PERIYAR UNIVERSITY

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SALEM - 636 011, Tamil Nadu, India.

CENTRE FOR DISTANCE AND ONLINE EDUCATION (CDOE)

MASTER OF SCIENCE IN MATHEMATICS

SEMESTER - II

CORE COURSE: TOPOLOGY

(Candidates admitted from 2024 onwards)

PERIYAR UNIVERSITY

CENTRE FOR DISTANCE AND ONLINE EDUCATION (CDOE)

M.Sc. MATHEMATICS 2024 admission onwards

CORE – VI
Topology

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SYLLABUS: TOPOLOGY

Objectives:

The objective of this course is to study topological spaces, continuous functions, connectedness, compactness, countability and separation axioms.

Unit I: Topological spaces Topological spaces – Basis for a topology – The order topology – The product topology on $X \times Y$ – The subspace topology – Closed sets and limit points.

Unit II: Continuous functions Continuous functions – The product topology – The metric topology.

Unit III: Connectedness Connected spaces- Connected subspaces of the Real line – Components and local connectedness.

Unit IV: Compactness Compact spaces – Compact subspaces of the Real line – Limit Point Compactness – Local Compactness.

Unit V: Countability and Separation Axioms The Countability Axioms – The separation Axioms – Normal spaces – The Urysohn Lemma – The Urysohn metrization Theorem – The Tietz extension theorem.

References:

1. James R. Munkres, Topology (2nd Edition), Prentice Hall of India, New Delhi, 2011.

Suggested Reading:

1. J. Dugundji, Topology, Prentice Hall of India, New Delhi, 1975.

2. George F. Simmons, Introduction to Topology and Modern Analysis, McGraw Hill Book Co., 1963.

3. J.L. Kelley, General Topology, Van Nostrand, Reinhold Co., New York, 1955.

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Unit 1

Topological Spaces

Objectives:

This unit defines what a topological space is, and discuss some of the elementary concepts associated with topological spaces, namely, basis for a topology, open sets, closed sets and limit points. Further, it gives different ways of constructing a topology on a set so as to make it into a topological space.

1.1 Topological Spaces

Definition 1.1.1. *A **topology** on a set X is a collection τ of subsets of X with the following properties:*
(i) $\emptyset, X \in \tau$.
(ii) The union of the elements of any subcollection of τ is in τ.
(iii) The intersection of the elements of any finite subcollection of τ is in τ.
*The ordered pair (X, τ) is called a **topological space**.*

Definition 1.1.2. *If (X, τ) is a topological space, then a subset U of X is called an **open set** if $U \in \tau$.*

Definition 1.1.3. *If X is any set, the collection of all subsets of X is a topology on X, and is called the **discrete topology** on X. The collection consisting of X and $\emptyset$ only is also a topology on X, and is called the **indiscrete topology** (or) **trivial topology**.*

Example 1.1.4. *Consider a set $X = \{1, 2\}$. Then*

$$\tau_1 = \{\emptyset, X\},$$

$$\tau_2 = \Big\{\emptyset, X, \{1\}\Big\},$$
$$\tau_3 = \Big\{\emptyset, X, \{2\}\Big\}$$

and

$$\tau_4 = \Big\{\emptyset, X, \{1\}, \{2\}\Big\}$$

are all topologies on X.

Example 1.1.5. *Let X be a set and let τ_f be the collection of all subsets U of X such that $X - U$ either is finite or is all of X. Then τ_f is a topology on X, called the **finite complement topology**.*

Example 1.1.6. *Let X be a set and let τ_c be the collection of all subsets U of X such that $X - U$ either is countable or is all of X. Then τ_c is a topology on X.*

Definition 1.1.7. *Let τ and $\tau^{'}$ be two topologies on a given set X. If $\tau^{'} \supset \tau$, then we say that $\tau^{'}$ is **finer** than τ. If $\tau^{'}$ properly contains τ, then we say that $\tau^{'}$ is **strictly finer** than τ. We also say that τ is **coarser** than $\tau^{'}$, or **strictly coarser**, in those two respective situations.*

*We say that τ is **comparable** with $\tau^{'}$ if either $\tau^{'} \supset \tau$ or $\tau \supset \tau^{'}$.*

