dG(x)$$

$$= \int_a^b g(x)\ dF(x).$$

Putting $\alpha(x) = x$, we have

$$\int_a^b f(x)g(x)\ dx = \int_a^b f(x)\ dG(x)$$

$$= \int_a^b g(x)\ dF(x).$$

□

Note: When $\alpha(x) = x$, part (iii) of Theorem 3.5.1 is sometimes called the $first\ fundamental\ theorem\ of\ integral\ calculus.$ It states that $F'(x) = f(x)$ at each point of continuity of f.

Let us sum up

- We have discussed the sufficient conditions for the existence of Riemann-Stieltjes Integrals.

3.6 Second Fundamental Theorem of Integral Calculus

Theorem 3.6.1. ***(Second fundamental theorem of integral calculus).*** *Assume that $f \in \mathcal{R}$ on $[a, b]$. Let g be a function defined on $[a, b]$ such that the derivative g' exists in (a, b) and has the value*

$$g'(x) = f(x),\ \forall\ x \in (a, b).$$

At the endpoints assume that $g(a+)$ and $g(b-)$ exist and satisfy

$$g(a) - g(a+) = g(b) - g(b-).$$

Then we have

$$\int_a^b f(x)\ dx = \int_a^b g'(x)\ dx = g(b) - g(a).$$

Proof. For any partition of $[a, b]$, we have

$$\begin{aligned} g(b) - g(a) &= g(x_n) - g(x_0) \\ &= g(x_n) - g(x_{n-1}) + g(x_{n-1}) - \ldots.. - g(x_1) + g(x_1) - g(x_0) \\ &= \sum_{k=1}^{n} [g(x_k) - g(x_{k-1})]. \end{aligned}$$

By mean value theorem, there is a point $t_k \in (x_{k-1}, x_k)$ such that

$$\begin{aligned} g(x_k) - g(x_{k-1}) &= g'(t_k)(x_k - x_{k-1}) \\ g(b) - g(a) &= \sum_{k=1}^{n} g'(t_k)\Delta x_k \\ &= \sum_{k=1}^{n} f(t_k)\Delta x_k. \end{aligned}$$

Let $\epsilon > 0$ be given. Since $f \in \mathcal{R}$ on $[a, b]$, there is a partition P_ϵ of $[a, b]$ such that P finer than P_ϵ implies

$$\left| S(P, f, \alpha) - \int_a^b f(x)\, dx \right| < \epsilon$$

$$\text{i.e.,} \left| \sum_{k=1}^{n} f(t_k)\Delta x_k - \int_a^b f(x)\, dx \right| < \epsilon.$$

$$\implies \quad \left| g(b) - g(a) - \int_a^b f(x)\, dx \right| < \epsilon.$$

Since ϵ was arbitrary,

$$\int_a^b f(x)\, dx = g(b) - g(a).$$

□

Theorem 3.6.2. *Assume $f \in \mathcal{R}$ on $[a, b]$. Let α be a function which is continuous on $[a, b]$ and whose derivative α' is Riemann integrable on $[a, b]$. Then the following integrals exist and are equal:*

$$\int_a^b f(x)\, d\alpha(x) = \int_a^b f(x)\alpha'(x)\, dx.$$

Proof. By the second fundamental theorem of calculus we have, for each x in $[a, b]$,

$$\alpha(x) - \alpha(a) = \int_a^x \alpha'(t)\, dt.$$

By Theorem 3.5.2, we have

$$\int_a^b f(x)g(x)\ dx = \int_a^b f(x)\ dG(x),$$

where $G(x) = \int_a^x g(t)\ dt$.

Putting $g = \alpha'$, we get

$$\int_a^b f(x)\alpha'(x)dx = \int_a^b f(x)d\Big(\int_a^x g(t)dt\Big)$$

$$\implies \int_a^b f(x)\ \alpha'(x)dx = \int_a^b f(x)g(x)\ dx.$$

□

Let us sum up

- We have prove the second fundamental theorem of integral calculus.

3.7 Change of Variable in a Riemann Integral

The formula $\int_a^b f\ d\alpha = \int_c^d h\ d\beta$ of Theorem 2.6.1 for changing the variable in an integral assumes the form

$$\int_{g(c)}^{g(d)} f(x)\ dx = \int_c^d f[g(t)]g'(t)\ dt,$$

when $\alpha(x) = x$ and when g is a strictly monotonic function with a continuous derivative g'. It is valid if $f \in \mathcal{R}$ on $[a, b]$. When f is continuous, we can use Theorem 3.5.1 to remove the restriction that g be monotonic. In fact, we have the following theorem:

Theorem 3.7.1. ***(Change of variable in a Riemann integral).*** *Assume that g has a continuous derivative g' on an interval $[c, d]$. Let f be continuous on $g([c, d])$ and define F by the equation*

$$F(x) = \int_{g(c)}^{x} f(t)\ dt, \quad if\ x \in g([c, d]).$$

Then, for each x in $[c, d]$ the integral $\int_c^x f[g(t)]g'(t)\ dt$ exists and has the value $F[g(x)]$. In particular, we have

$$\int_{g(c)}^{g(d)} f(x)\ dx = \int_c^d f[g(t)]g'(t)\ dt.$$

Proof. Assume that g has a continuous derivative g' on an interval $[c, d]$. Let f be continuous on $g([c, d])$.

Since both g' and the composite function $f \circ g$ are continuous on $[c, d]$, the product $g'.(f \circ g)$ is also continuous on $[c, d]$. Therefore, $\int_c^x f[g(t)]g'(t)\, dt$ exists for each x in $[c, d]$.

Define G on $[c, d]$ as follows:

$$G(x) = \int_c^x f[g(t)]g'(t)\, dt.$$

Claim: $G(x) = F[g(x)]$, for all x in $[c, d]$.

By Theorem 3.5.1, we have

$$G'(x) = f[g(x)]g'(x).$$

By chain rule, we have

$$\begin{aligned} [F(g(x))]' &= F'[g(x)].g'(x) \\ &= f(g(x))g'(x) \quad (\because F'(x) = f(x).) \\ G'(x) &= [F(g(x))]' \end{aligned}$$

$$\implies [G(x) - F(g(x))]' = 0.$$

$$\implies G(x) - F(g(x)) = \text{constant}.$$

But, when $x = c$, we get $G(c) = 0$ and $F[g(c)] = 0$, so this constant must be 0.

Hence $G(x) = F[g(x)]$, for all x in $[c, d]$.

In particular, when $x = d$, we get $G(d) = F[g(d)]$. Therefore,

$$\int_c^d f(g(t))g'(t)\, dt = \int_{g(c)}^{g(d)} f(t)\, dt.$$

□

Let us sum up

- We have discussed the change of variable in a Riemann integral.

3.8 Second Mean-Value Theorem for Riemann Integrals

Theorem 3.8.1. *Let g be continuous and assume that $f \nearrow$ on $[a,b]$. Let A and B be two real numbers satisfying the inequalities*

$$A \leq f(a+) \text{ and } B \geq f(b-).$$

Then there exists a point x_0 in $[a,b]$ such that

$$i) \int_a^b f(x)g(x)\,dx = A\int_a^{x_0} g(x)\,dx + B\int_{x_0}^b g(x)\,dx.$$

In particular, if $f(x) \geq 0$ for all x in $[a,b]$, we have

$$ii) \int_a^b f(x)g(x)\,dx = B\int_{x_0}^b g(x)\,dx,$$

where $x_0 \in [a,b]$.

Proof. To prove (i):

Let $\alpha(x) = \int_a^x g(t)\,dt$. Then $\alpha' = g$.

Since g is continuous, α is continuous.

Since f is increasing on $[a,b]$, by Theorem 3.4.2, there exists a point x_0 in $[a,b]$ such that

$$\int_a^b f(x)\,d\alpha(x) = f(a)\int_a^{x_0} d\alpha(x) + f(b)\int_{x_0}^b d\alpha(x)$$

$$\int_a^b f(x)\,\alpha'(x)dx = f(a)\int_a^{x_0} \alpha'(x)dx + f(b)\int_{x_0}^b \alpha'(x)dx$$

$$\int_a^b f(x)g(x)dx = f(a)\int_a^{x_0} g(x)dx + f(b)\int_{x_0}^b g(x)dx$$

$$= A\int_a^{x_0} g(x)dx + B\int_{x_0}^b g(x)dx,$$

where $A = f(a)$ and $B = f(b)$.

Now let A and B be any two real numbers satisfying $A \leq f(a+)$ and $f(b-) \leq B$.

Then, we can redefine f at the endpoints a and b to have the values $f(a) = A$ and $f(b) = B$.

The modified f is still increasing on $[a, b]$.
Also we know that changing the value of f at a finite number of points does not affect the value of a Riemann integral.
To prove (ii):
Consider

$$\int_a^b f(x)g(x)dx = A\int_a^{x_0} g(x)dx + B\int_{x_0}^b g(x)dx.$$

Putting $A = 0$, we have

$$\int_a^b f(x)g(x)dx = B\int_{x_0}^b g(x)dx.$$

□

Note: Part (ii) is known as Bonnet's Theorem.

Let us sum up

- We have discussed the second Mean-Value Theorem for Riemann Integrals.

3.9 Riemann-Stieltjes Integrals Depending on a Parameter

Theorem 3.9.1. *Let f be continuous at each point (x, y) of a rectangle*

$$Q = \{(x, y) : a \leq x \leq b,\ c \leq y \leq d\}.$$

Assume that α is of bounded variation on $[a, b]$ and let F be the function defined on $[c, d]$ by the equation

$$F(y) = \int_a^b f(x, y)\ d\alpha(x).$$

Then F is continuous on $[c, d]$. In other words, if $y_0 \in [c, d]$, we have

$$\lim_{y \to y_0} \int_a^b f(x, y)\ d\alpha(x) = \int_a^b \lim_{y \to y_0} f(x, y)\ d\alpha(x)$$
$$= \int_a^b f(x, y_0)\ d\alpha(x).$$

Proof. Let f be continuous at each point (x, y) of a rectangle

$$Q = \{(x, y) : a \leq x \leq b,\ c \leq y \leq d\}.$$

Assume that $\alpha \nearrow$ on $[a, b]$.

Since f is continuous and Q is compact, f is uniformly continuous on Q.

i.e., given $\epsilon > 0$, there exists a $\delta > 0$ such that for every pair of points $z = (x, y)$ and $z' = (x', y')$ in Q with $|z - z'| < \delta$, we have

$$|f(x, y) - f(x', y')| < \epsilon.$$

Let $|y - y'| < \delta$, we have $|(x, y) - (x, y')| = \sqrt{(x - x)^2 + (y - y')^2} = \sqrt{(y - y')^2} = |y - y'| < \delta$. Now,

$$\begin{aligned} |F(y) - F(y')| &= \left| \int_a^b f(x, y)\, d\alpha(x) - \int_a^b f(x, y')\, d\alpha(x) \right| \\ &= \left| \int_a^b [f(x, y) - f(x, y')] d\alpha(x) \right| \\ &\leq \int_a^b |f(x, y) - f(x, y')|\, d\alpha(x) \\ &< \epsilon \int_a^b d\alpha(x) = \epsilon[\alpha(b) - \alpha(a)]. \end{aligned}$$

This shows that F is continuous on $[c, d]$.

$$\text{Hence, } \lim_{y \to y_0} F(y) = F(y_0) \tag{23}$$

Since f is continuous, $\lim_{y \to y_0} f(x, y) = f(x, y_0)$.

Therefore, (23) $\Longrightarrow$

$$\begin{aligned} \lim_{y \to y_0} \int_a^b f(x, y)\, d\alpha(x) &= \int_a^b f(x, y_0)\, d\alpha(x) \\ &= \int_a^b \lim_{y \to y_0} f(x, y)\, d\alpha(x), \end{aligned}$$

where $y_0 \in [c, d]$. □

Theorem 3.9.2. *If f is continuous on the rectangle $[a, b] \times [c, d]$, and $g \in \mathcal{R}$ on $[a, b]$, then the function F defined by the equation*

$$F(y) = \int_a^b g(x) f(x, y)\, dx,$$

is continuous on $[c, d]$. That is, if $y_0 \in [c, d]$, we have

$$\lim_{y \to y_0} \int_a^b g(x) f(x, y)\, dx = \int_a^b g(x) f(x, y_0)\, dx.$$

Proof. Let $G(x) = \int_a^x g(t)\, dt$, then, by Theorem 3.1.3 we have

$$F(y) = \int_a^b f(x, y)\, dG(x).$$

Now applying Theorem 3.9.1, it follows that F is continuous on $[c, d]$ and we have

$$\lim_{y \to y_0} \int_a^b f(x, y)\, dG(x) = \int_a^b f(x, y_0)\, dG(x).$$

$$\therefore \lim_{y \to y_0} \int_a^b f(x, y)\, G'(x) dx = \int_a^b f(x, y_0)\, G'(x) dx$$

$$\implies \lim_{y \to y_0} \int_a^b f(x, y)\, g(x) dx = \int_a^b f(x, y_0)\, g(x) dx.$$

□

Let us sum up

- We have derived the Riemann-Stieltjes integrals depending on a parameter.

3.10 Differentiation Under the Integral Sign

Theorem 3.10.1. *Let* $Q = \{(x, y) : a \le x \le b,\ c \le y \le d\}$. *Assume that* α *is of bounded variation on* $[a, b]$, *and for each fixed* y *in* $[c, d]$, *assume that the integral*

$$F(y) = \int_a^b f(x, y)\, d\alpha(x),$$

exists. If the partial derivative $D_2 f$ *is continuous on* Q, *the derivative* $F'(y)$ *exists for each* y *in* (c, d) *and is given by*

$$F'(y) = \int_a^b D_2 f(x, y)\, d\alpha(x).$$

Proof. Let $y_0 \in (c, d)$ and $y \neq y_0$.
Consider

$$F(y) - F(y_0) = \int_a^b f(x, y)\, d\alpha(x) - \int_a^b f(x, y_0)\, d\alpha(x)$$

$$\implies \frac{F(y) - F(y_0)}{y - y_0} = \int_a^b \frac{f(x, y) - f(x, y_0)}{y - y_0}\, d\alpha(x)$$

$$= \int_a^b D_2 f(x, \bar{y})\, d\alpha(x),$$

where $\bar{y}$ is between y and y_0.

Since $D_2 f$ is continuous on Q, by using Theorem 3.9.1, we have

$$\begin{aligned}
\lim_{y \to y_0} \frac{F(y) - F(y_0)}{y - y_0} &= \lim_{y \to y_0} \int_a^b D_2 f(x, \bar{y})\, d\alpha(x) \\
&= \int_a^b D_2 f(x, \bar{y})\, d\alpha(x) \\
\therefore\ F'(y) &= \int_a^b D_2 f(x, \bar{y})\, d\alpha(x).
\end{aligned}$$

□

Let us sum up

- We have discussed differentiation under the integral sign.

3.11 Interchanging the Order of Integration

Theorem 3.11.1. *Let $Q = \{(x, y) : a \le x \le b,\ c \le y \le d\}$. Assume that α is of bounded variation on $[a, b]$, β is of bounded variation on $[c, d]$, and f is continuous on Q. If $(x, y) \in Q$, define*

$$F(y) = \int_a^b f(x, y)\, d\alpha(x), \quad G(x) = \int_c^d f(x, y)\, d\beta(y).$$

Then $F \in \mathcal{R}(\beta)$ on $[c, d]$, $G \in \mathcal{R}(\alpha)$ on $[a, b]$, and we have

$$\int_c^d F(y)\, d\beta(y) = \int_a^b G(x)\, d\alpha(x).$$

In other words, we may interchange the order of integration as follows:

$$\int_a^b \left[\int_c^d f(x, y)\, d\beta(y) \right] d\alpha(x) = \int_c^d \left[\int_a^b f(x, y)\, d\alpha(x) \right] d\beta(y).$$

Proof. Let $Q = \{(x, y) : a \le x \le b,\ c \le y \le d\}$ and assume that α is of bounded variation on $[a, b]$, β is of bounded variation on $[c, d]$, and f is continuous on Q.

For $(x, y) \in Q$, define

$$F(y) = \int_a^b f(x, y)\, d\alpha(x) \text{ and } G(x) = \int_c^d f(x, y)\, d\beta(y).$$

Hence, by Theorem 3.9.1, F is continuous on $[c, d]$ and hence $F \in \mathcal{R}(\beta)$ on $[c, d]$.
Similarly, $G \in \mathcal{R}(\alpha)$ on $[a, b]$.
To prove: $\int_c^d F(y)\, d\beta(y) = \int_a^b G(x)\, d\alpha(x)$.
Assume $\alpha \nearrow$ on $[a, b]$ and $\beta \nearrow$ on $[c, d]$.
Since f is continuous on Q and Q is compact, f is uniformly continuous on Q.
i.e., Given $\epsilon > 0$, there is a $\delta > 0$ such that for every pair of points $z = (x, y)$ and $z' = (x', y')$ in Q, with $|z - z'| < \delta$, we have

$$|f(x, y) - f(x', y')| < \epsilon.$$

Let us now subdivide Q into n^2 equal rectangles by subdividing $[a, b]$ and $[c, d]$ each into n equal parts, where n is chosen so that

$$\frac{b-a}{n} < \frac{\delta}{\sqrt{2}}, \qquad \frac{d-c}{n} < \frac{\delta}{\sqrt{2}}.$$

Let

$$x_k = a + \frac{k(b-a)}{n}, \qquad y_k = c + \frac{k(d-c)}{n},\ (k = 0, 1, 2, ..., n).$$

For $k = 0, 1, 2, ..., n$, we have

$$\int_a^b \left(\int_c^d f(x, y)\, d\beta(y) \right) d\alpha(x) = \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} \int_{x_k}^{x_{k+1}} \left(\int_{y_j}^{y_{j+1}} f(x, y)\, d\beta(y) \right) d\alpha(x).$$

We apply Theorem 3.4.1 twice on the right, we have

$$\int_a^b \left(\int_c^d f(x, y)\, d\beta(y) \right) d\alpha(x) = \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} \int_{x_k}^{x_{k+1}} f(x, y_j')[\beta(y_{j+1}) - \beta(y_j)] d\alpha(x),$$

where $y_j' \in [y_j, y_{j+1}]$.

$$= \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} f(x_k', y_j')[\beta(y_{j+1}) - \beta(y_j)][\alpha(x_{k+1}) - \alpha(x_k)],$$

where $(x_k', y_j') \in Q_{k,j}$ having (x_k, y_j) and (x_{k+1}, y_{j+1}) as opposite vertices.
Similarly, we have

$$\int_c^d \left(\int_a^b f(x, y)\, d\alpha(x) \right) d\beta(y) = \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} \int_{x_k}^{x_{k+1}} \left(\int_{y_j}^{y_{j+1}} f(x, y)\, d\alpha(y) \right) d\beta(x).$$

We apply Theorem 3.4.1 twice on the right, we have

$$\int_c^d \left(\int_a^b f(x,y)\ d\alpha(x) \right) d\beta(y) = \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} f(x_k^{''}, y_j^{''})[\beta(y_{j+1}) - \beta(y_j)][\alpha(x_{k+1}) - \alpha(x_k)],$$

where $(x_k^{''}, y_j^{''}) \in Q_{k,j}$.

By uniform continuity of f, we have

$$|f(x_k', y_k') - f(x_k^{''}, y_k^{''})| < \epsilon.$$

Put $F(y) = \int_a^b f(x,y)\ d\alpha(x)$ and $G(x) = \int_c^d f(x,y)\ d\beta(y)$. Consider

$$\begin{aligned}
\left| \int_a^b G(x)\ d\alpha(x) - \int_c^d F(y)\ d\beta(y) \right| &= \left| \int_a^b \int_c^d f(x,y)\ d\alpha(x) d\beta(y) - \int_c^d \int_a^b f(x,y) d\alpha(x)\ d\beta(y) \right| \\
&= \Bigg| \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} f(x_k', y_j')[\beta(y_{j+1}) - \beta(y_j)][\alpha(x_{k+1}) - \alpha(x_k)] \\
&\quad - \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} f(x_k^{''}, y_j^{''})[\beta(y_{j+1}) - \beta(y_j)][\alpha(x_{k+1}) - \alpha(x_k)] \Bigg|
\end{aligned}$$

$$\begin{aligned}
&= \left| \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} [f(x_k', y_k') - f(x_k^{''}, y_k^{''})][\beta(y_{j+1}) - \beta(y_j)][\alpha(x_{k+1}) - \alpha(x_k)] \right| \\
&\le \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} |f(x_k', y_k') - f(x_k^{''}, y_k^{''})| [\beta(y_{j+1}) - \beta(y_j)][\alpha(x_{k+1}) - \alpha(x_k)] \\
&< \epsilon \sum_{j=0}^{n-1} [\beta(y_{j+1}) - \beta(y_j)] \sum_{k=0}^{n-1} [\alpha(x_{k+1}) - \alpha(x_k)] \\
&= \epsilon[\beta(d) - \beta(c)][\alpha(b) - \alpha(a)].
\end{aligned}$$

Since ϵ was arbitrary, we have

$$\int_a^b G(x)\ d\alpha(x) = \int_c^d F(y)\ d\beta(y).$$

$$\therefore \quad \int_a^b \left[\int_c^d f(x,y)\ d\beta(y) \right] d\alpha(x) = \int_c^d \left[\int_a^b f(x,y)\ d\alpha(x) \right] d\beta(y).$$

□

Theorem 3.11.2. *Let f be continuous on the rectangle $[a,b] \times [c,d]$. If $g \in \mathcal{R}$ on $[a,b]$ and if $h \in \mathcal{R}$ on $[c,d]$, then we have*

$$\int_a^b \left[\int_c^d g(x)h(y)f(x,y)\ dy \right] dx = \int_c^d \left[\int_a^b g(x)h(y)f(x,y)\ dx \right] dy.$$

Proof. Let $\alpha(x) = \int_a^x g(u)\ du$ and $\beta(y) = \int_c^y h(v)\ dv$. Apply Theorem 3.1.3 and 3.11.1, we have

$$\int_a^b \left[\int_c^d g(x)h(y)f(x,y)\ dy \right] dx = \int_c^d \left[\int_a^b g(x)h(y)f(x,y)\ dx \right] dy.$$

□

Let us sum up

- We have derived the interchanging the order of integration.

Summary

- Discussed α is of bounded variation on $[a,b]$ and assume that $f \in \mathcal{R}(\alpha)$ on $[a,b]$. Then $f \in \mathcal{R}(\alpha)$ on every subinterval $[c,d]$ of $[a,b]$.
- Necessary and sufficient conditions for existence of Riemann-Stieltjes Integrals.
- Derived first and second Mean-Value Theorem for Riemann-Stieltjes integrals.
- Discussed the first and second fundamental theorem of integral calculus.
- Discussed the change of variable in a Riemann integral.
- Discussed Second Mean-Value Theorem for Riemann Integrals.
- Derived Riemann-Stieltjes integrals Depending on a Parameter.
- Discussed differentiation Under the integral sign.
- Derived interchanging the order of integration.

Exercises

1. Give an example of a bounded function f and an increasing function α defined on $[a,b]$ such that $|f| \in R(\alpha)$ but for which $\int_a^b f d\alpha$ does not exist.
2. Let x_n be a sequence of functions of bounded variation on $[a,b]$. Suppose there exists a function α defined on $[a,b]$ such that the total variation of $\alpha - \alpha_n$ on $[a,b]$

tends to 0 as $n \to \infty$. Assume also that $\alpha(a) = \alpha_n(a) = 0$ for each $n = 1, 2, \cdots$. If f is continuous on $[a, b]$, prove that

$$\lim_{n\to\infty} \int_a^b f(x) d\alpha_(x) = \int_a^b f(x) d\alpha(x).$$

3. Assume $g \in R$ on $[a, b]$ and define $f(x) = \int_a^x g(t)dt$ if $x \in [a, b]$. Prove that the integral $\int_a^x |g(t)|dt$ gives the total variation of f on $[a, x]$.

4. Let f be a positive continuous function in $[a, b]$. Let M denote the maximum value of f on $[a, b]$. Show that

$$\lim_{n\to\infty} \left(\int_a^b f(x)^n dx \right)^{1/n} = M.$$

5. Use Lebesgue's theorem to prove that if $f \in R$ and $g \in R$ on $[a, b]$ and if $f(x) \geq m > 0$ for all x in$[a, b]$, then the function h defined by $h(x) = f(x)^{g(x)}$ is Riemann-integrable on $[a, b]$.

References

Tom M. Apostol, Mathematical Analysis, Second Edition, Addison-Wesley Publishing Company Inc., New York, 1974.

Suggested Readings

1. T.M. Apostol, Calculus Vol.2, Multi-Variable Calculus and Linear Algebra with Applications to Differential Equations and Probability, Second Edition - Reprint, John Wiley & Sons, 2016.

2. R.G. Bartle, Real Analysis, John Wiley and Sons Inc., 1976.

3. W. Rudin, Principles of Mathematical Analysis, Third Edition, Mc-Graw Hill Company, New York, 1976.

4. S.C. Malik and S. Arora. Mathematical Analysis, Wiley Eastern Limited, New Delhi, 1991.

5. B.R. Gelbaum and J. Olmsted, Counter Examples in Analysis, Holden day, San Francisco, 1964.

6. A.L. Gupta and N.R. Gupta, Principles of Real Analysis, Pearson Education, (Indian print) 2003.

Unit 4

INFINITE SERIES AND INFINITE PRODUCTS

Objectives

After the successful completion of this unit; the students are expected

- To recall the basic concepts of limit property, series and products.
- To analyse a given infinite series converge (or) diverge.
- To understand the fundamental concepts of infinite series and infinite products.
- To analyse and work problem related to double sequence and double series.

4.1 Double Sequences

Definition 4.1.1. *A function f whose domain is $\mathbb{Z}^+ \times \mathbb{Z}^+$ is called a double sequence.*

Definition 4.1.2. *If $a \in \mathbb{C}$, we write $\lim\limits_{p,q\to\infty} f(p,q) = a$ and we say that the double sequence f converges to a, provided that the following condition is satisfied. For every $\epsilon > 0$, there exists an N such that $|f(p,q) - a| < \epsilon$, whenever both $p > N$ and $q > N$.*

Theorem 4.1.3. *Assume that $\lim\limits_{p,q\to\infty} f(p,q) = a$. For each fixed p, assume that the limit $\lim\limits_{q\to\infty} f(p,q)$ exists. Then the limit $\lim\limits_{p\to\infty}\left(\lim\limits_{q\to\infty} f(p,q)\right)$ also exists and has the value a.*

Proof. For each fixed p, assume that the limit $\lim\limits_{q\to\infty} f(p,q)$ exists.
Let $F(p) = \lim\limits_{q\to\infty} f(p,q)$.

Given $\epsilon > 0$, choose N_1 such that

$$|f(p,q) - a| < \frac{\epsilon}{2}, \qquad if\ p > N_1 \quad and \ \ q > N_1. \qquad\qquad (a)$$

Since, $\lim_{q\to\infty} f(p,q) = F(p), \forall\ p$ we can choose N_2 such that

$$|F(p) - f(p,q)| < \frac{\epsilon}{2}, \quad if\ q > N_2, \qquad\qquad (b)$$

where N_2 depends on p as well as on ϵ. For each $p > N_1$, choose N_2 and then choose a fixed q such that $q = \max\{N_1, N_2\}$.

For such p and q

$$\begin{aligned} |F(p) - a| &= |F(p) - f(p,q) + f(p,q) - a| \\ &\leq |F(p) - f(p,q)| + |f(p,q) - a| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Thus, we have

$$|F(p) - a| < \epsilon, \qquad\qquad p > N_1.$$

which implies that

$$\lim_{p\to\infty} F(p) = a.$$

□

Note : To distinguish $\lim_{p,q\to\infty} f(p,q)$ from $\lim_{p\to\infty}\left(\lim_{q\to\infty} f(p,q)\right)$, the first is called a double limit, the second an iterated limit.

The following example shows that the converse of the above theorem is not true .

Example. Let

$$\begin{aligned} f(p,q) &= \frac{pq}{p^2+q^2}, \quad (p = 1, 2, \cdots,\ q = 1, 2, \cdots) \\ &= \frac{pq}{q^2\left(\frac{p^2}{q^2}+1\right)} \\ \lim_{q\to\infty} f(p,q) &= \lim_{q\to\infty} \frac{p}{q\left(\frac{p^2}{q^2}+1\right)} = 0 \end{aligned}$$

$$\therefore \lim_{p\to\infty}\left(\lim_{q\to\infty} f(p,q)\right) = 0.$$

If $p = q$, then

$$f(p,q) = \frac{p^2}{2p^2} = \frac{1}{2}.$$

If $p = 2q$, then

$$f(p,q) = \frac{2q^2}{5q^2} = \frac{2}{5}.$$

Hence, $\lim\limits_{p,q\to\infty} f(p,q)$ does not exist.

Let us sum up

The existence of the double limit $\lim\limits_{p,q\to\infty} f(p,q)$ and of $\lim\limits_{q\to\infty} f(p,q)$ implies the existence of the iterated limit $\lim\limits_{p\to\infty}\left(\lim\limits_{q\to\infty} f(p,q)\right)$.

Check your progress

1. Let $f(m,n) = \dfrac{m}{m+n}$, $m,n = 1,2,\ldots$. Show that $\lim\limits_{m,n\to\infty} f(m,n)$ does not exist.

4.2 Double Series

Definition 4.2.1. *Let f be a double sequence and let s be the double sequence defined by the equation*

$$s(p,q) = \sum_{m=1}^{p}\sum_{n=1}^{q} f(m,n).$$

The pair (f,s) is called a double series and is denoted by the symbol $\sum\limits_{m,n} f(m,n)$ or, more briefly, by $\sum f(m,n)$. The double series is said to converge to the sum a if

$$\lim_{p,q\to\infty} s(p,q) = a.$$

Note : Each pair $f(m,n)$ is called a term of the double series and each $s(p,q)$ is a partial sum.
If $\sum f(m,n)$ has only positive terms, it is easy to show that it converges if and only if the set of partial sums is bounded.

Definition 4.2.2. *We say that $\sum f(m,n)$ converges absolutely if $\sum |f(m,n)|$ converges.*

Let us sum up

- We introduced the concept of double series and its convergence.

4.3 Rearrangement Theorem for Double Series

Definition 4.3.1. *Let f be a double sequence and let g be a one-to-one function defined on $\mathbb{Z}^+$ with range $\mathbb{Z}^+ \times \mathbb{Z}^+$. Let G be the sequence defined by*

$$G(n) = f[g(n)] \qquad if\ n \in \mathbb{Z}^+.$$

Then g is said to be an arrangement of the double sequence f into the sequence G.

Note : Let $\{f_1, f_2, ...\}$ be a countable collection of functions, each defined on $\mathbb{Z}^+$, having the following properties:

a) Each f_n is one-to-one on $\mathbb{Z}^+$.

b) The range $f_n(\mathbb{Z}^+)$ is a subset Q_n of $\mathbb{Z}^+$.

c) $\{Q_1, Q_2, ...\}$ is a collection of disjoint sets whose union is $\mathbb{Z}^+$.

Let $\sum a_n$ be an absolutely convergent series and define

$$b_k(n) = a_{f_k(n)}, \quad if\ n \in \mathbb{Z}^+,\ k \in \mathbb{Z}^+.$$

Then:

i) For each k, $\sum_{n=1}^{\infty} b_k(n)$ is an absolutely convergent subseries of $\sum a_n$.

ii) If $s_k = \sum_{n=1}^{\infty} b_k(n)$, the series $\sum\limits_{k=1}^{\infty} s_k$ converges absolutely and has the same sum as $\sum\limits_{k=1}^{\infty} a_k$.

Theorem 4.3.2. *Let $\sum f(m,n)$ be a given double series and let g be an arrangement of the double sequence f into a sequence G. Then*

a) $\sum G(n)$ converges absolutely if and only if $\sum f(m,n)$ converges absolutely. Assuming that $\sum f(m,n)$ does converge absolutely, with sum S, we have further:

b) $\sum\limits_{n=1}^{\infty} G(n) = S$.

c) $\sum\limits_{n=1}^{\infty} f(m,n)$ *and* $\sum\limits_{m=1}^{\infty} f(m,n)$ *both converge absolutely.*

d) If $A_m = \sum\limits_{n=1}^{\infty} f(m,n)$ *and* $B_n = \sum\limits_{m=1}^{\infty} f(m,n)$, *both series* A_m *and* B_n *converge absolutely and both have sum* S. *That is,*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f(m,n) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} f(m,n) = S.$$

Proof. To prove (a): Assume $\sum G(n)$ converges absolutely.

Claim : $\sum f(m,n)$ converges absolutely.

Let $T_k = |G(1)| + ... + |G(k)|$ and let

$$S(p,q) = \sum_{m=1}^{p}\sum_{n=1}^{q} |f(m,n)|.$$

We know that, a double series of positive terms converges if and only if the set of partial sums is bounded.

Hence, for each k there exists a pair (p,q) such that $T_k \le S(p,q)$.

Hence, $\sum |G(n)|$ has bounded partial sums.

Hence $\sum f(m,n)$ converges absolutely.

Conversely, assume $\sum f(m,n)$ converges absolutely.

Claim : $\sum G(n)$ converges absolutely.

For each pair (p,q), there exists an integer r such that $S(p,q) \le T_r$.

Hence, $\sum |f(m,n)|$ has bounded partial sums.

Hence $\sum G(n)$ converges absolutely.

To prove (b): Now assume that $\sum |f(m,n)|$ converges.

Before we prove (b), we will show that the sum of the series $\sum G(n)$ is independent of the function g used to construct G from f, where g is an arrangement of the double sequence f into the sequence G.

Let h be another arrangement of the double sequence f into a sequence H.

Then we have

$$G(n) = f[g(n)] \text{ and } H(n) = f[h(n)].$$

$\implies G(n) = H[k(n)]$, where $k(n) = h^{-1}[g(n)]$.

Since k is a one-to-one mapping of $\mathbb{Z}^+$ onto $\mathbb{Z}^+$, the series $\sum H(n)$ is a rearrangement of $\sum G(n)$.

Hence, $\sum H(n)$ and $\sum G(n)$ has the same sum.

Let us denote this common sum by S'.

To prove : $S = S'$.

Let $T = \lim_{p,q\to\infty} S(p,q)$.

Given $\epsilon > 0$, choose N such that

$$|T - S(p,q)| < \frac{\epsilon}{2} \quad if\ p > N \ \ and \ \ q > N.$$

Now write

$$t_k = \sum_{n=1}^{k} G(n), \qquad s(p,q) = \sum_{m=1}^{p}\sum_{n=1}^{q} f(m,n).$$

Choose M such that t_M includes all terms $f(m,n)$ with

$$1 \le m \le N+1, \qquad 1 \le n \le N+1.$$

Then $t_M - s(N+1,N+1)$ is a sum of terms $f(m,n)$ with either $m > N$ or $n > N$.

If $n \ge M$, we have

$$\begin{aligned} |t_M - s(N+1,N+1)| &\le |t_n - s(N+1,N+1)| \\ &\le T - S(N+1,N+1) < \frac{\epsilon}{2}. \end{aligned}$$

Similarly,

$$|S - s(N+1,N+1)| \le T - S(N+1,N+1) < \frac{\epsilon}{2}.$$

$$\begin{aligned} |t_n - S| &= |t_n - s(N+1,N+1) + s(N+1,N+1) - S| \\ &\le |t_n - s(N+1,N+1)| + |S - s(N+1,N+1)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Hence, $|t_n - S| < \epsilon$ whenever $n \ge M$.

Hence, $t_n \to S$ as $n \to \infty$.

Since $\lim_{n\to\infty} t_n = S'$,

$$S' = S.$$

To prove (c):

Now the series $\sum_{n=1}^{\infty} f(m,n)$ and $\sum_{m=1}^{\infty} f(m,n)$ are subseries of $\sum G(n)$.
By (a), we have
$\sum_{n=1}^{\infty} f(m,n)$ and $\sum_{m=1}^{\infty} f(m,n)$ both converge absolutely.
To prove (d):

Let $A_m = \sum_{n=1}^{\infty} f(m,n)$ and $B_n = \sum_{m=1}^{\infty} f(m,n)$.
By note, we have
The series $A_m = \sum_{n=1}^{\infty} f(m,n)$ and $B_n = \sum_{m=1}^{\infty} f(m,n)$ both converge absolutely, and both have the same sum S.

$$i.e., \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f(m,n) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} f(m,n) = S.$$

□

Note : The series $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f(m,n)$ and $\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} f(m,n)$ are called **iterated series**. Convergence of both iterated series does not imply their equality. For example, suppose

$$f(m,n) = \begin{cases} 1, & if\ m = n+1,\ n = 1,2,...., \\ -1, & if\ m = n-1,\ n = 1,2,...., \\ 0, & otherwise. \end{cases}$$

Then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f(m,n) = -1, \quad but \quad \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} f(m,n) = 1.$$

Let us sum up

- We have derived the necessary and sufficient conditions for the rearrangement of a double series.

4.4 A Sufficient Condition for Equality of Iterated Series

Theorem 4.4.1. *Let f be a complex-valued double sequence. Assume that $\sum_{n=1}^{\infty} f(m,n)$ converges absolutely for each fixed m and that*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |f(m,n)|,$$

converges. Then:

a) *The double series* $\sum\limits_{m,n} f(m,n)$ *converges absolutely.*

b) *The series* $\sum\limits_{m=1}^{\infty} f(m,n)$ *converges absolutely for each* n.

c) *Both iterated series* $\sum\limits_{n=1}^{\infty}\sum\limits_{m=1}^{\infty} f(m,n)$ *and* $\sum\limits_{m=1}^{\infty}\sum\limits_{n=1}^{\infty} f(m,n)$ *converge absolutely and we have*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} f(m,n) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f(m,n) = \sum_{m,n} f(m,n).$$

Proof. To prove (a):

Let g be an arrangement of the double sequence f into a sequence G.