Example 1.1.8. *On $X = \{1, 2, 3\}$, consider the topologies*
$\tau_1 = \Big\{\emptyset, X, \{1\}, \{2\}, \{1, 2\}, \{1, 3\}\Big\}$ *and*
$\tau_2 = \Big\{\emptyset, X, \{1\}, \{1, 3\}\Big\}$.
Then τ_1 is strictly finer than τ_2.
But $\tau_3 = \Big\{\emptyset, X, \{1\}\Big\}$ and $\tau_4 = \Big\{\emptyset, X, \{2\}\Big\}$ are not comparable.

Remark 1.1.9. *From the above example, it is clear that, two topologies on X need not be comparable.*

Let Us Sum Up:

In this section, we have seen the definitions of topology, topological space and open set with some examples. We have also discussed the comparison of topologies on a given set.

Check your Progress:

1. Which of the following is not a topology on $X = \{a, b, c\}$?

   (A) $\{\phi, X\}$ (B) $\{\phi, X, \{a, b\}, \{c\}\}$

   (C) $\{\phi, X, \{a, b\}, \{a, c\}, \{a\}\}$ (D) None of these

2. If $\tau = \{\phi, X, \{a, c\}, \{b\}\}$ is a topology on $X = \{a, b, c\}$, then which of the following is not open?

   (A) ϕ (B) $\{a\}$ (C) $\{b\}$ (D) None of these

3. If $\tau_1 = \Big\{\emptyset, X, \{1\}, \{1, 3\}\Big\}$ and $\tau_2 = \Big\{\emptyset, X, \{1\}\Big\}$ are topologies on $X = \{a, b, c\}$, then which of the following is true?

   (A) τ_1 is strictly coarser than τ_2 (B) τ_1 is finer than τ_2

   (C) τ_1 is strictly finer than τ_2 (D) τ_1 is coarser than τ_2

1.2 Basis for a Topology

Definition 1.2.1. *If X is a set, a **basis** for a topology on X is a collection $\mathscr{B}$ of subsets of X (called **basis elements**) such that*
(1) For each $x \in X$, there is atleast one basis element B containing x.
(2) If x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$.

Definition 1.2.2. *If $\mathscr{B}$ is a basis for a topology on X, then we define the **topology** τ **generated by** $\mathscr{B}$ as follows:*

"A subset U of X is said to be open in X (that is, to be an element of τ) if for each $x \in U$, there is a basis element $B \in \mathscr{B}$ such that $x \in B$ and $B \subset U$".
Each member of $\mathscr{B}$ is in the topology τ generated by $\mathscr{B}$.
i.e) Each member of $\mathscr{B}$ is open in X.

Example 1.2.3. *The collection $\mathscr{B}$ of all circular regions (interiors of circles) in the plane forms a basis.*

Example 1.2.4. *The collection $\mathscr{B}'$ of all rectangular regions (interiors of rectangles) having sides parallel to the coordinate axes is also a basis for some topology on the plane.*

Example 1.2.5. *Let X be any set and $\mathscr{B}$ be the collection of all one-point subsets of X. That is, $\mathscr{B} = \Big\{\{x\} : x \in X\Big\}$. Then it is easy to see that $\mathscr{B}$ is a basis for the discrete topology on X.*

Lemma 1.2.6. *Let X be a set and let $\mathscr{B}$ be a basis for a topology τ on X. Then τ equals the collection of all unions of elements of $\mathscr{B}$.*

Proof. Given a collection of elements of $\mathscr{B}$, they are also elements of τ. Because τ is a topology, their union is in τ. Conversely, given $U \in \tau$, choose for each $x \in U$ an element B_x of $\mathscr{B}$ such that $x \in B_x \subset U$. Then $U = \bigcup_{x \in U} B_x$, so U equals a union of elements of $\mathscr{B}$. □

Remark 1.2.7. *By the previous lemma, every open set U in X can be expressed as a union of basis elements. However, the expression for U is not unique.*