All the partial sums of $\sum |G(n)|$ are bounded by $\sum\limits_{m=1}^{\infty}\sum\limits_{n=1}^{\infty} |f(m,n)|$.

Hence, $\sum G(n)$ is absolutely convergent.

By Theorem 4.3.1(a), the double series $\sum\limits_{m,n} f(m,n)$ converges absolutely.

To prove (b):

The series $\sum\limits_{m=1}^{\infty} f(m,n)$ is a subseries of $\sum G(n)$.

By (a), we have $\sum\limits_{m=1}^{\infty} f(m,n)$ converges absolutely for each n.

To prove (c):

By Theorem 4.3.1(d), we have $\sum\limits_{n=1}^{\infty}\sum\limits_{m=1}^{\infty} f(m,n)$ and $\sum\limits_{m=1}^{\infty}\sum\limits_{n=1}^{\infty} f(m,n)$both converge absolutely, and

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} f(m,n) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f(m,n) = \sum_{m,n} f(m,n).$$

□

Theorem 4.4.2. *Let* $\sum a_m$ *and* $\sum b_n$ *be two absolutely convergent series with sums* A *and* B, *respectively. Let* f *be the double sequence defined by the equation*

$$f(m,n) = a_m b_n, \quad if\ (m,n) \in \mathbb{Z}^+ \times \mathbb{Z}^+.$$

Then $\sum\limits_{m,n} f(m,n)$ *converges absolutely and has the sum* AB.

Proof. Let f be the double sequence defined by the equation

$$f(m,n) = a_m b_n, \qquad if\ (m,n) \in \mathbb{Z}^+ \times \mathbb{Z}^+.$$

Consider,

$$\begin{aligned}\sum_{m=1}^{\infty}|a_m|\sum_{n=1}^{\infty}|b_n| &= \sum_{m=1}^{\infty}\left(|a_m|\sum_{n=1}^{\infty}|b_n|\right)\\ &= \sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty}|a_m||b_n|\right)\\ &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}|a_m||b_n|\\ &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}|a_m b_n|.\end{aligned}$$

Therefore,

$$AB = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}|a_m b_n|.$$

By Theorem 4.4.1, the double series $\sum\limits_{m,n} a_m b_n$ converges absolutely and has the sum AB.

Hence, $\sum\limits_{m,n} f(m,n)$ converges absolutely and has the sum AB. □

Let us sum up

- A sufficient condition for the equality of iterated series has been derived.

4.5 Multiplication of Series

Definition 4.5.1. *Given two series* $\sum\limits_{n=0}^{\infty} a_n$ *and* $\sum\limits_{n=0}^{\infty} b_n$, *define*

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}, \quad if\ n = 0, 1, 2, ...$$

The series $\sum\limits_{n=0}^{\infty} c_n$ *is called the Cauchy product of* $\sum a_n$ *and* $\sum b_n$.

Note : This definition may be motivated as follows. If we take two power series $\sum a_n z^n$ and $\sum b_n z^n$, multiply them term by term, and collect terms containing the same power of z, we get

$$\begin{aligned}\sum_{n=0}^{\infty} a_n z^n . \sum_{n=0}^{\infty} b_n z^n &= (a_0 + a_1 z + a_2 z^2 + \cdots)(b_0 + b_1 z + b_2 z^2 + \cdots)\\ &= a_0 b_0 + (a_0 b_1 + a_1 b_0)z + \cdots\end{aligned}$$

$$= c_0 + c_1 z + c_2 z^2 + \cdots$$

Setting $z = 1$, we arrive at the above definition.

Absolute convergence of both $\sum a_n$ and $\sum b_n$ implies convergence of the Cauchy product to the value

$$\sum_{n=0}^{\infty} c_n = \Big(\sum_{n=0}^{\infty} a_n\Big)\Big(\sum_{n=0}^{\infty} b_n\Big).$$

This equation may fail to hold if both $\sum a_n$ and $\sum b_n$ are conditionally convergent. It is valid if at least one of $\sum a_n$, $\sum b_n$ is absolutely convergent.

Theorem 4.5.2. ***(Mertens).*** *Assume that* $\sum_{n=0}^{\infty} a_n$ *converges absolutely and has sum* A, *and suppose* $\sum_{n=0}^{\infty} b_n$ *converges with sum* B. *Then the Cauchy product of these two series converges and has the sum* AB.

Proof. Define

$$A_n = \sum_{k=0}^{n} a_k, \quad B_n = \sum_{k=0}^{n} b_k, \quad \text{and} \quad C_n = \sum_{k=0}^{n} c_k,$$

where $c_k = a_k b_{n-k}$, if $n = 0, 1, 2, \cdots$.

Let

$$d_n = B - B_n \quad \text{and} \quad e_n = \sum_{k=0}^{n} a_k d_{n-k}.$$

Then

$$C_p = \sum_{k=0}^{p} c_k = \sum_{n=0}^{p}\sum_{k=0}^{n} a_k b_{n-k} = \sum_{n=0}^{p}\sum_{k=0}^{p} f_n(k), \tag{25}$$

where

$$f_n(k) = \begin{cases} a_k b_{n-k}, & n \geq k, \\ 0, & n < k. \end{cases}$$

Then (25) becomes

$$\begin{aligned} C_p &= \sum_{k=0}^{p}\sum_{n=0}^{p} f_n(k) \\ &= \sum_{k=0}^{p}\sum_{n=k}^{p} a_k b_{n-k} \end{aligned}$$

$$= \sum_{k=0}^{p} a_k \sum_{m=0}^{p-k} b_m$$

$$= \sum_{k=0}^{p} a_k B_{p-k} \quad (\because B_n = \sum_{k=0}^{n} b_k)$$

$$= \sum_{k=0}^{p} a_k (B - d_{p-k}) \quad (\because d_n = B - B_n)$$

$$= \sum_{k=0}^{p} a_k B - \sum_{k=0}^{p} a_k d_{p-k}$$

$$= B \sum_{k=0}^{p} a_k - \sum_{k=0}^{p} a_k d_{p-k}$$

$$= A_p B - e_p \quad (\because e_n = \sum_{k=0}^{n} a_k d_{n-k}).$$

It suffices to show that $e_p \to 0$ as $p \to \infty$.

We know that if $\sum a_n$ is convergent then $a_n \to 0$ as $n \to \infty$.

Hence, the sequence $\{d_n\}$ converges to 0, since $B = \sum b_n$.

Since every converges sequence is bounded, $\{d_n\}$ is bounded.

Hence, We can choose $M > 0$ such that $|d_n| \leq M$ for all n.

Let $K = \sum_{n=0}^{\infty} |a_n|$.

By our assumption, $\sum a_n$ converges absolutely.

Hence, given $\epsilon > 0$, choose N such that $n > N$ implies

$$|d_n| < \tfrac{\epsilon}{2K} \text{ } (\because \ \{d_n\} \text{ converges to } 0)$$

and also

$$\sum_{n=N+1}^{\infty} |a_n| < \tfrac{\epsilon}{2M}. \text{(by Cauchy criterion)}$$

Then, for $p > 2N$, we can write

$$|e_p| = \left| \sum_{k=0}^{p} a_k d_{p-k} \right|$$

$$= \left| \sum_{k=0}^{N} a_k d_{p-k} + \sum_{k=N+1}^{p} a_k d_{p-k} \right|$$

$$\leq \left| \sum_{k=0}^{N} a_k d_{p-k} \right| + \left| \sum_{k=N+1}^{p} a_k d_{p-k} \right|$$

$$\leq \sum_{k=0}^{N} |a_k||d_{p-k}| + \sum_{k=N+1}^{p} |a_k||d_{p-k}|$$

$$
\begin{aligned}
&\le \frac{\epsilon}{2K}\sum_{k=0}^{N}|a_k| + M\sum_{k=N+1}^{p}|a_k| \\
&\le \frac{\epsilon}{2K}\sum_{k=0}^{\infty}|a_k| + M\sum_{k=N+1}^{\infty}|a_k| \\
&\le \frac{\epsilon}{2K}K + M\frac{\epsilon}{2M} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}
$$

$$\implies \quad e_p \to 0 \quad as \quad p \to \infty.$$

Now $C_p = A_pB - e_p$

$$\implies \lim_{p\to\infty} C_p = \lim_{p\to\infty} A_pB - \lim_{p\to\infty} e_p.$$

$$\implies C = AB.$$

Hence the proof. □

Let us sum up

- In this section, we have defined the notion of Cauchy product of two series and discussed the possibility of finding the sum of such series.

4.6 Cesaro Summability

Definition 4.6.1. *Let s_n denote the n^{th} partial sum of the series $\sum a_n$, and let $\{\sigma_n\}$ be the sequence of arithmetic means defined by*

$$\sigma_n = \frac{s_1 + ... + s_n}{n}, \quad if\ n = 1, 2, ...$$

The series $\sum a_n$ is said to be Cesaro summable (or$(C,\ 1)$summable) if $\{\sigma_n\}$ converges. If $\lim_{n\to\infty} \sigma_n = S$, then S is called the Cesaro sum (or$(C,\ 1)$sum) of $\sum a_n$, and we write

$$\sum a_n = S \quad (C,\ 1).$$

Example 1. Let $a_n = z^n$, $|z| = 1$, $z \neq 1$.

$$\sigma_n = \frac{s_1 + ... + s_n}{n}.$$

Let s_n be the n^{th} partial sum of $\sum a_n$.

$$\begin{aligned} i.e., \quad s_n &= a_1 + a_2 + ... + a_n \\ &= z + z^2 + ... + z^n \\ &= z(1 + z + ... + z^{n-1}) \\ &= z\left(\frac{1-z^n}{1-z}\right) \\ &= z\left(\frac{1}{1-z} - \frac{z^n}{1-z}\right) \end{aligned}$$

$$\therefore \quad s_n = \frac{z}{1-z} - \frac{z^{n+1}}{1-z}$$

Put $n = 1, 2, ..., n$. We have

$$s_1 = \frac{z}{1-z} - \frac{z^2}{1-z}, \quad s_2 = \frac{z}{1-z} - \frac{z^3}{1-z}, \cdots, s_n = \frac{z}{1-z} - \frac{z^{n+1}}{1-z}$$

$$\begin{aligned} \sigma_n &= \frac{1}{n}\left[\left(\frac{z}{1-z} - \frac{z^2}{1-z}\right) + \left(\frac{z}{1-z} - \frac{z^3}{1-z}\right) + ... + \left(\frac{z}{1-z} - \frac{z^{n+1}}{1-z}\right)\right] \\ &= \frac{1}{n}\left[\frac{nz}{1-z} - \left(\frac{z^2}{1-z} + \frac{z^3}{1-z} + ... + \frac{z^{n+1}}{1-z}\right)\right] \\ &= \frac{1}{n}\left[\frac{nz}{1-z} - \frac{z^2}{1-z}(1 + z + ... + z^{n-1})\right] \\ &= \frac{1}{n}\left[\frac{nz}{1-z} - \frac{z^2}{1-z}\left(\frac{1-z^n}{1-z}\right)\right] \end{aligned}$$

$$\therefore \quad \sigma_n = \frac{z}{1-z} - \frac{z^2(1-z^n)}{n(1-z)^2}$$

$$\lim_{n\to\infty} \sigma_n = \frac{z}{1-z} - \lim_{n\to\infty} \frac{z^2(1-z^n)}{n(1-z)^2}$$

We know that $\lim\limits_{n\to\infty} \frac{1}{n} = 0$.

$$\therefore \lim_{n\to\infty} \sigma_n = \frac{z}{1-z}.$$

If $a_n = z^{n-1}$, then

$$\sum_{n=1}^{\infty} z^{n-1} = \frac{1}{1-z} \quad (C,\ 1).$$

Putting $z = -1$, we have

$$\sum_{n=1}^{\infty} (-1)^{n-1} = \frac{1}{2} \quad (C,\ 1).$$

Example 2. Let $a_n = (-1)^{n+1}n$.

$$\sigma_n = \frac{s_1 + \cdots + s_n}{n}.$$

Let $s_n = a_1 + a_2 + \cdots + a_n$ be the n^{th} partial sum of $\sum a_n$.

For $n = 1, 2, \cdots$, we have

$$a_1 = (-1)^{1+1}.1 = 1,\ a_2 = (-1)^{2+1}.2 = -2,\ a_3 = 3, \cdots$$

Then

$$s_1 = 1,\ \ s_2 = a_1 + a_2 = 1 - 2 = -1,\ \ s_3 = a_1 + a_2 + a_3 = 1 - 2 + 3 = 2, \cdots$$

$$s_n = \begin{cases} -\frac{n}{2} & if\ n\ is\ even \\ \frac{n+1}{2} & if\ n\ is\ odd \end{cases}$$

$$\therefore\ s_1 + s_2 + ... + s_n = \begin{cases} 0 & if\ n\ is\ even \\ \frac{n+1}{2} & if\ n\ is\ odd \end{cases}$$

$$\implies\ \ \sigma_n = \begin{cases} 0 & if\ n\ is\ even \\ \frac{n+1}{2.n} & if\ n\ is\ odd \end{cases}$$

$$\lim_{n\to\infty} \frac{n+1}{2n} = \lim_{n\to\infty} \frac{n\left(1 + \frac{1}{n}\right)}{2n} = \frac{1}{2}.$$

$$\therefore \lim_{n\to\infty} inf\ \sigma_n = 0\ \ \ and\ \ \ \lim_{n\to\infty} \sup\ \sigma_n = \frac{1}{2}.$$

Hence $\sum(-1)^{n+1}n$ is not $(C,\ 1)$ summable.

Theorem 4.6.2. *If a series is convergent with sum S, then it is also $(C,\ 1)$ summable with Cesaro sum S.*

Proof. Given $\sum a_n$ convergent with sum S.

Let s_n denote the nth partial sum of the series $\sum a_n$.

i.e., $\lim\limits_{n\to\infty} s_n = S$.

Define

$$\sigma_n = \frac{s_1 + s_2 + \cdots + s_n}{n},\ \ if\ \ n = 1, 2, \cdots$$

and introduce

$$t_n = s_n - S \text{ and } \tau_n = \sigma_n - S.$$

To prove : $\lim_{n\to\infty} \sigma_n = S$.

It is sufficient to prove that sequence $\tau_n \to 0$ as $n \to \infty$.

Now,

$$\begin{aligned}
\tau_n &= \sigma_n - S \\
&= \frac{s_1 + s_2 + \cdots + s_n}{n} - S \quad \left(\because \sigma_n = \frac{s_1 + s_2 + \cdots + s_n}{n}\right) \\
&= \frac{t_1 + S + t_2 + S + \cdots + t_n + S}{n} - S \quad (\because t_n = s_n - S) \\
&= \frac{nS + t_1 + t_2 + \cdots + t_n}{n} - S \\
&= \frac{nS}{n} + \frac{t_1 + t_2 + \cdots + t_n}{n} - S \\
&= S + \frac{t_1 + t_2 + \cdots + t_n}{n} - S \\
&= \frac{t_1 + t_2 + \cdots + t_n}{n}.
\end{aligned}$$

$$\therefore \tau_n = \frac{t_1 + t_2 + ... + t_n}{n}. \tag{26}$$

Since $\{s_n\}$ converges to S and $t_n = s_n - S$.

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} (s_n - S) = \lim_{n\to\infty} s_n - S = S - S = 0.$$

$$\therefore \lim_{n\to\infty} t_n = 0$$

i.e., $\{t_n\}$ is converges to zero.

Hence, t_n is bounded and there exist a constant $A > 0$ such that $|t_n| \leq A$.

Since $t_n \to 0$ as $n \to \infty$, given $\epsilon > 0$, choose N such that $n > N$ implies $|t_n| < \epsilon$.

Taking $n > N$ in (26), we obtain

$$\begin{aligned}
|\tau_n| &= \left|\frac{t_1 + t_2 + ... + t_n}{n}\right| \\
&= \left|\frac{t_1 + t_2 + ... + t_N + t_{N+1} + ... + t_n}{n}\right| \\
&\leq \left|\frac{t_1 + t_2 + ... + t_N}{n}\right| + \left|\frac{t_{N+1} + ... + t_n}{n}\right|
\end{aligned}$$

$$\leq \frac{|t_1| + |t_2| + ... + |t_N|}{n} + \frac{|t_{N+1}| + ... + |t_n|}{n}$$
$$< \frac{A + A + ... + A}{n} + \frac{\epsilon + \epsilon + ... + \epsilon}{n}$$
$$= \frac{N.A}{n} + \frac{n.\epsilon}{n} = \frac{N.A}{n} + \epsilon$$

Therefore,

$$|\tau_n| < \frac{N.A}{n} + \epsilon.$$

Since ϵ was arbitrary, we have

$$|\tau_n| \leq \frac{N.A}{n}.$$

We know that $\frac{1}{n} \to 0$ as $n \to \infty$.

$$\therefore \lim_{n\to\infty} \tau_n = 0.$$

$$\tau_n = \sigma_n - S \implies \lim_{n\to\infty} \tau_n = \lim_{n\to\infty} \sigma_n - S$$

$$\implies 0 = \lim_{n\to\infty} \sigma_n - S \implies \lim_{n\to\infty} \sigma_n = S.$$

Hence $\sum a_n = S \quad (C,\ 1)$. □

Note : We have really proved that if a sequence $\{s_n\}$ converges, then the sequence $\{\sigma_n\}$ of arithmetic means also converges and, in fact, to the same limit.

Let us sum up

- In this section, we have defined the notion of Cesaro summability of a given series and we have proved that every convergent series is Cesaro summabale.

Check your progress

1. Show the series $\cos x + \cos 3x + \cos 5x + \cdots \ (x\ real,\ x \neq m\pi)$ has $(c, 1)$ sum 0.

4.7 Infinite Products

This section gives a brief introduction to the theory of infinite products.

Definition 4.7.1. *Given a sequence $\{u_n\}$ of real or complex numbers, let*

$$P_1 = u_1, \quad P_2 = u_1 u_2, \; P_n = u_1 u_2 \cdots u_n = \prod_{k=1}^{n} u_k. \tag{27}$$

The ordered pair of sequences $(\{u_n\}, \{P_n\})$ is called an infinite product (or simply, a product). The P_n is called the nth partial product and u_n is called the nth factor of the product. The following symbols are used to denote the product defined by (27):

$$u_1 u_2 \cdots u_n \cdots, \qquad \prod_{n=1}^{\infty} u_n \tag{28}$$

Note : The symbol $\prod\limits_{n=N+1}^{\infty} u_n$ means $\prod\limits_{n=1}^{\infty} u_{N+n}$. We also write $\prod$ when there is no danger of misunderstanding.

By analogy with infinite series, it would seem natural to call the product (28) convergent if $\{P_n\}$ converges. However, this definition would be inconvenient since every product having one factor equal to zero would converge, regardless of the behavior of the remaining factors. The following definition turns out to be more useful:

Definition 4.7.2. *Given an infinite product $\prod\limits_{n=1}^{\infty} u_n$, let $P_n = \prod\limits_{k=1}^{n} u_k$.*

a) If infinitely many factors u_n are zero, we say the product diverges to zero.

b) If no factor u_n is zero, we say the product converges if there exists a number $P \neq 0$ such that $\{P_n\}$ converges to P. In this case, P is called the value of the product and we write $\prod\limits_{n=1}^{\infty} u_n$. If $\{P_n\}$ converges to zero, we say the product diverges to zero.

c) If there exists an N such that $n > N$ implies $u_n \neq 0$, we say $\prod\limits_{n=1}^{\infty} u_n$ converges, provided that $\prod\limits_{n=N+1}^{\infty} u_n$ converges as described in (b). In this case, the value of the product $\prod\limits_{n=1}^{\infty} u_n$ is

$$u_1 u_2 \cdots u_N \prod_{n=N+1}^{\infty} u_n.$$

d) $\prod\limits_{n=1}^{\infty} u_n$ is called divergent if it does not converge as described in (b) or (c).

Example : $\prod\limits_{n=1}^{\infty} \left(1 + \dfrac{1}{n}\right)$ and $\prod\limits_{n=2}^{\infty} \left(1 - \dfrac{1}{n}\right)$ are both divergent.
In the first case, $P_n = n + 1$, and in the second case, $P_n = \frac{1}{n}$.

Theorem 4.7.3. ***(Cauchy condition for product).*** *The infinite product $\prod u_n$ converges if and only if for every $\epsilon > 0$ there exists an N such that $n > N$ implies*

$$|u_{n+1}u_{n+2}.....u_{n+k} - 1| < \epsilon, \qquad for\ k = 1, 2, 3, \cdots \tag{29}$$

Proof. Assume that the product $\prod u_n$ converges.

To prove : For every $\epsilon > 0$ there exist an N such that $n > N$ implies

$|u_{n+1}u_{n+2}.....u_{n+k} - 1| < \epsilon$, for $k = 1, 2, 3, ..$

We can assume that $u_n \neq 0$ for all n.

Let $P_n = u_1u_2.....u_n$ and $\lim\limits_{n\to\infty} P_n = P$.

Since $u_n \neq 0$ for all n, $P_n \neq 0$.

Then $P \neq 0$ and there exists a $M > 0$ such that $|P_n| > M$.

Now $\{P_n\}$ satisfies the Cauchy condition for sequences.

Hence, given $\epsilon > 0$, there is an N such that $n > N$ implies

$$|P_{n+k} - P_n| < \epsilon M, \qquad for \quad k = 1, 2, \cdots$$

Now dividing by $|P_n|$ on both sides, we get

$$\begin{aligned} \frac{|P_{n+k} - P_n|}{|P_n|} &< \frac{M\epsilon}{|P_n|} \\ \implies \left|\frac{P_{n+k} - P_n}{P_n}\right| &< \frac{M\epsilon}{M} \\ \implies \left|\frac{P_{n+k}}{P_n} - 1\right| &< \epsilon \\ \implies \left|\frac{u_1u_2\cdots u_{n+k}}{u_1u_2\cdots u_n} - 1\right| &< \epsilon \end{aligned}$$

$$\therefore |u_{n+1}u_{n+2}\cdots u_{n+k} - 1| < \epsilon \ \text{ for } k = 1, 2, \cdots$$

Conversely, assume that for every $\epsilon > 0$ there exists an N such that $n > N$ implies

$$|u_{n+1}u_{n+2}\cdots u_{n+k} - 1| < \epsilon, \qquad for\ k = 1, 2, 3, \cdots$$

To prove : $\prod u_n$ converges.

Now $n > N$ implies $u_n \neq 0$.

Suppose $u_n = 0$ when $n > N$.

By our assumption

$$|u_{n+1}u_{n+2}\cdots u_{n+k} - 1| < \epsilon \implies |0 - 1| < \epsilon,$$

i.e., $\epsilon > 1$ which is impossible.

$$\therefore \quad u_n \neq 0.$$

For $\epsilon = \frac{1}{2}$, by (29), there exists N_0 such that

$$|u_{n+1}u_{n+2}\cdots u_{n+k} - 1| < \frac{1}{2} \;\; for \; k = 1, 2, 3.....$$

Let $q_n = u_{N_0+1}u_{N_0+2}\cdots u_n$ if $n > N_0$.

From (29), we have

$$\begin{aligned} |u_{N_0+1}u_{N_0+2}\cdots u_n - 1| &< \frac{1}{2} \\ \implies |q_n - 1| &< \frac{1}{2} \end{aligned}$$

$$\implies \frac{-1}{2} < q_n - 1 < \frac{1}{2}$$

$$\implies \frac{1}{2} < q_n < \frac{3}{2}$$

Hence, if $\{q_n\}$ converges it cannot converge to zero.

To show that $\{q_n\}$ converges.

Let $\epsilon > 0$ be arbitrary.

From (29), we have

$$\begin{aligned} \left| \frac{u_{N_0+1}, u_{N_0+2}, ...u_n, u_{n+1}, ..., u_{n+k}}{u_{N_0+1}, u_{N_0+2}, ...u_n} - 1 \right| &< \epsilon \\ \implies \left| \frac{q_{n+k}}{q_n} - 1 \right| &< \epsilon \\ \implies \left| \frac{q_{n+k} - q_n}{q_n} \right| &< \epsilon \\ \implies \frac{|q_{n+k} - q_n|}{|q_n|} &< \epsilon \\ \implies |q_{n+k} - q_n| < \epsilon|q_n| &< \frac{3\epsilon}{2}. \end{aligned}$$

Hence, $\{q_n\}$ satisfies the Cauchy condition for sequences.

Hence, $\{q_n\}$ is convergent.

Hence the product $\prod u_n$ converges. □

Note : Taking $k = 1$ in (29), we find that convergence of $\prod u_n$ implies $\lim_{n\to\infty} u_n = 1$. For this reason, the factors of a product are written as $u_n = 1 + a_n$. Thus convergence of $\prod(1 + a_n)$ implies $\lim_{n\to\infty} a_n = 0$.

Theorem 4.7.4. *Assume that each $a_n > 0$. Then the product $\prod(1 + a_n)$ converges if and only if the series $\sum a_n$ converges.*

Proof. Assume that each $a_n > 0$ and the series $\sum a_n$ converges.
Consider $e^x = 1 + x + \frac{x^2}{2!} + ...$

$\implies e^x \geq 1 + x.$ for all $x \in \mathbb{R}$

It is enough to consider

$$1 + x \leq e^x, \text{ for all } x \geq 0. \tag{30}$$

Putting $x = a_1$ in (30), we get

$$1 + a_1 \leq e^{a_1}$$

Putting $x = a_2$ in (30), we get

$$1 + a_2 \leq e^{a_2}$$

.

.

.

Putting $x = a_n$ in (30), we get

$$1 + a_n \leq e^{a_n}$$

Multiplying above inequalities, we get

$$(1 + a_1)(1 + a_2)......(1 + a_n) < e^{a_1}.e^{a_2}.......e^{a_n}$$

$$\implies (1 + a_1)(1 + a_2)......(1 + a_n) < e^{a_1+a_2+.....+a_n}$$

But

$$a_1 + a_2 + ... + a_n < (1 + a_1)(1 + a_2)......(1 + a_n)$$

$$\therefore a_1 + a_2 + ... + a_n < (1+a_1)(1+a_2)......(1+a_n) < e^{a_1+a_2+.....+a_n}$$

Putting $s_n = a_1 + a_2 + ... + a_n$ and $P_n = (1+a_1)(1+a_2)......(1+a_n)$, we get

$$\therefore s_n < P_n \leq e^{s_n} \quad \forall n \tag{31}$$

Since $a_n > 0$, the sequences $\{s_n\}$ and $\{P_n\}$ are monotonically increasing.

We need only show that $\{s_n\}$ is bounded if and only if $\{P_n\}$ is bounded.

If $\sum\limits_{n=1}^{\infty} a_n$ is convergent to S, then $\lim\limits_{n\to\infty} s_n = S$.

From (31), we have $P_n < e^S$ for all n.

Hence, $\{P_n\}$ is bounded above by e^S.

$\implies$ $\{P_n\}$ is convergent.

Hence the infinite product $\prod(1+a_n)$ converges.

Conversely, assume that $\prod(1+a_n)$ converges.

To prove : the series $\sum a_n$ converges.

$\prod(1+a_n)$ converges $\implies$ the sequence $\{P_n\}$ is convergent.

i.e., $\lim\limits_{n\to\infty} P_n = P$(say).

$\because s_n < P_n$, $s_n < P$ for all n.

Therefore, s_n is bounded above and hence convergent.

Hence the series $\sum a_n$ converges. □

Definition 4.7.5. *The product $\prod(1+a_n)$ is said to converge absolutely if $\prod(1+|a_n|)$ converges.*

Theorem 4.7.6. *Absolute convergence of $\prod(1+a_n)$ implies convergence.*

Proof. Assume that $\prod(1+a_n)$ is absolutely convergent.

i.e., $\prod(1+|a_n|)$ is convergent.

To prove : $\prod(1+a_n)$ is convergent.

Let $P_n = (1+a_1)(1+a_2)......(1+a_n)$.

Let $\epsilon > 0$ be given. Choose $n > N$ such that $\left|\frac{P_{n+k}}{P_n} - 1\right| < \epsilon \quad \forall \ k \geq 1$,

$$\implies \left|\frac{(1+a_1)(1+a_2).....(1+a_n)(1+a_{n+1})...(1+a_{n+k})}{(1+a_1).....(1+a_n)} - 1\right| < \epsilon$$

$$\implies |(1+a_{n+1})(1+a_{n+2})......(1+a_{n+k}) - 1| < \epsilon.$$

But

$$|(1+a_{n+1})(1+a_{n+2})......(1+a_{n+k})-1| \leq (1+|a_{n+1}|)(1+|a_{n+2}|)......(1+|a_{n+k}|)-1.$$

Hence, by Cauchy's general principle of convergence, the infinite product $\prod_{n=1}^{\infty}(1+a_n)$ is convergent.

□

Theorem 4.7.7. *Assume that each $a_n \geq 0$. Then the product $\prod(1-a_n)$ converges if and only if the series $\sum a_n$ converges.*

Proof. Assume $\sum a_n$ converges and $a_n \geq 0$.

Let $s_n = a_1 + a_2 + ... + a_n$ be the nth partial sum of the series $\sum a_n$.

Since $\sum a_n$ converges, $\lim_{n\to\infty} s_n = S$(say).

Let $P_n = (1-a_1).....(1-a_n)$.

To prove : P_n converges.

Now

$$(1-a_1)(1-a_2).....(1-a_n) \leq a_1 + a_2 + ... + a_n.$$

$$\implies P_n < S.$$

Hence, P_n is bounded by S, P_n converges.

Hence the product $\prod(1-a_n)$ converges.

Conversely, assume that the product $\prod(1-a_n)$ converges.

To prove : $\sum a_n$ converges.

Suppose $\sum a_n$ diverges.

If $\{a_n\}$ does not converge to zero, then $\prod(1-a_n)$ also diverges.

Therefore, we can assume that $a_n \to 0$ as $n \to \infty$.

Discarding a few terms if necessary, we can assume that each $a_n \leq \frac{1}{2}$.

Hence, $1-a_n \geq \frac{1}{2} \implies 1-a_n \neq 0$ for all n.

Let

$$P_n = (1-a_1)(1-a_2)......(1-a_n) \quad and \quad q_n = (1+a_1)(1+a_2)......(1+a_n).$$

Now

$$\begin{aligned}(1-a_k)(1+a_k) &= 1-a_k^2 \leq 1\\ \implies (1-a_k)(1+a_k) &\leq 1\\ \implies p_n q_n \leq 1 &\implies p_n \leq \frac{1}{q_n}\end{aligned}$$

If $q_n \to +\infty$ as $n \to \infty$, then $P_n \to 0$ as $n \to \infty$.

By the part (b) of Definition 4.7.2, it follows that $\prod(1-a_n)$ diverges to 0.

This is a contradiction to our assumption that $\prod(1-a_n)$ converges.

Hence $\sum a_n$ converges. □

4.8 Power Series

An infinite series of the form

$$a_0 + \sum_{n=1}^{\infty} a_n (z-z_0)^n$$

written more briefly as

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n \tag{32}$$

is called a power series in $z - z_0$. Here z, z_0, and $a_n (n = 0, 1, 2, \ldots)$ are complex numbers. With every power series (32) there is associated a disk, called the disk of convergence, such that the series converges absolutely for every z interior to this disk and diverges for every z outside this disk. The center of the disk is at z_0 and its radius is called the radius of convergence of the power series. (The radius may be 0 or $+\infty$ in extreme cases.) The next theorem establishes the existence of the disk of convergence and provides us with a way of calculating its radius.

Theorem 4.8.1. *Given a power series* $\sum_{n=0}^{\infty} a_n (z-z_0)^n$*, let*

$$\lambda = \limsup_{n\to\infty} \sqrt[n]{|a_n|}, \quad r = \frac{1}{\lambda},$$

(where $r = 0$ *if* $\lambda = +\infty$ *and* $r = +\infty$ *if* $\lambda = 0$*). Then the series converges absolutely if* $|z - z_0| < r$ *and diverges if* $|z - z_0| > r$*. Furthermore, the series converges uniformly on every compact subset interior to the disk of convergence.*

Proof. Now

$$\limsup_{n\to\infty} \sqrt[n]{\left|a_n\left(z-z_0\right)^n\right|} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}\sqrt[n]{|z-z_0|^n}$$

$$= \limsup_{n\to\infty} \sqrt[n]{|a_n|}|z-z_0|$$

$$= \frac{1}{r}|z-z_0|.$$

Hence, by root test the series $\sum a_n\left(z-z_0\right)^n$ converges absolutely if $|z-z_0|<r$ and diverges

if $|z-z_0|>r$.

Next we prove the series converges uniformly on every compact subset interior to the disk of convergence.

Let T be a compact subset of the disk of convergence. Then there is a point p in T such that $z\in T$ implies

$$|z-z_0|\leq|p-z_0|<r$$

$$\implies |z-z_0|^n\leq|p-z_0|^n\leq r^n$$

Hence, $\left|a_n\left(z-z_0\right)^n\right|\leq\left|a_n\left(p-z_0\right)^n\right|$ for each z in T.

By Weierstrass M-test, the series $\sum\limits_{n=0}^{\infty} a_n\left(z-z_0\right)^n$ converges uniformly. □

Note : If the limit $\lim\limits_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$ exists, its value is also equal to the radius of convergence of (32).

Example 1. Find the radius of convergence of $\sum\limits_{n=0}^{\infty} z^n$.

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n = \sum_{n=0}^{\infty} z^n$$

here $a_n=1$ and $z_0=0$. Now

$$r = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{1}{1}\right| r = 1$$

$$\therefore r=1.$$

If $|z| = 1$, the series diverges, since $\{z^n\}$ does not tend to 0 as $n \to \infty$.

Example 2. Find the radius of convergence of $\sum\limits_{n=0}^{\infty} \frac{z^n}{n^2}$.

here $a_n = \frac{1}{n^2}$.

$$\begin{aligned} r = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| &= \lim_{n\to\infty} \left| \frac{1/n^2}{1/(n+1)^2} \right| \\ &= \lim_{n\to\infty} \left| \frac{(n+1)^2}{n^2} \right| \\ &= \lim_{n\to\infty} \left(1 + \frac{1}{n^2} + \frac{2}{n} \right) = 1 \end{aligned}$$

$$\therefore r = 1.$$

It converges for all z with $|z| = 1$, by the comparison test, since $|z^n/n^2| = 1/n^2$.

Example 3. Find the radius of convergence of $\sum\limits_{n=0}^{\infty} \frac{z^n}{n}$.

here $a_n = \frac{1}{n}$.

$$\begin{aligned} r = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| &= \lim_{n\to\infty} \left| \frac{1/n}{1/(n+1)} \right| \\ &= \lim_{n\to\infty} \left| \frac{(n+1)}{n} \right| \\ &= \lim_{n\to\infty} \left(1 + \frac{1}{n} \right) = 1 \end{aligned}$$

$$\therefore r = 1.$$

The series diverges if $z = 1$. It converges for all other z with $|z| = 1$.

Theorem 4.8.2. *Assume that the power series* $\sum\limits_{n=0}^{\infty} a_n(z - z_0)^n$ *converges for each* z *in* $B(z_0; r)$. *Then the function* f *defined by the equation*

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n, \quad if\ z \in B(z_0; r), \tag{32}$$

is continuous on $B(z_0; r)$.

Proof. Assume that the power series $\sum\limits_{n=0}^{\infty} a_n(z - z_0)^n$ converges for each z in $B(z_0; r)$.
It is clear that each point in $B(z_0; r)$ belongs to some compact subset of $B(z_0; r)$.
Let $z_0 \in B(z_0; r)$.

$$\lim_{z\to z_0} f(z) - f(z_0) = \lim_{z\to z_0} \sum_{n=0}^{\infty} a_n(z - z_0)^n$$

$$\implies \lim_{z\to z_0} f(z) - f(z_0) = 0 \implies \lim_{z\to z_0} f(z) = f(z_0).$$

Hence, f is continuous at z_0.

Since z_0 is arbitrary, we have f is continuous on $B(z_0;r)$. □

Note : The series in (32) is said to represent f in $B(z_0;r)$. It is also called a power series expansion of f about z_0.

Theorem 4.8.3. *Assume that $\sum a_n(z-z_0)^n$ converges if $z \in B(z_0;r)$. Suppose that the equation*

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n,$$

is known to be valid for each z in some open subset S of $B(z_0;r)$. Then, for each point z_1 in S, there exists a neighborhood $B(z_1;R) \subseteq S$ in which f has a power series expansion of the form

$$f(z) = \sum_{k=0}^{\infty} b_k(z-z_1)^k, \tag{33}$$

where

$$b_k = \sum_{n=k}^{\infty} \binom{n}{k} a_n(z_1-z_0)^{n-k} \qquad (k = 0,1,2,....). \tag{34}$$

Proof. Let $z \in S$.

Now

$$\begin{aligned} f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n &= \sum_{n=0}^{\infty} a_n(z-z_1+z_1-z_0)^n \\ &= \sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} \binom{n}{k} (z-z_1)^k(z_1-z_0)^{n-k} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} c_n(k), \end{aligned}$$

where

$$c_n(k) = \begin{cases} \binom{n}{k} a_n(z-z_1)^k(z_1-z_0)^{n-k}, & if\ k \le n, \\ 0, & if\ k > n. \end{cases}$$

Now choose R such that $B(z_1;R) \subseteq S$ and assume that $z \in B(z_1;R)$.