Lemma 1.2.8. *Let X be a topological space. Suppose that $\mathscr{C}$ is a collection of open sets of X such that for each open set U of X and each x in U, there is an element C of $\mathscr{C}$ such that $x \in C \subset U$. Then $\mathscr{C}$ is a basis for the topology of X.*

Proof. First, let us prove that $\mathscr{C}$ is a basis.
(i) Let $x \in X$. Since X itself is an open set, by hypothesis, there exists an element $C \in \mathscr{C}$ such that $x \in C \subset X$.
(ii) Let $x \in C_1 \cap C_2$ where $C_1, C_2 \in \mathscr{C}$.
Since C_1 and C_2 are open, we have $C_1 \cap C_2$ is also open. Then, by hypothesis, there is an element $C_3 \in \mathscr{C}$ such that $x \in C_3 \subset C_1 \cap C_2$. Thus $\mathscr{C}$ is a basis for some topology on X.

Next, let τ be the given topology on X and let τ' be the topology generated by the basis $\mathscr{C}$. Then, it is enough to prove that $\tau = \tau'$.
Let $U \in \tau$ and let $x \in U$. Therefore, by hypothesis, there is an element $C \in \mathscr{C}$ such that $x \in C \subset U$.
$\Rightarrow U \in \tau'$.
$\therefore \tau' \supset \tau$.

Now, let $W \in \tau'$. Then by Lemma 1.2.6, W equals the union of elements of $\mathscr{C}$. Since every member of $\mathscr{C}$ belongs to τ and since τ is a topology on X, the above union is also a member of τ.

i.e) $W \in \tau$. Therefore $\tau \supset \tau'$.

Thus $\tau = \tau'$. □

The following lemma gives the criterion for comparing two topologies in terms of their bases.

Lemma 1.2.9. *Let $\mathscr{B}$ and $\mathscr{B}'$ be the bases for the topologies τ and τ' respectively, on a set X. Then the following are equivalent:*

(1) τ' is finer than τ.

(2) For each $x \in X$ and each basis element $B \in \mathscr{B}$ containing x, there is a basis element $B' \in \mathscr{B}'$ such that $x \in B' \subset B$.

Proof. $(1) \Rightarrow (2)$. Suppose that $\tau' \supset \tau$.

Let $x \in X$ and let $B \in \mathscr{B}$ such that $x \in B$. Since every member of $\mathscr{B}$ belongs to τ, we have $B \in \tau$. Then $B \in \tau'$ because $\tau' \supset \tau$. Since τ' is generated by the basis $\mathscr{B}'$, there is an element $B' \in \mathscr{B}'$ such that $x \in B' \subset B$.

$(2) \Rightarrow (1)$. Suppose that (2) holds.

We have to prove that $\tau' \supset \tau$.

Let $U \in \tau$ and let $x \in U$. Since τ is generated by the basis $\mathscr{B}$, there is an element $B \in \mathscr{B}$ such that $x \in B \subset U$.

Now, we have a basis element $B \in \mathscr{B}$ such that $x \in B$. Therefore, by (2), there is an element $B' \in \mathscr{B}'$ such that $x \in B' \subset B$. Since $B \subset U$, we have that there is a basis element $B' \in \mathscr{B}'$ such taht $x \in B' \subset U$.

$\Rightarrow U \in \tau'$.

Therefore, $\tau' \supset \tau$. □

Example 1.2.10. *By Lemma 1.2.9, one can easily see that the collection $\mathscr{B}$ of all circular regions in the plane generates the same topology as the collection $\mathscr{B}'$ of all rectangular regions in the plane whose sides parallel to the co-ordinate axes.*

Definition 1.2.11. *If $\mathscr{B}$ is the collection of all open intervals in the real line,*

$$(a, b) = \{x \in \mathbb{R} : a < x < b\},$$

then the topology generated by $\mathscr{B}$ is called the ***standard topology*** *on* $\mathbb{R}$. *Whenever we consider* $\mathbb{R}$*, it means that* $\mathbb{R}$ *is given the standard topology.*

If $\mathscr{B}'$ *is the collection of all half open intervals of the form*

$$[a, b) = \{x \in \mathbb{R} : a \leq x < b\},$$

where $a < b$*, the topology generated by* $\mathscr{B}'$ *is called the* ***lower limit topology*** *on* $\mathbb{R}$. *When* $\mathbb{R}$ *is given the lower limit topology, we denote it by* $\mathbb{R}_l$.

Let $K = \{1/n : n \in \mathbb{Z}_+\}$ *and let* $\mathscr{B}''$ *be the collection of all open intervals* (a, b)*, along with all sets of the form* $(a, b) - K$. *The topology generated by* $\mathscr{B}''$ *is called the* K***- topology*** *on* $\mathbb{R}$. *When* $\mathbb{R}$ *is given this topology, we denote it by* $\mathbb{R}_K$.

Lemma 1.2.12. *The topologies of* $\mathbb{R}_l$ *and* $\mathbb{R}_K$ *are strictly finer than the standard topology on* $\mathbb{R}$*, but are not comparable with one another.*

Proof. Let τ, τ' and τ'' be the topologies of $\mathbb{R}$, $\mathbb{R}_l$, and $\mathbb{R}_K$, respectively.
First, let us prove that the lower limit topology on $\mathbb{R}$ is strictly finer than the standard topology on $\mathbb{R}$. That is, to prove that $\tau' \supset \tau$ and $\tau \not\supset \tau'$.

Let $x \in \mathbb{R}$ and consider the basis element (a, b) for τ containing x. Then there is a basis element $[x, b)$ for τ' such that $x \in [x, b) \subset (a, b)$. Therefore, by Lemma 1.2.9, we have

$$\tau' \supset \tau. \tag{1.1}$$

Next, let $x \in \mathbb{R}$ and consider the basis element $[x, d)$ for τ'. But there is no basis element (a, b) for τ so that $x \in (a, b)$ and $(a, b) \subset [x, d)$. Therefore, by Lemma 1.2.9,

$$\tau \not\supset \tau'. \tag{1.2}$$

Thus, by (1.1) and (1.2), τ' is strictly finer than τ.

Next, let us prove that the K-topology on $\mathbb{R}$ is strictly finer than the standard topology on $\mathbb{R}$. To prove $\tau'' \supset \tau$, let $x \in \mathbb{R}$ and consider the basis element (a, b) for τ containing x. Then (a, b) itself is a basis element for τ'' so that $x \in (a, b) \subset (a, b)$, and so

$$\tau'' \supset \tau. \tag{1.3}$$

Next, to prove $\tau \not\supset \tau''$, let $0 \in \mathbb{R}$ and consider the basis element $(-1, 1) - K$ for τ'' containing 0. But there is no basis element (a, b) for τ containing 0 so that $(a, b) \subset (-1, 1) - K$. Therefore, by Lemma 1.2.9, we have

$$\tau \not\supset \tau''. \tag{1.4}$$

Thus, by (1.3) and (1.4), τ'' is strictly finer than τ.

Similarly, we can prove that the topologies of $\mathbb{R}_l$ and $\mathbb{R}_K$ are not comparable. □

Definition 1.2.13. *A* ***subbasis*** $\mathcal{S}$ *for a topology on* X *is a collection of subsets of* X *whose union equals* X*.*

Example 1.2.14. $\mathcal{S} = \{\{1\}, \{2, 3\}, \{4\}\}$ *is a subbasis for some topology on* $X = \{1, 2, 3, 4\}$.

Definition 1.2.15. *The* ***topology generated by the subbasis*** $\mathcal{S}$ *is defined to be the collection* τ *of all unions of finite intersections of elements of* $\mathcal{S}$*.*

Example 1.2.16. *On* $X = \{1, 2, 3, \}$*, consider the subbasis* $\mathcal{S} = \{\{1\}, \{2, 3\}\}$.
Let us find the topology τ *generated by* $\mathcal{S}$ *.*
Let $\mathscr{B}$ *be the collection of all finite intersection of member of* $\mathcal{S}$*. Then* $\mathscr{B} = \{\{1\}, \{2, 3\}, \emptyset\}$.
Therefore, the topology generated by $\mathcal{S}$ *is given by collecting all unions of member of* $\mathscr{B}$*.*
i.e) $\tau = \{\{1\}, \{2, 3\}, \emptyset, \{1, 2, 3\}\}$.