$z \in B(z_1;R) \implies |z - z_1| < R$.

Then the iterated series $\sum\limits_{n=0}^{\infty}\sum\limits_{k=0}^{\infty} c_n(k)$ converges absolutely, since

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} |c_n(k)| = \sum_{n=0}^{\infty} |a_n|(|z - z_1| + |z_1 - z_0|)^n$$

$$= \sum_{n=0}^{\infty} |a_n|(z_0 + |z - z_1| + |z_1 - z_0| - z_0)^n$$

$$= \sum_{n=0}^{\infty} |a_n|(z_2 - z_0)^n \tag{35}$$

where

$$z_2 = z_0 + |z - z_1| + |z_1 - z_0| \implies z_2 - z_0 = |z - z_1| + |z_1 - z_0|$$

$$\implies |z_2 - z_0| < R + |z_1 - z_0| \leq r,$$

and hence the series in (35) converges.

Hence, by theorem 4.4.1, we can interchange the order of summation to obtain

$$f(z) = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} c_n(k) = \sum_{k=0}^{\infty}\sum_{n=k}^{\infty} \binom{n}{k} a_n (z - z_1)^k (z_1 - z_0)^{n-k}$$
$$= \sum_{k=0}^{\infty} b_k (z - z_1)^k,$$

where

$$b_k = \sum_{n=k}^{\infty} \binom{n}{k} a_n (z_1 - z_0)^{n-k}.$$

□

Note : For any $R > 0$ that satisfies the condition

$$B(z_1;R) \subseteq S.$$

Theorem 4.8.4. *Assume that $\sum a_n(z - z_0)^n$ converges for each z in $B(z_0;r)$. Then the function f defined by the equation*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad if \;\; z \in B(z_0;r), \tag{36}$$

has a derivative $f'(z)$ for each z in $B(z_0;r)$, given by

$$f'(z) = \sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1}. \tag{37}$$

Proof. Assume that $z_1 \in B(z_0;r)$.

Hence, by theorem 4.8.3, f has a power series expansion of the form

$$f(z) = \sum_{k=0}^{\infty} b_k (z-z_1)^k,$$

where

$$b_k = \sum_{n=k}^{\infty} \binom{n}{k} a_n (z_1-z_0)^{n-k}. \quad (k=0,1,2,...) \tag{38}$$

Let $z \in B(z_1;R)$ and $z \neq z_1$, we have

$$f(z) - f(z_1) = b_1(z-z_1) + \sum_{k=1}^{\infty} b_{k+1}(z-z_1)^{k+1}$$

$$\frac{f(z)-f(z_1)}{z-z_1} = b_1 + \sum_{k=1}^{\infty} b_{k+1}(z-z_1)^k.$$

By theorem 4.8.2, f is continuous on $B(z_0;r)$.

$$\therefore \lim_{z\to z_1} \frac{f(z)-f(z_1)}{z-z_1} = b_1 + \lim_{z\to z_1} \sum_{k=1}^{\infty} b_{k+1}(z-z_1)^k$$

$$\implies f'(z_1) = b_1$$

To find b_1:

Putting $k=1$ in (38), we get

$$b_1 = \sum_{n=1}^{\infty} \binom{n}{1} a_n (z_1-z_0)^{n-1}$$

$$= \sum_{n=1}^{\infty} n a_n (z_1-z_0)^{n-1}.$$

Since z_1 is an arbitrary point of $B(z_0;r)$,

$$\therefore \quad f'(z) = \sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1}.$$

□

Note : The series in (36) and (37) have the same radius of convergence.

By repeated application of (37), we find that for each $k = 1, 2, ...$, the derivative $f^{(k)}(z)$ exists in $B(z_0; r)$ and is given by the series

$$f^{(k)}(z) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n (z - z_0)^{n-k}. \tag{39}$$

If we put $z = z_0$ in (39), we obtain the important formula

$$f^{(k)}(z_0) = k! a_k \quad (k = 1, 2, ...).$$

This equation tells us that if two series $\sum a_n(z - z_0)^n$ and $\sum b_n(z - z_0)^n$ both represent the same function in a neighborhood $B(z_0; r)$, then $a_n = b_n$ for every n.

i.e., the power series expansion of a function f about a given point z_0 is uniquely determined, and it is given by the formula

$$f(z) = \sum_{n=0}^{\infty} \frac{f^n(z_0)}{n!} (z - z_0)^n,$$

valid for each z in the disk of convergence.

Let us sum up

- We have discussed the concept of power series.
- We have discussed radius of convergence.
- We have solved example of radius of convergence.

4.8.1 Multiplication of Power Series

Theorem 4.8.5. *Given two power series expansion about the origin, say*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad if \ z \in B(0; r),$$

and

$$g(z) = \sum_{n=0}^{\infty} b_n z^n, \quad if \ z \in B(0; R).$$

Then the product $f(z)g(z)$ *is given by the power series*

$$f(z)g(z) = \sum_{n=0}^{\infty} c_n z^n, \quad if \ z \in B(0; r) \cap B(0; R),$$

where

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} \qquad (n = 0, 1, 2, \dots).$$

Proof. The Cauchy product of the two given series is

$$\begin{aligned} f(z)g(z) &= \sum_{n=0}^{\infty} a_n z^n \sum_{n=0}^{\infty} b_n z^n \\ &= (a_0 + a_1 z + a_2 z^2 + \cdots)(b_0 + b_1 z + b_2 z^2 + \cdots) \\ &= a_0 b_0 + (a_0 b_1 + a_1 b_0) z + (a_0 b_2 + a_1 b_1 + a_2 b_0) z^2 + \dots\dots \\ &= c_0 + c_1 z + c_2 z^2 + \cdots \\ &= \sum_{n=0}^{\infty} c_n z^n, \text{ where } c_n = \sum_{k=0}^{n} a_k b_{n-k}. \end{aligned}$$

□

Note : If the two series are identical, we get

$$f(z)^2 = \sum_{n=0}^{\infty} c_n z^n,$$

where $c_n = \sum\limits_{k=0}^{n} a_k a_{n-k} = \sum\limits_{m_1+m_2=n} a_{m_1} a_{m_2}$.

The symbol $\sum\limits_{m_1+m_2=n}$ indicates that the summation is to be extended over all nonnegative integers m_1 and m_2 whose sum is n.

Similarly, for any integer $p > 0$, we have

$$f(z)^p = \sum_{n=0}^{\infty} c_n(p) z^n,$$

where

$$c_n(p) = \sum_{m_1+m_2+\dots\dots+m_p=n} a_{m_1} a_{m_2} \dots\dots a_{m_p}.$$

4.8.2 The Taylor's Series Generated By a Function

Definition 4.8.6. *Let f be a real-valued function defined on an interval I in R. If f has derivatives of every order at each point of I, we write $f \in \mathbb{C}^\infty$ on I.*

If $f \in \mathbb{C}^\infty$ on some neighborhood of a point c, the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x - c)^n,$$

is called the Taylor's series about c generated by f.

Note : Taylor's formula states that if $f \in \mathbb{C}^\infty$ on the closed interval $[a, b]$ and if $c \in [a, b]$, then for every x in $[a, b]$ and for every n, we have

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!}(x-c)^k + \frac{f^{(n)}(x_1)}{n!}(x-c)^n,$$

where x_1 is some point between x and c. The point x_1 depends on x, c and on n.
A necessary and sufficient condition for the Taylor's series to converge to $f(x)$ is that

$$\lim_{n\to\infty} \frac{f^{(n)}(x_1)}{n!}(x-c)^n = 0.$$

Theorem 4.8.7. *Assume that $f \in \mathbb{C}^\infty$ on $[a, b]$ and let $c \in [a, b]$. Assume that there is a neighborhood $B(c)$ and a constant M (which might depend on c) such that $|f^{(n)}(x)| \leq M^n$ for every x in $B(c) \cap [a, b]$ and every $n = 1, 2, ...$ Then, for each x in $B(c) \cap [a, b]$, we have*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n.$$

4.8.3 Bernstein's Theorem

Theorem 4.8.8. *Assume f has a continuous derivative of order $n+1$ in some open interval I containing c, and define $E_s(x)$ for x in I by the equation*

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k + E_n(x). \tag{40}$$

Then $E_n(x)$ is also given by the integral

$$E_n(x) = \frac{1}{n!}\int_c^x (x-t)^n f^{(n+1)}(t)dt. \tag{41}$$

Proof. The proof is by induction on n.
Putting $n = 1$ in (40), we have

$$\begin{aligned} f(x) &= \sum_{k=0}^{1} \frac{f^{(k)}(c)}{k!}(x-c)^k + E_1(x) \\ &= \frac{f^{(0)}(c)}{0!}(x-c)^0 + \frac{f^{(1)}(c)}{1!}(x-c)^1 + E_1(x) \\ &= f(c) + f^{(\prime)}(c)(x-c) + E_1(x) \\ \implies E_1(x) &= f(x) - f(c) - f'(c)(x-c) \end{aligned}$$

$$= \int_c^x [f'(t) - f'(c)]\, dt$$
$$= \int_c^x u(t)\; dv(t),$$

where $u(t) = f'(t) - f'(c)$ and $v(t) = t - x$.

Integration by parts gives

$$\int_c^x u(t)\; dv(t) = u(x)v(x) - u(c)v(c) - \int_c^x v(t)\; du(t)$$

$$= \int_c^x (x-t)f''(t)dt$$

$$\therefore \quad E_1(x) = \int_c^x (x-t)f''(t)dt.$$

The result is true for $n = 1$.

Now we assume that the result is true for n.

$$i.e., \quad E_n(x) = \frac{1}{n!}\int_c^x (x-t)^n f^{(n+1)}(t)dt.$$

To prove the result is true for $n+1$:

From (40) we have

$$E_n(x) = f(x) - \sum_{k=0}^{n} \frac{f^k(c)}{k!}(x-c)^k$$
$$E_{n+1}(x) = f(x) - \sum_{k=0}^{n+1} \frac{f^k(c)}{k!}(x-c)^k$$
$$= f(x) - \sum_{k=0}^{n} \frac{f^k(c)}{k!}(x-c)^k - \frac{f^{(n+1)}(c)}{(n+1)!}(x-c)^{n+1}$$
$$= E_n(x) - \frac{f^{(n+1)}(c)}{(n+1)!}(x-c)^{n+1}$$

$$\therefore \quad E_{n+1}(x) = E_n(x) - \frac{f^{(n+1)}(c)}{(n+1)!}(x-c)^{n+1}. \tag{42}$$

Now

$$(n+1)\int_c^x (x-t)^n dt = (n+1)\left[\frac{(x-t)^{n+1}}{n+1}(-1)\right]_c^x = \Big[-(x-t)^{n+1}\Big]_c^x$$
$$= -(x-x)^{n+1} + (x-c)^{n+1}$$

$$= (x-c)^{n+1}.$$

Hence, putting $E_n(x) = \frac{1}{n!}\int_c^x (x-t)^n f^{(n+1)}(t)dt$ and using $(x-c)^{n+1} = (n+1)\int_c^x (x-t)^n dt$ in (42), we get

$$\begin{aligned} E_{n+1}(x) &= \frac{1}{n!}\int_c^x (x-t)^n f^{(n+1)}(t)dt - \frac{f^{(n+1)}(c)}{n!}\int_c^x (x-t)^n dt \\ &= \frac{1}{n!}\int_c^x (x-t)^n \left[f^{(n+1)}(t) - f^{(n+1)}(c)\right] dt \\ &= \frac{1}{n!}\int_c^x u(t)dv(t), \end{aligned}$$

where $u(t) = f^{(n+1)}(t) - f^{(n+1)}(c)$ and $v(t) = -(x-t)^{n+1}/(n+1)$.

Integration by parts gives us

$$\begin{aligned} E_{n+1}(x) &= -\frac{1}{n!}\int_c^x v(t)du(t) \\ &= \frac{1}{n!}\int_c^x \frac{(x-t)^{n+1}}{(n+1)} f^{(n+2)}(t)\ dt \\ &= \frac{1}{(n+1)!}\int_c^x (x-t)^{n+1} f^{(n+2)}(t)\ dt. \end{aligned}$$

This proves (41). □

Note : The change of variable $t = x + (c-x)u$ transforms the integral in (41) to the form

$$E_n(x) = \frac{(x-c)^{n+1}}{n!}\int_0^1 u^n f^{(n+1)}[x + (c-x)u]du.$$

Theorem 4.8.9. ***(Bernstein).*** *Assume f and all its derivatives are nonnegative on a compact interval $[b, b+r]$. Then, if $b \le x < b+r$, the Taylor's series*

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(b)}{k!}(x-b)^k,$$

converges to $f(x)$.

Proof. By a translation we can assume $b = 0$.

The result is trivial if $x = 0$.

Hence, we assume $0 < x < r$.

We use Taylor's formula with remainder and write

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k + E_n(x). \tag{43}$$

We will prove that the error term satisfies the inequalities

$$0 \le E_n(x) \le \left(\frac{x}{r}\right)^{n+1} f(r) \tag{44}$$

This implies that $E_n(x) \to 0$ as $n \to \infty$, since $(x/r)^{n+1} \to 0$ if $0 < x < r$.

To prove (44):

Consider

$$E_n(x) = \frac{(x-c)^{n+1}}{n!} \int_0^1 u^n f^{(n+1)}[x + (c-x)u] du. \tag{45}$$

Putting $c = 0$ in (45), we get

$$E_n(x) = \frac{x^{n+1}}{n!} \int_0^1 u^n f^{(n+1)}(x - xu) du,$$

for each x in $[0, r]$.

If $x \neq 0$, let

$$F_n(x) = \frac{E_n(x)}{x^{n+1}} = \frac{1}{n!} \int_0^1 u^n f^{(n+1)}(x - xu) du.$$

The function $f^{(n+1)}$ is monotonic increasing on $[0, r]$ since its derivative is nonnegative. Therefore we have

$$f^{(n+1)}(x - xu) = f^{(n+1)}[x(1-u)] \le f^{(n+1)}[r(1-u)], \quad if \ 0 \le u \le 1.$$

$$\implies F_n(x) \le F_n(r) \qquad if \ \ 0 < x \le r$$

$$\implies \frac{E_n(x)}{x^{n+1}} \le \frac{E_n(r)}{r^{n+1}}$$

$$\implies E_n(x) \le \left(\frac{x}{r}\right)^{n+1} E_n(r).$$

Putting $x = r$ in (43), we get $E_n(r) \le f(r)$ since each term in the sum is nonnegative.

$$\therefore \ E_n(x) \le \left(\frac{x}{r}\right)^{n+1} f(r).$$

Hence the proof. □

4.8.4 Abel's Limit Theorem

If $-1 < x < 1$, integration of the geometric series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

gives us the series expansion

$$log(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n},$$

also valid for $-1 < x < 1$.

Theorem 4.8.10. ***(Abel's limit theorem).*** *Assume that we have*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \text{ if } -r < x < r. \tag{46}$$

If the series also converges at $x = r$, then the limit $\lim_{x \to r^-} f(x)$ exists and we have

$$\lim_{x \to r^-} f(x) = \sum_{n=0}^{\infty} a_n r^n.$$

Proof. Assume that $r = 1$ and $\sum a_n$ converges.
Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ for $-1 < x < 1$.
If $x = 1$, let $f(1) = \sum_{n=0}^{\infty} a_n$.
To prove : $\lim_{x \to r^-} f(x) = f(1)$.
i.e., we prove f is continuous from the left at $x = 1$.
Consider

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Multiply the series for $f(x)$ by the geometric series, and by theorem 4.8.5, we have

$$\frac{1}{1-x} f(x) = \sum_{n=0}^{\infty} x^n \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} c_n x^n,$$

where

$$c_n = \sum_{k=0}^{n} a_k.$$

Hence we have

$$f(x) - f(1) = (1-x)\sum_{n=0}^{\infty}[c_n - f(1)]x^n, \quad \text{if } -1 < x < 1. \tag{47}$$

By hypothesis, $\lim_{n\to\infty} c_n = f(1)$.

Hence, given $\epsilon > 0$, we can find that N such that $n \geq N$ implies $|c_n - f(1)| < \frac{\epsilon}{2}$.

If we split the sum (46) into two parts, we get

$$f(x) - f(1) = (1-x)\sum_{n=0}^{N-1}[c_n - f(1)]x^n + (1-x)\sum_{n=N}^{\infty}[c_n - f(1)]x^n. \tag{48}$$

Let M denote the largest of the N numbers $|c_n - f(1)|$, $n = 0, 1, 2, ..., N-1$.

If $0 < x < 1$, (41) gives

$$\begin{aligned}
|f(x) - f(1)| &\leq (1-x)NM + (1-x)\frac{\epsilon}{2}\sum_{n=N}^{\infty} x^n \\
&= (1-x)NM + (1-x)\frac{\epsilon}{2}(x^N + x^{N+1} + \cdots) \\
&= (1-x)NM + (1-x)\frac{\epsilon}{2}x^N(1 + x + x^2 + \cdots) \\
&= (1-x)NM + (1-x)\frac{\epsilon}{2}\frac{x^N}{1-x} \\
&< (1-x)NM + \frac{\epsilon}{2}.
\end{aligned}$$

Now let us choose $\delta = \frac{\epsilon}{2NM}$. Then $0 < 1 - x < \delta$ implies $|f(x) - f(1)| < \epsilon$. Therefore,

$$\lim_{x\to 1-} f(x) = f(1).$$

□

Theorem 4.8.11. *Let* $\sum\limits_{n=0}^{\infty} a_n$ *and* $\sum\limits_{n=0}^{\infty} b_n$ *be two convergent series and let* $\sum\limits_{n=0}^{\infty} c_n$ *denote their Cauchy product. If* $\sum\limits_{n=0}^{\infty} c_n$ *converges, we have*

$$\sum_{n=0}^{\infty} c_n = \left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right).$$

Proof. The two power series $\sum\limits_{n=0}^{\infty} a_n x^n$ and $\sum\limits_{n=0}^{\infty} b_n x^n$ both converge for $x = 1$.

Hence, they converge in the neighborhood $B(0; 1)$.

Assume $|x| < 1$, Then by using theorem 4.8.5 we have

$$\sum_{n=0}^{\infty} c_n x^n = \left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right).$$

Now let $x \to 1-$ and apply Abel's theorem, we get

$$\sum_{n=0}^{\infty} c_n = \left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right).$$

□

Let us sum up

- We have proved the Abel limit theorem.

4.8.5 Tauber's Theorem

The converse of Abel's limit theorem is false in general. That is, if $\int$ is given by (46), the limit $f(r-)$ may exist but yet the series $\sum a_n r^n$ may fail to converge.

Example. Let $a_n = (-1)^n$. Then

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} (-1)^n x^n \quad if \quad -1 < x < 1 \\ \Longrightarrow f(x) &= \frac{1}{1+x} \quad if \quad -1 < x < 1 \\ \Longrightarrow f(x) &\to \frac{1}{2} \quad as \quad x \to 1- . \end{aligned}$$

However, the series $\sum (-1)^n$ diverges.