Let Us Sum Up:

In this section, we have studied the following concepts:

(1) Basis for a topology with examples

(2) Generating the topology from the given basis

(3) The criterion for comparing two topologies in terms of their bases

(4) Some topologies on the real line

(5) Subbasis for a topology

Check your Progress:

1. Which of the following is not open in $\mathbb{R}$?

   (A) $(0, \infty)$ (B) the set of rationals (C) $(1, 3) \cup [2, 5)$ (D) None of these

2. Which of the following is correct?

   (A) Every basis is a topology

   (B) Any two topologies are comparable

   (C) Every basis is a subbasis for a topology

   (D) None of these

3. Which of the following is not true for $(1, \infty)$?

   (A) open in $\mathbb{R}$ (B) open in $\mathbb{R}_l$

   (C) open in $\mathbb{R}_K$ (D) None of these

1.3 The Order Topology

Definition 1.3.1. *A **relation** on a set A is a subset C of the cartesian product $A \times A$. If C is a relation on A, we use notation xCy to mean the same thing as $(x, y) \in C$. We read it "x is in the relation C to y."*

Definition 1.3.2. *An **equivalance relation** on a set A is a relation C on A with the following properties:*
(i) (Reflexivity) xCx, $\forall x \in A$.
(ii) (Symmetry) If xCy then yCx.
(iii) (Transitivity) If xCy and yCz, then xCz.

Definition 1.3.3. *A relation C on a set A is called an **order relation** (or) **simple order** (or) a **linear order** if it has the following properties:*
(i)(Comparability) For every $x, y \in A$ for which $x \neq y$ either xCy (or) yCx.
(ii) (Non refelxivity) For no x in A does the relation xCx hold.
(iii) (Transitivity) If xCy and yCz, then xCz.

Definition 1.3.4. *Suppose A and B are two sets with order relations $<_A$ and $<_B$ respectively. Define an order relation $<$ on $A \times B$ by defining*

$$a_1 \times b_1 < a_2 \times b_2$$

*if $a_1 <_A a_2$, or if $a_1 = a_2$ and $b_1 <_B b_2$. It is called the **dictionary order relation** on $A \times B$.*

Definition 1.3.5. *Suppose that A is a set ordered by the relation $<$. Let A_0 be a subset of A. We say that the element b is the **largest element** of A_0 if $b \in A_0$ and $x \leq b$, $\forall x \in A_0$. Similarly, we say that a is the **smallest element** of A_0 if $a \in A_0$ and $a \leq x$, $\forall x \in A_0$.*

Definition 1.3.6. *Let X be a set with a simple order relation and assume that X has more than one element. Let $\mathscr{B}$ denote the collection of all sets of the following types:*
(i) All open intervals (a, b) in X.
(ii) All intervals of the form $[a_0, b)$, where a_0 is the smallest element (if any) of X.
(iii) All intervals of the form $(a, b_0]$, where b_0 is the largest element (if any) of X.
*The collection $\mathscr{B}$ is a basis for a topology on X, which is called the **order topology**.*

Let us prove that $\mathscr{B}$ is a basis.
(1) Let $x \in X$. If x is the smallest element of X, then every element in $\mathscr{B}$ of type (ii) contains x. If x is the largest element of X, then every element in $\mathscr{B}$ of type (iii) contains x. If x is neither a smallest nor a largest element of X, then x belongs to a member in $\mathscr{B}$ of type (i). So, the first condition for a basis is satisfied.
(2) Since intersection of any two members of $\mathscr{B}$ is again a member of $\mathscr{B}$, the second condition for a basis is also satisfied. Therefore, $\mathscr{B}$ is a basis.