Theorem 4.8.12. ***(Tauber).*** *Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ for $-1 < x < 1$, and assume that $\lim_{n\to\infty} n a_n = 0$. If $f(x) \to S$ as $x \to 1-$, then $\sum_{n=0}^{\infty} a_n$ converges and has sum S.*

Proof. Let

$$n\sigma_n = \sum_{k=0}^{n} k\,|a_k| \implies \sigma_n = \frac{1}{n}\sum_{k=0}^{n} k\,|a_k|\,.$$

Then

$$\sigma_n \to 0 \quad as \quad n \to \infty.$$

Also,

$$\lim_{n\to\infty} f(x_n) = S \quad if \quad x_n = 1 - \frac{1}{n}$$

Hence, given $\epsilon > 0$, we can choose N so that $n \geq N$ implies

$$|f(x_n) - S| < \frac{\varepsilon}{3}, \quad \sigma_n < \frac{\varepsilon}{3}, \quad n |a_n| < \frac{\varepsilon}{3}.$$

Now let $s_n = \sum_{k=0}^{n} a_k$.
Then, for $-1 < x < 1$, we can write

$$s_n - S = f(x) - S + \sum_{k=0}^{n} a_k \left(1 - x^k\right) - \sum_{k=n+1}^{\infty} a_k x^k$$

Now let $x \in (0,1)$. Then for each k,

$$\left(1 - x^k\right) = (1 - x)\left(1 + x + \cdots + x^{k-1}\right) \leq k(1 - x)$$

Therefore, if $n \geq N$ and $0 < x < 1$, we have

$$|s_n - S| \leq |f(x) - S| + (1 - x) \sum_{i=0}^{\pi} k |a_k| + \frac{\varepsilon}{3n(1 - x)}$$

Taking $x = x_n = 1 - 1/n$, we get

$$|s_n - S| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

Hence the proof. □

Let us sum up

- We have studied about multiplication of power series.
- We have discussed the Taylor's series generated by a function.
- We have derived Bernstein's Theorem.
- We have proved the Abel limit theorem and the Tauber theorem.
- We have discussed Cauchy product of the series.

Summary

- Introduced the concept of double sequence and series discussed their convergence with some examples.
- Discuss the rearrangement theorem for Double series.

- Discussed the Cesaro Summability and some examples
- Discussed about infinite products.
- Derived Cauchy condition for product.
- Discussed Absolute convergence of infinite product implies convergence.
- Introduced the concept of power series and radius of convergence with its properties
- Discussed the Taylor's series generated by a function.
- Derived Bernstein's Theorem.
- Derived Abel's limit theorem.
- Discussed Cauchy product of the series.

Exercises

1. Investigate the existence of the two iterated limits and the double lint of the double sequence f defined by
$a)\ f(p,q) = \dfrac{1}{p+q}$ $\qquad b)\ f(p,q) = \dfrac{\cos p}{q}$
2. Prove that a double series of positive terms converges if and only if the set of partial sums is bounded.
3. Show that a double series converges if it converges absolutely.
4. Show that the sum of the series $1-1-1+1+1-1-1+1+1\cdots$ has $(C,1)$ sum 0.
5. Prove the following statements:
    a) A double series of positive terms converges if and only if the set of partial sums is bounded.
    b) A double series converges if it converges absolutely.
    c) $\sum_{m,n} e^{-(m^2+n^2)}$ converges.

6. A series of the form $\sum_{n=1}^{\infty} a_n/n^s$ is called a Dirichlet series. Given two absolutely convergent Dirichlet series, say $\sum_{n=1}^{\infty} a_n/n^s$ and $\sum_{n=1}^{\infty} b_n/n^s$, having sums $A(s)$ and $B(s)$, respectively. Show that $\sum_{n=1}^{\infty} c_n/n^s = A(s)B(s)$ where $c_n = \sum_{d|n} a_d b_{n/d}$.

7. Given a series $\sum a_n$, let

$$s_n = \sum_{k=1}^{n} a_k \qquad t_n = \sum_{k=1}^{n} k a_k \qquad \sigma_n = \sum_{k=1}^{n} s_k.$$

Prove that

a) $t_n = (n+1)s_n - n\sigma_n$.

b) If $\sum a_n$ is $(C, 1)$ summable, then $\sum a_n$ converges if and only if $t_n = o(n)$ as $n \to \infty$.

c) $\sum a_n$ is $(C, 1)$ summable if and only if $\sum_{n=1}^{\infty} t_n/n(n+1)$ converges.

8. a) Let $a_n = (-1)^n/\sqrt{n}$ for $n = 1, 2, \cdots$. Show that $\prod(1 + a_n)$ diverges but that $\sum a_n$ converges.

b) Let $a_{2n-1} = -1/\sqrt{n}, a_{2n} = 1/\sqrt{n} + 1/n$ for $n = 1, 2, \cdots$. Sow that $\prod(1 + a_n)$ converges but that $\sum a_n$ diverges.

References

Tom M. Apostol, Mathematical Analysis, Second Edition, Addison-Wesley Publishing Company Inc., New York, 1974.

Suggested Readings

1. T.M. Apostol, Calculus Vol.2, Multi-Variable Calculus and Linear Algebra with Applications to Differential Equations and Probability, Second Edition - Reprint, John Wiley & Sons, 2016.

2. R.G. Bartle, Real Analysis, John Wiley and Sons Inc., 1976.

3. W. Rudin, Principles of Mathematical Analysis, Third Edition, Mc-Graw Hill Company, New York, 1976.

4. B.R. Gelbaum and J. Olmsted, Counter Examples in Analysis, Holden day, San Francisco, 1964.

5. A.L. Gupta and N.R. Gupta, Principles of Real Analysis, Pearson Education, (Indian print) 2003.

Unit 5

Sequences of Functions

Objectives

After the successful completion of this unit, the learners are expected to

- recall the basic concepts of sequence of functions and series of functions.
- analyse whether a given sequence or series of functions converge pointwise (or) uniformly.
- derive necessary conditions under which the continuity, differentiability or integrability may be transferred to the limit function.

5.1 Introduction

In mathematical analysis, sequences and series of functions are important concepts, particularly when exploring the behavior of functions as they approach certain limits. They finds applications in Fourier series and power series. Understanding sequences and series of functions is crucial in areas such as functional analysis, partial differential equations, and approximation theory.

In this chapter, we will introduce two different notion of convergence of sequence of functions namely, pointwise convergence and uniform convergence. We will see that uniform convergence is a stronger form of convergence for sequences of functions compared to pointwise convergence.

5.2 Pointwise Convergence of Sequence of Functions

Let $\{f_n\}$ be a sequence whose terms are real or complex-valued functions defined on the same domain on the real line $\mathbb{R}$ or in the complex plane $\mathbb{C}$.

For each x in the domain, we see that the sequence $\{f_n(x)\}$ is a sequence of real numbers whose terms are the corresponding function values. Let S denote the set of x for which this second sequence converges.

The function f defined by the equation

$$f(x) = \lim_{n\to\infty} f_n(x), \text{ if } x \in S,$$

is called the limit function of the sequence $\{f_n\}$, and we say that $\{f_n\}$ converges pointwise to f on the set S.

If $\{f_n\}$ converges pointwise on E, then there exists a function f such that, for every $\epsilon > 0$ and for every $x \in E$, there is an integer N (depending on ϵ and x) such that

$$n > N \implies |f_n(x) - f(x)| < \epsilon.$$

5.3 Examples of Sequences of Real-Valued Functions

We will provide an example of a sequence of continuous functions with a discontinuous limit function.

Example 1. Let

$$f_n(x) = \frac{x^{2n}}{(1 + x^{2n})}, \text{ if } x \in \mathbb{R}, \ n = 1, 2, \dots$$

In this case, $\lim\limits_{n\to\infty} f_n(x)$ exists for every real x, and the limit function f is given by

$$f(x) = \begin{cases} 0 & if\ |x| < 1, \\ \frac{1}{2} & if\ |x| = 1, \\ 1 & if\ |x| > 1. \end{cases}$$

We see that each f_n is continuous on $\mathbb{R}$, but the limit function f is discontinuous at $x = 1$ and $x = -1$.

Next, we provide an example of a sequence of functions for which

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx \neq \int_0^1 \lim_{n\to\infty} f_n(x)dx.$$

Example 2. Let

$$f_n(x) = n^2 x(1-x)^n, \text{ if } x \in \mathbb{R},\ n = 1, 2, \ldots$$

If $0 \le x \le 1$ the limit $f(x) = \lim\limits_{n\to\infty} f_n(x)$ exists and equals 0.

$$\int_0^1 f(x)\, dx = 0.$$

$$\int_0^1 f_n(x)\, dx = n^2 \int_0^1 x(1-x)^n\, dx = n^2 \int_0^1 (1-t)t^n\, dt$$

$$= \frac{n^2}{n+1} - \frac{n^2}{n+2} = \frac{n^2}{(n+1)(n+2)}$$

$$\therefore \lim_{n\to\infty} \int_0^1 f_n(x)\, dx = 1.$$

We will provide an example of a sequence of differentiable functions $\{f_n\}$ with limit as 0 for which $\{f_n'\}$ diverges.

Example 3. Let

$$f_n(x) = \frac{\sin nx}{\sqrt{n}}, \text{ if } x \in \mathbb{R},\ n = 1, 2, \ldots$$

Then

$$\lim_{n\to\infty} f_n(x) = 0 \quad \forall x.$$

But

$$f_n'(x) = \sqrt{n}\, cos\, nx,$$

so $\lim\limits_{n\to\infty} f_n'(x)$ does not exist for any x.

Example 4. Let $f_n(x) = x^n$ for $x \in (0,1)$ and $n \in \mathbb{N}$. Then we see that $f_n(x) \to 0$ for every $x \in (0,1)$. Thus, (f_n) converges pointwise to the zero function on $(0,1)$. Note that if there exists $N \in \mathbb{N}$ such that $|x^n| < \epsilon$ for all $n \ge N$ and for all $x \in (0,1)$, then, letting $x \to 1$, we would get $1 \le \epsilon$, which is not possible, had we chosen $\epsilon < 1$. Thus, the convergence is not uniform.

In fact, for a given $\epsilon > 0$,

$$|f_n(x) - f(x)| < \epsilon \Leftrightarrow x^n < \epsilon \Leftrightarrow \frac{1}{\epsilon} < \left(\frac{1}{x}\right)^n \Leftrightarrow n > \frac{\ln(1/\epsilon)}{\ln(1/x)}.$$

Hence, we are not in a position find an integer N independent of x such that $|f_n(x)| < \epsilon$ for all $x \in (0,1)$ and for all $n \ge N$.

Let us sum up

- We have defined the concept of pointwise convergence of a sequence of functions
- We have seen several examples of sequence of functions that converge pointwise.

5.4 Definition of Uniform Convergence

Definition 5.4.1. *A sequence of functions $\{f_n\}$ is said to converge uniformly to f on a set S if, for every $\epsilon > 0$, there exists an N(depending only on ϵ) such that $n > N$ implies*

$$|f_n(x) - f(x)| < \epsilon, \textit{for every } x \textit{ in } S.$$

We symbolically write this as

$$f_n \to f \textit{ uniformly on } S.$$

Definition 5.4.2. *A sequence $\{f_n\}$ is said to be uniformly bounded on S if there exists a constant $M > 0$ such that $|f_n(x)| \leq M$ for all x in S and for all n. The number M is called a uniform bound for $\{f_n\}$.*

Let us sum up

- We have defined the notion of uniform convergence of a sequence of functions.

Check your progress

1. If $\{f_n\}$ and $\{g_n\}$ converge uniformly on a set S, prove that $\{f_n + g_n\}$ converges uniformly on S. If, in addition, $\{f_n\}$, and $\{g_n\}$, are sequences of bounded functions, prove that $\{f_n g_n\}$ converges uniformly on S.

5.5 Uniform Convergence and Continuity

Theorem 5.5.1. *Assume that $\{f_n\} \to f$ uniformly on S. If each $\{f_n\}$ is continuous at a point c of S, then the limit function f is also continuous at c.*

Proof. Assume that $f_n \to f$ uniformly on S.
If c is an isolated point of S, then f is automatically continuous at c.

Assume that c is an accumulation point of S.

By our assumption, for every $\epsilon > 0$ there is an N such that $n \geq N$ implies

$$|f_n(x) - f(x)| < \frac{\epsilon}{3}, \text{ for every } x \text{ in } S.$$

Since f_N is continuous at c, there is a neighborhood $B(c)$ such that $x \in B(c) \cap S$

$$\implies |f_N(x) - f_N(c)| < \frac{\epsilon}{3}.$$

Consider

$$\begin{aligned} |f(x) - f(c)| &= |f(x) - f_N(x) + f_N(x) - f_N(c) + f_N(c)| \\ &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(c)| + |f_N(c) - f(c)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

$$\therefore |f(x) - f(c)| < \epsilon.$$

Hence f is continuous at c. □

Let us sum up

We have seen that the limit function of a uniformly convergent sequence of continuous functions is continuous.

Check your progress

1. Let $\{f_n\}$ be a sequence of continuous functions which converges uniformly to a function f on a set E. Prove that $\lim_{n\to\infty} f_n(x_n) = f(x)$ for every sequence of points $x_n \in E$ such that $x_n \to x$, and $x \in E$.

5.6 The Cauchy Condition for Uniform Convergence

Theorem 5.6.1. *Let $\{f_n\}$ be a sequence of functions defined on a set S. There exists a function f such that $\{f_n\} \to f$ uniformly on S if and only if the following condition (called the Cauchy condition) is satisfied: For every $\epsilon > 0$ there exists an N such that $m > N$ and $n > N$ implies*

$$|f_m(x) - f_n(x)| < \epsilon, \ \forall \ x \in S.$$

Proof. Assume that $f_n \to f$ uniformly on S.

Then, for given $\epsilon > 0$, there exists N such that $n > N$ implies

$$|f_n(x) - f(x)| < \frac{\epsilon}{2}, \quad for\ all\ x \in S.$$

Let $m > N$.

$$\therefore \quad |f_m(x) - f(x)| < \frac{\epsilon}{2}, \quad for\ all\ x \in S.$$

Consider

$$\begin{aligned} |f_n(x) - f_m(x)| &= |f_n(x) - f(x) + f(x) - f_m(x)| \\ &\leq |f_n(x) - f(x)| + |f(x) - f_m(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Conversely, suppose the Cauchy condition is satisfied.

Hence, for each x in S, the sequence $\{f_n(x)\}$ converges.

Let $f(x) = \lim_{m \to \infty} f_n(x)$ if $x \in S$.

To prove : $f_n \to f$ uniformly on S.

Let $\epsilon > 0$ be given. Then by Cauchy condition

$$|f_n(x) - f_m(x)| < \epsilon \quad if \ \ n,\ m > N.$$

Keeping n fixed and letting $m \to \infty$, we have

$$\lim_{m \to \infty} |f_n(x) - f_m(x)| = |f_n(x) - f(x)| < \epsilon$$

$$\therefore \quad |f_n(x) - f(x)| < \epsilon \quad if \ \ n > N.$$

This proves that $f_n \to f$ uniformly on S. □

Note : If $\{f_n\}$ and f are functions from a nonempty set S to a metric space (T, d_T), we say that $f_n \to f$ uniformly on S, if, for every $\epsilon > 0$, there is an N(depending only on ϵ) such that $n \geq N$ implies

$$d_T(f_n(x), f(x)) < \epsilon, \quad \forall \ \ x \in S.$$

Let us sum up

- We have proved the Cauchy criterion for uniform convergence of a sequence of functions. It is a test for uniform convergence of a given sequence of functions.

5.7 Uniform Convergence of Infinite Series of Functions

Definition 5.7.1. *Given a sequence $\{f_n\}$ of functions defined on a set S. For each x in S, let*

$$s_n(x) = \sum_{k=1}^{n} f_k(x) \ \ (n = 1, 2, \dots). \tag{49}$$

If there exists a function f such that $s_n \to f$ uniformly on S, we say the series $\sum f_n(x)$ converges uniformly on S and we write

$$\sum_{n=1}^{\infty} f_n(x) = f(x) \text{ (uniformly on S).}$$

Theorem 5.7.2. ***(Cauchy condition for uniform convergence of series).*** *The infinite series $\sum f_n(x)$ converges uniformly on S if and only if for every $\epsilon > 0$ there is an N such that $n > N$ implies*

$$\left| \sum_{k=n+1}^{n+p} f_k(x) \right| < \epsilon,$$

for each $p = 1, 2, \dots$ and every x in S.

Proof. Assume that $\sum f_n(x)$ converges uniformly on S.
Let $s_n(x) = \sum_{k=1}^{n} f_k(x) \ \ (n = 1, 2, \dots)$ be the partial sum of the series $\sum f_n(x)$.
Hence, $s_n(x)$ converges uniformly on S.
By theorem 5.5.1, for given $\epsilon > 0$, there is an N such that $n > N, \ m > N$ implies

$$|s_n(x) - s_m(x)| < \epsilon \quad \forall \, x \in S.$$

For $m > n$, let $m = n + p, \ p = 1, 2, \dots$

$$\therefore \quad \left| \sum_{k=1}^{n} f_k(x) - \sum_{k=1}^{n+p} f_k(x) \right| < \epsilon$$

$$\therefore \quad \left| \sum_{k=n+1}^{n+p} f_k(x) \right| < \epsilon,$$

for each $p = 1, 2, \cdots$ and every x in S.
Conversely, assume the Cauchy condition for uniform convergence of series is satisfied.
i.e., for every $\epsilon > 0$ there is an N such that $n > N$ implies

$$\left| \sum_{k=n+1}^{n+p} f_k(x) \right| < \epsilon,$$

for each $p = 1, 2, \ldots$ and every x in S.

$\implies |s_n(x) - s_{n+p}(x)| < \epsilon$ for all $p = 1, 2, \ldots$ and for all $x \in S$.

This implies the partial sum $s_n(x)$ of the series $\sum f_n(x)$ converges uniformly on S.

Hence, $\sum f_n(x)$ converges uniformly on S. □

Theorem 5.7.3. ***(Weierstrass M-test).*** *Let $\{M_n\}$ be a sequence of nonnegative numbers such that*

$$0 \leq |f_n(x)| \leq M_n, \quad for\ n = 1, 2, \cdots\ \textit{and}\ \ \forall\, x \in S.$$

Then $\sum f_n(x)$ converges uniformly on S if $\sum M_n$ converges.

Proof. Assume $\sum M_n$ converges.

To prove : $\sum f_n(x)$ converges uniformly on S.

Let $\epsilon > 0$ be given.

Let $s_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x)$ and $P_n = M_1 + M_2 + \cdots + M_n$.

Since $\{P_n\}$ converges in $\mathbb{R}$, we have $\{P_n\}$ is a Cauchy sequence in $\mathbb{R}$.

Hence, there exists $N \in \mathbb{N}$ such that

$$|P_n - P_m| < \epsilon \text{ for all } n, m \geq N.$$

$$\text{i.e., } |M_{m+1} + M_{m+2} + \cdots + M_n| < \epsilon \text{ for all } n, m \geq N \text{ with } n > m$$

$$\text{i.e., } M_{m+1} + M_{m+2} + \cdots + M_n < \epsilon \text{ for all } n, m \geq N \text{ with } n > m\ (M_n \geq 0\ \forall\, n).$$

Now for $n > m$ we have

$$\begin{aligned} |s_n(x) - s_m(x)| &= |f_{m+1}(x) + f_{m+2}(x) + \cdots + f_n(x)| \\ &\leq |f_{m+1}(x)| + |f_{m+2}(x)| + \cdots + |f_n(x)| \\ &\leq M_{m+1} + M_{m+2} + \cdots + M_n < \epsilon. \end{aligned}$$

$$\therefore\ \ |s_n(x) - s_m(x)| < \epsilon\ \ \forall\, n, m \geq N\ \ \textit{and}\ \ \forall\, x \in S.$$

By theorem 5.5.1, $\{s_n\}$ converges uniformly on S.

Hence, $\sum f_n(x)$ converges uniformly on S.

Hence the proof. □

Theorem 5.7.4. *Assume that $\sum f_n(x) = f(x)$ (uniformly on S). If each f_n is continuous at a point x_0 of S, then f is also continuous at x_0.*

Proof. Assume that $\sum f_n(x) = f(x)$ uniformly on S, and each f_n is continuous at a point x_0 of S.

Define

$$s_n(x) = \sum_{k=1}^{n} f_k(x) \quad (n = 1, 2, \cdots).$$

For each n, since each f_n is continuous at x_0, s_n is also continuous at x_0.

By our assumption $s_n \to f$ uniformly on S.

Hence, by using theorem 5.4.1, we have f is continuous at x_0. □

Let us sum up

- Introduced the notion of uniform convergence of series of functions
- Developed a test for uniform convergence, namely Weierstrass M-test.

5.8 Uniform Convergence and Riemann-Stieltjes Integration

Theorem 5.8.1. *Let α be of bounded variation on $[a, b]$. Assume that each term of the sequence $\{f_n\}$ is a real-valued function such that $f_n \in \mathcal{R}(\alpha)$ on $[a, b]$ for each $n = 1, 2, ...$ Assume that $f_n \to f$ uniformly on $[a, b]$ and define $g_n(x) = \int_a^x f_n(t)d\alpha(t)$ if $x \in [a, b]$, $n = 1, 2, ...$ Then we have*

a) $f \in \mathcal{R}(\alpha)$ on $[a, b]$.

b) $g_n \to g$ uniformly on $[a, b]$, where $g(x) = \int_a^x f(t)d\alpha(t)$.

Proof. Assume that α is increasing on $[a, b]$ and $\alpha(a) < \alpha(b)$.

Assume that each term of the sequence $\{f_n\}$ is a real-valued function such that $f_n \in \mathcal{R}(\alpha)$ on $[a, b]$ for all $n = 1, 2, \ldots$

Assume that $f_n \to f$ uniformly on $[a, b]$.

Define

$$g_n(x) = \int_a^x f_n(t)d\alpha(t) \;\; if \; x \in [a, b].$$

To prove : $f \in \mathcal{R}(\alpha)$ on $[a, b]$.

We show that f satisfies Riemann's condition with respect to α on $[a, b]$.

Since $f_n \to f$ uniformly on $[a, b]$, for given $\epsilon > 0$, we can choose N such that

$$|f(x) - f_N(x)| < \frac{\epsilon}{3[\alpha(b) - \alpha(a)]}. \quad \forall \ x \in [a, b].$$

Let P be a partition of $[a, b]$.

Now

$$\begin{aligned} |U(P, f - f_N, \alpha)| &= \sum_{k=1}^{N} |f(k) - f_N(k)| \Delta\alpha_k \\ &< \frac{\epsilon}{3[\alpha(b) - \alpha(a)]} \sum_{k=1}^{N} \Delta\alpha_k \\ &= \frac{\epsilon}{3[\alpha(b) - \alpha(a)]}.[\alpha(b) - \alpha(a)] < \frac{\epsilon}{3}. \end{aligned}$$

$$\therefore \ |U(P, f - f_N, \alpha)| < \frac{\epsilon}{3}.$$

Similarly, we have

$$L(P, f - f_N, \alpha)| < \frac{\epsilon}{3}.$$

For this N, choose P_ϵ be a partition of $[a, b]$ such that P finer than P_ϵ implies

$$U(P, f_N, \alpha) - L(P, f_N, \alpha) < \frac{\epsilon}{3}.$$

For such P, we have

$$\begin{aligned} U(P, f, \alpha) - L(P, f, \alpha) &\leq U(P, f - f_N, \alpha) - L(P, f - f_N, \alpha) + U(P, f_N, \alpha) - L(P, f_N, \alpha) \\ &< |U(P, f - f_N, \alpha)| + |L(P, f - f_N, \alpha)| + \frac{\epsilon}{3} \\ &= \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

$$\therefore \ U(P, f, \alpha) - L(P, f, \alpha) < \epsilon.$$

To prove : (b)

Let $\epsilon > 0$ be given.

Since $f_n \to f$ uniformly on $[a, b]$, for given $\epsilon > 0$ we can choose N such that $n \geq N$ implies

$$|f_n(t) - f(t)| < \frac{\epsilon}{2[\alpha(b) - \alpha(a)]},$$

for all $n > N$ and every t in $[a, b]$.

Let $x \in [a, b]$, we have

$$\begin{aligned}
|g_n(x) - g(x)| &= \left| \int_a^x f_n(t)d\alpha(t) - \int_a^x f(t)d\alpha(t) \right| \\
&= \left| \int_a^x [f_n(t) - f(t)]d\alpha(t) \right| \\
&\leq \int_a^x |f_n(t) - f(t)|d\alpha(t) \\
&< \frac{\epsilon}{2[\alpha(b) - \alpha(a)]} \int_a^x d\alpha(t) \\
&= \frac{\epsilon}{2[\alpha(b) - \alpha(a)]} \int_a^x \alpha'(t)dt \\
&= \frac{\epsilon}{2[\alpha(b) - \alpha(a)]}[\alpha(x) - \alpha(a)] \leq \frac{\epsilon}{2} < \epsilon. \quad (\because\ \alpha(x) < \alpha(b))
\end{aligned}$$

$$\therefore \quad |g_n(x) - g(x)| < \epsilon.$$

This proves that $g_n \to g$ uniformly on $[a, b]$. □

Note : The conclusion implies that, for each x in $[a, b]$, we can write

$$\lim_{n\to\infty} \int_a^x f_n(t)d\alpha(t) = \int_a^x \lim_{n\to\infty} f_n(t)d\alpha(t).$$

This property is often described by saying that a uniformly convergent sequence can be integrated term by term.

Theorem 5.8.2. *Let α be of bounded variation on $[a, b]$ and assume that $\sum f_n(x) = f(x)$ (uniformly on $[a, b]$), where each f_n is a real-valued function such that $f_n \in \mathcal{R}(\alpha)$ on $[a, b]$. Then we have*

a) $f \in \mathcal{R}(\alpha)$ on $[a, b]$.

b) $\int_a^x \sum\limits_{n=1}^{\infty} f_n(t)d\alpha(t) = \sum\limits_{n=1}^{\infty} \int_a^x f_n(t)d\alpha(t)$ (uniformly on $[a, b]$).

Proof. Assume that α is increasing on $[a, b]$ and $\alpha(a) < \alpha(b)$.

Assume that $\sum f_n(x) = f(x)$ uniformly on $[a, b]$, where each f_n is a real-valued function such that $f_n \in \mathcal{R}(\alpha)$ on $[a, b]$.

To prove : $f \in \mathbb{R}(\alpha)$.

Define

$$s_n(x) = \sum_{k=1}^{n} f_k(x)$$

Since $\sum f_n(x) = f(x)$ uniformly on $[a, b]$, $s_n \to f$ uniformly on $[a, b]$.

By theorem 5.7.1. part (a), we have $f \in \mathcal{R}(\alpha)$.

To prove : (b)

By theorem 5.7.1. part (b), we have

$$\begin{aligned}
\lim_{n\to\infty} \int_a^x s_n(x)\, d\alpha(x) &= \int_a^x f(x)\, d\alpha(x) \\
\implies \lim_{n\to\infty} \int_a^x \sum_{k=1}^{n} f_k(x)\, d\alpha(x) &= \int_a^x f(x)\, d\alpha(x) \\
\implies \lim_{n\to\infty} \sum_{k=1}^{n} \int_a^x f_k(x)\, d\alpha(x) &= \int_a^x f(x)\, d\alpha(x) \\
\implies \sum_{n=1}^{\infty} \int_a^x f_k(x)\, d\alpha(x) &= \int_a^x \sum f_n(x)\, d\alpha(x) \quad (\because \sum f_n(x) = f(x)).
\end{aligned}$$

□

5.9 Nonuniformly Convergent Sequences that can be Integrated Term by Term

Uniform convergence is a sufficient but not a necessary condition for term by term integration.

Example. Let

$$f_n(x) = x^n, \quad if \quad 0 \le x \le 1.$$

The limit function of the sequence $\{f_n\}$ is given by

$$f(x) = \begin{cases} 0 & 0 \le x < 1, \\ 1 & x = 1. \end{cases}$$

$$\int_0^1 f_n(x)dx = \int_0^1 x^n dx = \left[\frac{x^{n+1}}{n+1}\right]_0^1 = \frac{1}{n+1}.$$

$$\therefore \lim_{n\to\infty} \int_0^1 f_n(x)dx = \lim_{n\to\infty} \frac{1}{n+1} = 0.$$

and

$$\int_0^1 f(x)dx = 0.$$

Definition 5.9.1. *A sequence of functions $\{f_n\}$ is said to be boundedly convergent on T if $\{f_n\}$ is pointwise convergent and uniformly bounded on T.*

Theorem 5.9.2. *Let $\{f_n\}$ be a boundedly convergent sequence on $[a, b]$. Assume that each $f_n \in \mathcal{R}$ on $[a, b]$, and that the limit function $f \in \mathcal{R}$ on $[a, b]$. Assume also that there is a partition P of $[a, b]$, say*

$$P = \{x_0, x_1, \dots, x_n\},$$

such that, on every subinterval $[c, d]$ not containing any of the points x_k, the sequence $\{f_n\}$ converges uniformly to f. Then we have

$$\lim_{n\to\infty} \int_a^b f_n(t)dt = \int_a^b \lim_{n\to\infty} f_n(t)dt = \int_a^b f(t)dt.$$

Proof. Assume that $\{f_n\}$ be a boundedly convergent sequence on $[a, b]$.
Assume that each $f_n \in \mathcal{R}$ on $[a, b]$, and that the limit function $f \in \mathcal{R}$ on $[a, b]$.
Assume also that there is a partition P of $[a, b]$, say

$$P = \{x_0, x_1, \dots, x_n\},$$

such that, on every subinterval $[c, d]$ not containing any of the points x_k, the sequence $\{f_n\}$ converges uniformly to f.
By our assumption f is bounded and $\{f_n\}$ is uniformly bounded, there is a positive number M such that

$$|f(x)| \le M \quad and \quad |f_n(x)| \le M \;\; \forall \;\, x \in [a, b], \quad \forall \;\, n \ge 1.$$

Let $\epsilon > 0$ be given such that $2\epsilon < \|P\|$, let $h = \frac{\epsilon}{2m}$, where m is the number of subintervals of P.
Consider a new partition P' of $[a, b]$ given by

$$P' = \{x_0, x_0 + h, x_1 - h, x_1 + h, ..., x_{m-1} - h, x_{m-1} + h, x_m - h, x_m\}.$$

Now $\forall\; x \in [a, b]$,

$$|f(x) - f_n(x)| \le |f(x)| + |f_n(x)| \le M + M = 2M.$$

$$\therefore |f(x) - f_n(x)| \le 2M.$$

Since $f_n \in \mathcal{R}$ on $[a,b]$ and $f \in \mathcal{R}$ on $[a,b]$, $f - f_n \in \mathcal{R}$ on $[a,b]$.

Hence, $|f - f_n|$ is integrable on $[a,b]$.

$$\begin{aligned}\int_a^b |f-f_n|dx &= \int_{x_0}^{x_0+h} |f-f_n|dx + \int_{x_1-h}^{x_1+h} |f-f_n|dx + \cdots + \int_{x_m-h}^{x_m} |f-f_n|dx \\ &\leq 2M\int_{x_0}^{x_0+h} dx + 2M\int_{x_1-h}^{x_1+h} dx + \cdots + 2M\int_{x_m-h}^{x_m} dx \\ &= 2M[2h + 2h + \cdots + 2h(m \text{ times})] \\ &= 2M(2mh).\end{aligned}$$

Hence, the sum of the integrals of $|f - f_n|$ taken over the intervals

$$[x_0, x_0+h], \quad [x_1-h, x_1+h], \ldots, [x_{m-1}-h, x_{m-1}+h], \quad [x_m-h, x_m,]$$

is at most $2M(2mh) = 2M\epsilon$.

The remaining portion of $[a,b]$(say S) is the union of a finite number of closed intervals, in each of which $\{f_n\}$ is uniformly convergent to f.

Hence, there is an integer N (depending only on ϵ) such that for all x in S we have

$$|f(x) - f_N(x)| < \epsilon \quad if\ n \geq N.$$

Hence the sum of the integrals of $|f - f_n|$ over the intervals of S is at most $\epsilon(b-a)$.

$$\therefore \quad \int_a^b |f(x) - f_n(x)|\ dx \leq (2M + b - a)\epsilon \quad if\ n \geq N.$$

This proves that

$$\int_a^b f_n(x)\ dx \to \int_a^b f(x)\ dx \quad as\ n \to \infty.$$

□

Theorem 5.9.3. ***(Arzela).*** *Assume that $\{f_n\}$ is boundedly convergent on $[a,b]$ and suppose each $\{f_n\}$ is Riemann-integrable on $[a,b]$. Assume also that the limit function f is Riemann-integrable on $[a,b]$. Then*

$$\lim_{n\to\infty} \int_a^b f_n(x)\ dx = \int_a^b \lim_{n\to\infty} f_n(x)\ dx = \int_a^b f(x)\ dx.$$

Proof. Assume that $\{f_n\}$ is boundedly convergent on $[a,b]$ and $\{f_n\}$ is Riemann-integrable on $[a,b]$.

i.e., $\{f_n\}$ is uniformly convergent to f, where f is is Riemann-integrable on $[a, b]$.
Hence, $\int_a^b f_n \, dx$ and $\int_a^b f \, dx$ exist.
Since $f_n \to f$ uniformly, there exists N such that

$$|f_n - f| < \epsilon \ \ \forall \, n \geq N.$$

$$f_n - \epsilon < f < f_n + \epsilon \quad for \ n \geq N.$$

$$\implies \int_a^b f_n \, dx - \epsilon(b-a) \leq \int_a^b f \, dx \leq \int_a^b f_n \, dx + \epsilon(b-a) \quad for \ n \geq N.$$

$$\implies -\epsilon(b-a) \leq \int_a^b f_n \, dx - \int_a^b f \, dx \leq \epsilon(b-a) \quad for \ n \geq N.$$

$$\implies \left| \int_a^b f_n \, dx - \int_a^b f \, dx \right| < \epsilon(b-a)$$

$$\therefore \int_a^b f_n \, dx \to \int_a^b f \, dx.$$

□

5.10 Uniform Convergence and Differentiation

Let us recall the following example from the previous section.
Example. If $f_n(x) = \dfrac{\sin nx}{\sqrt{n}}$, for $x \in \mathbb{R}, \ n = 1, 2, \ldots,$ then

$$\lim_{n\to\infty} f_n(x) = 0 \quad \forall x.$$

But

$$f_n'(x) = \sqrt{n} \cos nx.$$

So $\lim\limits_{n\to\infty} f_n'(x)$ does not exist for any x.

From the above example, we see that the uniform convergence of a sequence f_n defined on $\mathbb{R}$ does not even imply guarantee the pointwise convergence of the sequence $\{f_n'\}$. Thus we state the result in the following way:

Theorem 5.10.1. *Assume that each term of $\{f_n\}$ is a real-valued function having a finite derivative at each point of an open interval (a,b). Assume that for at least one point x_0 in (a,b) the sequence $\{f_n(x_0)\}$ converges. Assume further that there exists a function g such that $f_n' \to g$ uniformly on (a,b). Then :*

a) There exists a function f such that $f_n \to f$ uniformly on (a,b).

b) For each x in (a,b) the derivative $f'(x)$ exists and equals $g(x)$.

Proof. To prove :(a)

Assume that $c \in (a,b)$ and define a new sequence $\{g_n\}$ as follows:

$$g_n(x) = \begin{cases} \frac{f_n(x)-f_n(c)}{x-c} & if\ x \neq c, \\ f_n'(c) & if\ x = c. \end{cases}$$

Note that the sequence $\{g_n(c)\}$ converges, since $g_n(c) = f_n'(c)$ and the sequence $f_n'(c)$ converges.

Let $x \neq c$. Then,

$$\begin{aligned} g_n(x) - g_m(x) &= \frac{f_n(x)-f_n(c)}{x-c} - \frac{f_m(x)-f_m(c)}{x-c} \\ &= \frac{[f_n(x)-f_m(x)] - [f_n(c)-f_m(c)]}{x-c} \end{aligned}$$

Putting $h(x) = f_n(x) - f_m(x)$, we get

$$g_n(x) - g_m(x) = \frac{h(x)-h(c)}{x-c}.$$

Since each f_n is differentiable in (a,b), $h'(x)$ exists for each x in (a,b) and

$$h'(x) = f_n'(x) - f_m'(x).$$

By Mean-Value Theorem, there is a point $x_1 \in (x,c)$ such that

$$h(x) - h(c) = h'(x_1)(x-c)$$

$$\implies g_n(x) - g_m(x) = f_n'(x_1) - f_m'(x_1),$$

since the sequence $\{f_n'\}$ converges uniformly on (a,b) and by using Cauchy criterion, for given $\epsilon > 0$, there is an N such that $n,\ m \geq N$ implies

$$|f_n'(x_1) - f_m'(x_1)| < \epsilon \quad if\ \ n \geq N,\ m \geq N,\ \ x_1 \in (x,c)$$

$$\implies |g_n(x) - g_m(x)| < \epsilon \quad \forall\, x \in (a, b).$$

Hence, the sequence $\{g_n\}$ converges uniformly on (a, b).

Now, we show that $\{f_n\}$ converges uniformly on (a, b).