Remark 1.3.7. *If X has no smallest element, then there are no sets of type (ii), and if X has no largest element, then there are no sets of type (iii) in $\mathscr{B}$.*

Example 1.3.8. *Consider the real line $\mathbb{R}$ with usual order $(<)$. Since $\mathbb{R}$ has neither a smallest element nor a largest element, the basis for the order topology on $\mathbb{R}$ consists of only open intervals in $\mathbb{R}$. Thus, the order topology on $\mathbb{R}$ is same as the standard topology on $\mathbb{R}$.*

Example 1.3.9. *Consider the set $\mathbb{R} \times \mathbb{R}$ with the dictionary order relation. Let us denote any element of $\mathbb{R} \times \mathbb{R}$ by $x \times y$. Since the set $\mathbb{R} \times \mathbb{R}$ has neither a smallest element nor a largest element, the order topology on $\mathbb{R} \times \mathbb{R}$ has as basis the collection of all open intervals of the form $(a \times b, c \times d)$ for $a < c$, and for $a = c, b < d$.*

Example 1.3.10. *Consider the ordered set of positive integers $\mathbb{Z}_+$.*
Clearly, 1 is the smallest element of $\mathbb{Z}_+$ and has no largest element. Then, the basis elements for the order topology on $\mathbb{Z}_+$ are of type (i) and (ii).

Let us prove that the order topology on $\mathbb{Z}_+$ is the discrete topology.

We prove this by proving that every one point set is open in the order topology.

Let $n \in \mathbb{Z}_+$ with $n > 1$. Then $\{n\} = (n-1, n+1)$, which is a basis element for the order topology on $\mathbb{Z}_+$. Also, $\{1\} = [1, 2)$ is also a basis element.

We know that every basis element is open.

Thus, every one point set is open.

Example 1.3.11. *Consider the ordered set $X = \{1, 2\} \times \mathbb{Z}_+$ with the dictionary order relation. Then*

$$\begin{aligned} X &= \{1, 2\} \times \mathbb{Z}_+ \\ &= \{1, 2\} \times \{1, 2, 3, \cdots\} \\ &= \{1 \times 1, 1 \times 2, 1 \times 3, \cdots, 2 \times 1, 2 \times 2, 2 \times 3, \cdots\} \end{aligned}$$

Clearly, 1×1 is the smallest element of X.

If we denote $1 \times n$ by a_n and $2 \times n$ by b_n $\forall n \in \mathbb{Z}_+$, then

$$X = \{a_1, a_2, \cdots, b_1, b_2, \cdots\}.$$

Claim: *The order topology on X is not the discrete topology.*

We have

$$\begin{aligned} \{a_1\} &= [a_1, a_2), \\ \{a_2\} &= (a_1, a_3), \\ &\vdots \\ \{a_n\} &= (a_{n-1}, a_{n+1}), \\ &\vdots \end{aligned}$$

and

$$\begin{aligned} \{b_2\} &= (b_1, b_3), \\ \{b_3\} &= (b_2, b_4), \\ &\vdots \\ \{b_n\} &= (b_{n-1}, b_{n+1}), \\ &\vdots \end{aligned}$$

That is, the above 1-point sets are all written in the form of basis element for the order topology on X. So, they are all open. But $\{b_1\}$ can not be written in any of the basis element structure.

Also, every basis element containing b_1 consists of the points of the a_i sequence. So, it can not be a subset of $\{b_1\}$. Therefore $\{b_1\}$ is not open. Hence our claim.

Definition 1.3.12. *If X is an ordered set, and a is an element of X, there are four subsets of X that are called the* ***rays determined by*** *a. They are the following:*

$$\begin{aligned}(a, +\infty) &= \{x \in X : x > a\},\\ (-\infty, a) &= \{x \in X : x < a\},\\ [a, +\infty) &= \{x \in X : x \geq a\},\\ (-\infty, a] &= \{x \in X : x \leq a\}.\end{aligned}$$

Sets of the first two types are called ***open rays****, and sets of the last two types are called* ***closed rays****.*

Remark 1.3.13. *If X is an ordered set, then*
(1) open rays in X are open sets in the order topology.
(2) open rays form a subbasis for the order topology on X.