Let $c = x_0$. Then

$$g_n(x) = \frac{f_n(x) - f_n(x_0)}{x - x_0}$$

$$\implies f_n(x) = f_n(x_0) + (x - x_0)g_n(x)$$

For $n > N$, $m > N$, we have

$$f_n(x) - f_m(x) = f_n(x_0) - f_m(x_0) + (x - x_0)[g_n(x) - g_m(x_0)].$$

By Cauchy criterion, we have

$$|f_n(x) - f_m(x)| \leq |f_n(x_0) - f_m(x_0)| + |x - x_0||g_n(x) - g_m(x_0)|$$

$$< \epsilon + (b - a)\epsilon.$$

which implies that $\{f_n\}$ satisfies the Cauchy criterion.

Hence, $\{f_n\}$ converges uniformly on (a, b).

To prove (b): Let $c \in (a, b)$ be arbitrary.

Claim:$f'(c) = g(c)$.

Since $\{g_n\}$ converges uniformly on (a, b), let $\lim_{n\to\infty} g_n(x) = G(x)$.

By hypothesis, f'_n exists. Therefore,

$$\lim_{x\to c} g_n(x) = g_n(c)$$

i.e.,g_n is continuous at c.

Since $g_n \to G$ uniformly on (a, b), the limit function G is also continuous at c. Therefore,

$$G(c) = \lim_{x\to c} G(x).$$

For $x \neq c$,

$$G(x) = \lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} \frac{f_n(x) - f_n(c)}{x - c} = \frac{f(x) - f(c)}{x - c}$$

and

$$G(c) = \lim_{x \to c} G(x) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c).$$

But

$$G(c) = \lim_{n \to \infty} g_n(c) = \lim_{n \to \infty} f_n'(c) = g(c).$$

Therefore,

$$f'(c) = g(c).$$

□

Theorem 5.10.2. *Assume that each f_n is a real-valued function defined on (a,b) such that the derivative $f_n'(x)$ exists for each x in (a,b). Assume that, for at least one point x_0 in (a,b), the series $\sum f_n(x_0)$ converges. Assume further that there exists a function g such that $\sum f_n'(x) = g(x)$ (uniformly on (a,b)). Then*

(a) There exists a function f such that $\sum f_n(x) = f(x)$ (uniformly on (a,b)).

(b) If $x \in (a,b)$, the derivative $f'(x)$ exists and equals $\sum f_n'(x)$.

Let us sum up

- We have derived sufficient condition for the uniform convergence of the differentiable functions.

Check your progress

1. The sequence $f_n(x) = \dfrac{\cos nx}{\sqrt{n}}$ (x real, $n = 1, 2, 3, \ldots$)
   (A) converges uniformly on $\mathbb{R}$
   (B) $f_n' \to f'$ on $\mathbb{R}$
   (C) $\{f_n'\}$ does not converge on $\mathbb{R}$
   (D) converges uniformly on $[0,1]$ only

5.11 Sufficient Conditions for Uniform Convergence of a Series

Theorem 5.11.1. ***(Dirichlet's test for uniform convergence).*** *Let $F_n(x)$ denote the nth partial sum of the series $\sum f_n(x)$, where each f_n is a complex-valued function defined on* *a set S. Assume that $\{F_n\}$ is uniformly bounded on S. Let $\{g_n\}$ be a sequence of real-valued functions such that $g_{n+1}(x) \leq g_n(x)$ for each x in S and for every $n = 1, 2, \ldots$, and assume that $g_n \to 0$ uniformly on S. Then the series $\sum f_n(x)g_n(x)$ converges uniformly on S.*

Proof. Let

$$s_n(x) = \sum_{k=1}^{n} f_k(x)g_k(x)$$

denote the nth partial sum of the series $\sum f_n g_n$.

By partial summation formula, we have

$$s_n(x) = \sum_{k=1}^{n} F_k(x)\left[g_k(x) - g_{k+1}(x)\right] + g_{n+1}(x)F_n(x).$$

Hence, if $n > m$, we have

$$\begin{aligned} s_n(x) - s_m(x) &= \sum_{k=1}^{n} F_k(x)\left[g_k(x) - g_{k+1}(x)\right] + g_{n+1}(x)F_n(x) \\ &\quad - \sum_{k=1}^{m} F_k(x)\left[g_k(x) - g_{k+1}(x)\right] + g_{n+1}(x)F_n(x) \end{aligned}$$

For $n > m$, we have

$$s_n(x) - s_m(x) = \sum_{k=m+1}^{n} F_k(x)\left(g_k(x) - g_{k+1}(x)\right) + g_{n+1}(x)F_n(x) - g_{m+1}(x)F_m(x).$$

Since the sequence $\{F_n\}$ is uniformly bounded, there is $M > 0$ such that

$$|F_n(x)| \leq M.$$

Therefore,

$$\begin{aligned} |s_n(x) - s_m(x)| &= |\sum_{k=m+1}^{n} F_k(x)\left(g_k(x) - g_{k+1}(x)\right) + g_{n+1}(x)F_n(x) - g_{m+1}(x)F_m(x)| \\ &\leq \sum_{k=m+1}^{n} |F_k(x)|\left(g_k(x) - g_{k+1}(x)\right) + g_{n+1}(x)|F_n(x)| + g_{m+1}(x)|F_m(x)| \\ &\leq M(g_{m+1}(x) - g_{n+1}(x)) + Mg_{n+1}(x) + Mg_{m+1}(x) \\ &= Mg_{m+1}(x) + Mg_{m+1}(x) \\ &= 2Mg_{m+1}(x). \end{aligned}$$

Since $g_n \to 0$ uniformly on S, for given $\epsilon > 0$, there is $N > 0$ such that

$$|g_n(x) - 0| < \frac{\epsilon}{2M}, \ \forall \ n \geq N.$$

Therefore,

$$|s_n(x) - s_m(x)| < 2M. \frac{\epsilon}{2M} = \epsilon. \tag{5.1}$$

Thus, $\{s_n(x)\}$ satisfies the Cauchy criterion and so $\{s_n\}$ converges uniformly.

This implies that $\sum f_n g_n$ converges uniformly on S.

Hence the proof. □

Let us sum up

- Derived sufficient conditions for the uniform convergence of a given series.

5.12 Mean Convergence

Definition 5.12.1. *Let* $(f_n\}$ *be a sequence of Riemann-integrable functions defined on* $[a, b]$. *Assume that* $f \in \mathcal{R}$ *on* $[a, b]$. *The sequence* $\{f_n\}$ *is said to converge in the mean to* f *on* $[a, b]$, *and we write*

$$l.i.m_{n \to \infty} f_n = f \qquad \text{on} \qquad [a, b],$$

if

$$\lim_{n \to \infty} \int_a^b |f_n(x) - f(x)|^2 \, dx = 0.$$

If the inequality $|f(x) - f_0(x)| < \epsilon$ holds for every x in $[a, b]$, then we have

$$\int_a^b |f(x) - f_n(x)|^2 \, dx \leq \epsilon^2 (b - a).$$

Therefore, uniform convergence of $\{f_n\}$ to f on $[a, b]$ implies mean convergence, provided that each f_n is Riemann-integrable on $[a, b]$.

Remark: Mean convergence need not imply pointwise convergence at any point of the interval.

For example, for each integer $n \geq 0$, subdivide the interval $[0, 1]$ into 2^n equal subintervals and let I_{2^n+k} denote that subinterval whose right endpoint is $\frac{(k+1)}{2^n}$, where $k = 0, 1, 2, \ldots, 2^n - 1$.

This yields a collection $\{I_1, I_2, \ldots\}$ of subintervals of [0. 1], of which the first few are:

$$I_1 = [0,1],\ I_2 = \left[0, \frac{1}{2}\right],\ I_3 = \left[\frac{1}{2}, 1\right],\ I_4 = \left[0, \frac{1}{4}\right],\ I_5 = \left[\frac{1}{4}, \frac{1}{2}\right],\ I_6 = \left[\frac{1}{2}, \frac{3}{4}\right],$$

and so on.

Define f_n on $[0,1]$ as follows :

$$f_n(x) = \begin{cases} 1 & if\ x \in I_n, \\ 0 & if\ x \in [0,1] - I_n. \end{cases}$$

Then $\{f_n\}$ converges in the mean to 0, since $\int_0^1 |f_n(x)|^2\, dx$ is the length of I_n, and

$$\int_0^1 |f_n(x)|^2\, dx \to 0 \quad as \quad n \to \infty.$$

On the other hand, for each x in $[0,1]$, we have

$$\limsup_{n\to\infty} f_n(x) = 1 \quad and \quad \liminf_{n\to\infty} f_n(x) = 0.$$

Hence $\{f_n\}$ does not converge for any x in $[0,1]$.

Theorem 5.12.2. *Assume that* $\lim_{n\to\infty} f_n = f$ *on* $[a,b]$. *If* $g \in \mathcal{R}$ *on* $[a,b]$, *define*

$$h(x) = \int_a^x f(t)g(t)dt, \qquad h_n(x) = \int_a^x f_n(t)g(t)dt,$$

if $x \in [a,b]$. *Then* $h_n \to h$ *uniformly on* $[a,b]$.

Proof. By Cauchy-schwarz inequality, we have

$$0 \le \left(\int_a^x |f(t) - f_n(t)||g(t)|\right)^2 \le \left(\int_a^x |f(t) - f_n(t)|^2 dt\right)\left(\int_a^x |g(t)|^2 dt\right). \qquad (a)$$

Since $\lim_{n\to\infty} f_n = f$, for given $\epsilon > 0$, we can choose N such that

$$n > N \implies \int_a^x |f(t) - f_n(t)|^2 dt < \frac{\epsilon^2}{A}. \qquad (b)$$

Since $g \in \mathcal{R}$, we have $g^2 \in \mathcal{R}$.

$$\therefore \int_a^x |g(t)|^2 dt < \infty. \qquad (c)$$

Substituting (b) and (c) in (a), we have

$$\left(\int_a^x |f(t) - f_n(t)||g(t)|\right)^2 \le \frac{\epsilon^2}{A}\int_a^x |g(t)|^2 dt < \frac{\epsilon^2}{A}\int_a^b |g(t)|^2 dt,$$

where $A = 1 + \int_a^b |g(t)|^2 dt$.

For $n > N$, we have

$$|h(x) - h_n(x)| < \epsilon \quad \forall\, x \in [a, b],$$

which implises that $h_n \to h$ uniformly on $[a, b]$. □

Theorem 5.12.3. *Assume that* $\lim_{n\to\infty} f_n = f$ *and* $\lim_{n\to\infty} g_n = g$ *on* $[a, b]$. *Define*

$$h(x) = \int_a^x f(t)g(t)dt, \qquad h_n(x) = \int_a^x f_n(t)g_n(t)dt,$$

if $x \in [a, b]$. *Then* $h_n \to h$ *uniformly on* $[a, b]$.

Proof. We have

$$\begin{aligned} h_n(x) - h(x) &= \int_a^x [f_n(t)g_n(t) - f(t)g(t)]dt \\ &= \int_a^x [f_n g_n - f_n g + f_n g - f g_n + f g_n - f g + f g - f g]dt \\ &= \int_a^x (f - f_n)(g - g_n)dt + \left(\int_a^x f_n g\, dt - \int_a^x f g\, dt \right) + \left(\int_a^x f g_n\, dt - \int_a^x f g\, dt \right). \end{aligned}$$

Applying the Cauchy-Schwarz inequality, we have

$$0 \le \left(\int_a^x |f - f_n||g - g_n|dt \right)^2 \le \left(\int_a^x |f - f_n|^2 dt \right)\left(\int_a^x |g - g_n|^2 dt \right).$$

Hence,

$$\begin{aligned} |h_n(x) - h(x)| \le & \left(\int_a^x |f - f_n|^2 dt \right)^{1/2} \left(\int_a^x |g - g_n|^2 dt \right)^{1/2} \\ & + \left| \int_a^x f_n g\, dt - \int_a^x f g\, dt \right| + \left| \int_a^x f g_n\, dt - \int_a^x f g\, dt \right|. \end{aligned}$$

Since $\lim_{n\to\infty} f_n = f$ and $\lim_{n\to\infty} g_n = g$ on $[a, b]$, we have

$$\lim_{n\to\infty} |h_n(x) - h(x)| = 0,$$

which implies that $h_n \to h$ uniformly on $[a, b]$. Hence the proof. □

Let us sum up

1. Introduced the notion of mean convergence.

2. We have seen that uniform convergence implies mean convergence.

3. We also discussed some important properties of mean convergence.

Check your progress

1. If $f_n(x) = x^n$ on $[0,1]$ and $f = \lim f_n$, then

   (A) f is constant

   (B) $f_n \to f$ uniformly.

   (C) f is monotonic.

   (D) Both (A) and (B) are true.

2. The sum of the series $\sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n}$ for $x \in \mathbb{R}$ is

   (A) continuous on $\mathbb{R}$

   (B) $f(x) = 1 + x^2$ for all $x \in \mathbb{R}$

   (C) discontinuous at $x = 0$

   (D) $f(x) = 0$ for all $x \in \mathbb{R}$

3. For $n \geq 1$, let $f_n(x) = xe^{-nx^2}$, $x \in \mathbb{R}$. Then the sequence $\{f_n\}$ is

   (A) uniformly convergent on $\mathbb{R}$

   (B) uniformly convergent only on compact subsets of $\mathbb{R}$

   (C) bounded and not uniformly convergent on $\mathbb{R}$

   (D) a sequence of unbounded functions

4. For the sequence $f_n(x) = \frac{1}{1+x^n}$, $0 \leq x \leq 1$, which one is true?

   (A) converges uniformly

   (B) Converges to 1 pointwise

   (C) The limit function is continuous

   (D) The limit function is not continuous

5. If f is non-negative and $f_n(x) = \begin{cases} f(x) & \text{if} \quad f(x) \leq n \\ n & \text{if} \quad f(x) > n \end{cases}$, then $\lim f_n =$

   (A) $f(x)$ (B) 1 (C) 0 (D)

6. For sequences $f_n(x) = \frac{1}{nx+1}$ and $g_n(x) = \frac{x}{nx+1}$ defined on $(0,1)$, which of the following is true?.

   (A) $\{f_n\}$ converges pointwise and $\{g_n\}$ converges uniformly on $(0,1)$

   (B) Both $\{f_n\}$ and $\{g_n\}$ converge uniformly on $(0,1)$

   (C) $\{f_n\}$ converges uniformly and $\{g_n\}$ converges pointwise on $(0,1)$

   (D) Both $\{f_n\}$ and $\{g_n\}$ converge pointwise on $(0,1)$

7. Which of the following is true for the series (1) $\sum_{n=1}^{\infty} \frac{\sin(n^2 x)}{n^2}$ and (2) $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n}$?

   (A) The series (1) converges uniformly on $\mathbb{R}$

   (B) The series (2) converges uniformly on $\mathbb{R}$

(C) Both (1) and (2) converges uniformly on $\mathbb{R}$

(D) Both (1) and (2) converge pointwise on $\mathbb{R}$

8. If $f_n \to f$ uniformly and $g_n \to g$ uniformly on E, then which of the following is not true ?

   (A) $f_n + g_n \to f + g$ uniformly on E.

   (B) $f_n g_n \to fg$ uniformly on E.

   (C) $f_n g_n \to fg$ uniformly on E if $\{f_n\}$ and $\{g_n\}$ are bounded functions

   (D) $f_n g_n \to fg$ uniformly on E if $\{f_n\}$ and $\{g_n\}$ are uniformly bounded

Summary

- Discussed the notion of pointwise and uniform convergence of sequence of functions and their limit functions.
- Solved several examples of sequences of functions that converge either pointwise or uniformly.
- Studied certain test for uniform convergence.
- Analysed the conditions for which continuity, differentiability and integrability can be transferred to limit functions.
- Discussed certain test for convergence of series of functions like Weierstrass M-test.
- Introduced the notion of mean convergence.

Self-Assessment Questions

1. State and prove the Dirichlet's test for uniform convergence.

2. Let $\alpha \in BV[a,b]$. Assume that each term of the sequence $\{f_n\}$ is such that $f_n \in \mathcal{R}(\alpha)$ on $[a,b]$ for each $n = 1, 2, \ldots$. Assume that $f_n \to f$ uniformly on $[a,b]$ and define $g_n(x) = \int_a^x f_n(t)d\alpha(t)$ if $x \in [a,b],\ n = 1, 2, \ldots$. Prove that $f \in \mathcal{R}(\alpha)$ on $[a,b]$ and $g_n \to g$ uniformly on $[a,b]$, where $g(x) = \int_a^x f(t)d\alpha(t)$.

3. Assume that $\{f_n\}$ is sequence of differential functions defined on (a, b). Assume that $\{f_n(x_0)\}$ converges at x_0 in (a, b). Assume that there exists a function g such that $f_n' \to g$ uniformly on (a, b). Prove that there exists a function f such that $f_n \to f$ uniformly on (a, b) and $f'(x) = g(x)$ for each x in (a, b).

Exercises

1. Assume that $f_n \to f$ uniformly on S and each f_n is bounded on S. Prove that f_n is uniformly bounded on S.

2. $f_n(x) = \dfrac{1}{nx+1}$ if $0 < x < 1,\ n = 1, 2, \ldots$ Prove that $\{f_n\}$ converges pointwise but not uniformly on $(0, 1)$.

3. $g_n(x) = \dfrac{x}{nx+1}$ if $0 < x < 1,\ n = 1, 2, \ldots$ Prove that $\{f_n\} \to 0$ uniformly on $(0, 1)$.

4. Assume that $f_n \to f$ uniformly on S, and that each f_n is continuous on S. If $x \in S$, let $\{x_n\}$ be a sequence of points in S such that $x_n \to s$. Prove that $f_n(x_n) \to f(x)$.

5. Let $f_n(x) = \dfrac{1}{1+n^2x^2}$ if $0 \leq x \leq 1,\ n = 1, 2, \ldots.$ Prove that $\{f_n\}$ converges pointwise but not uniformly on $[0, 1]$. Is term-by-term integration possible?

6. Prove that $\sum_{n=1}^{\infty} a_n \sin nx$ and $\sum_{n=1}^{\infty} a_n \cos nx$ are uniformly convergent on $\mathbb{R}$ if $\sum |a_n|$ converges.

References

Tom M. Apostol, Mathematical Analysis, Second Edition, Addison-Wesley Publishing Company Inc., New York, 1974.

Suggested Readings

1. T.M. Apostol, Calculus Vol.2, Multi-Variable Calculus and Linear Algebra with Applications to Differential Equations and Probability, Second Edition - Reprint, John Wiley & Sons, 2016.

2. R.G. Bartle, Real Analysis, John Wiley and Sons Inc., 1976.
3. W. Rudin, Principles of Mathematical Analysis, Third Edition, Mc-Graw Hill Company, New York, 1976.
4. S.C. Malik and S. Arora. Mathematical Analysis, Wiley Eastern Limited, New Delhi, 1991.
5. B.R. Gelbaum and J. Olmsted, Counter Examples in Analysis, Holden day, San Francisco, 1964.
6. A.L. Gupta and N.R. Gupta, Principles of Real Analysis, Pearson Education, (Indian print) 2003.