Let Us Sum Up:

In this section, we have studied the following concepts:
(1) Order topology with examples
(2) Open rays and closed rays

Check your Progress:

1. Which of the following is a basis element for the dictionary order topology on $\mathbb{R} \times \mathbb{R}$?

   (A) $\Big((-1) \times (-1), (-1) \times 0\Big)$ (B) $\Big(0 \times 1, (-1) \times 0\Big)$

   (C) $\Big(0 \times (-1),\ (-1) \times 0\Big)$ (D) $\Big((-1) \times 0, (-1) \times (-1)\Big)$

2. For which of the following sets, all the three types of basis elements exist for the order topology?

   (A) $\mathbb{Z}_+$ (B) $\mathbb{R}_+$

   (C) $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\}$ (D) $[a, b]$ with $a, b \in \mathbb{R}$

3. The order topology on $\mathbb{Z}_+$ is ....

   (A) the standard topology (B) the indiscrete topology

   (C) the discrete topology (D) the trivial topology

1.4 The Product Topology on X × Y

Definition 1.4.1. *Let X and Y be topological spaces. The **product topology** on $X \times Y$ is the topology having as basis the collection $\mathscr{B}$ of all sets of the form $U \times V$, where U is an open subset of X and V is an open subset of Y.*

Theorem 1.4.2. *If $\mathscr{B}$ is a basis for a topology on X and $\mathscr{C}$ is a basis for a topology on Y, then the collection*

$$\mathscr{D} = \{B \times C | B \in \mathscr{B} \text{ and } C \in \mathscr{C}\}$$

is a basis for the topology on $X \times Y$.

Proof. Let us prove that $\mathscr{D}$ is a basis using Lemma 1.2.8. Let W be an open set in the product topology of $X \times Y$ and let a point $x \times y$ of W. Then, by the definition of the product topology, there exists a basis element $U \times V$ such that $x \times y \in U \times V \subset W$. Since U is open in X and the topology on X is generated by the basis $\mathscr{B}$, there is a member $B \in \mathscr{B}$ such that $x \in B \subset U$.

Similarly, there exists a member $C \in \mathscr{C}$ such that $y \in C \subset V$. Therefore, $x \times y \in B \times C \subset U \times V$. But already we have $U \times V \subset W$. Thus, we have a member $B \times C \in \mathscr{D}$ such that $x \times y \in B \times C \subset W$.
$\Rightarrow \mathscr{D}$ satisfies the hypothesis of Lemma 1.2.8. Therefore, $\mathscr{D}$ is a basis for the topology on $X \times Y$. □

Example 1.4.3. *Consider, the standard topology (order topology) on $\mathbb{R}$. The product of this topology with itself is called the standard topology on $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$. Its basis is given by the collection of all products $U \times V$ of open sets of $\mathbb{R}$.*

But the previous theorem shows that the much smaller collection of all products $(a, b) \times (c, d)$ of open intervals in $\mathbb{R}$, will also be a basis for the topology on $\mathbb{R}^2$. Also each such set can be pictured as the interior of a rectangle in $\mathbb{R}^2$.

Thus, the standard topology on $\mathbb{R}^2$ is same as the topology generated by the basis consists of all open rectangles in $\mathbb{R}^2$ with sides parallel to the co-ordinate axes.

Definition 1.4.4. *Let $\pi_1 : X \times Y \to X$ be defined as*

$$\pi_1(x, y) = x$$

and let $\pi_2 : X \times Y \to Y$ be defined as

$$\pi_2(x, y) = y.$$

The maps π_1 and π_2 are called ***projections*** *of $X \times Y$ onto X and Y, respectively.*

Remark 1.4.5. *If U is an open subset of X, then*

$$\begin{aligned} \pi_1^{-1}(U) &= \{(x, y) \in X \times Y : \pi_1(x, y) \in U\} \\ &= \{(x, y) \in X \times Y : x \in U\} \\ &= U \times Y. \end{aligned}$$

Since U is open in X and Y is open in Y, we have $U \times Y = \pi_1^{-1}(U)$ is open in $X \times Y$. Similarly, if V is an open subset of Y, then $\pi_2^{-1}(V) = X \times V$ is open in $X \times Y$. Thus,

$$\pi_1^{-1}(U) \cap \pi_2^{-1}(V) = (U \times Y) \cap (X \times V) = (U \cap X) \times (Y \cap V) = U \times V.$$

Theorem 1.4.6. *The collection*

$$\mathcal{S} = \left\{\pi_1^{-1}(U) | U \text{is open in} X \right\} \cup \left\{\pi_2^{-1}(V) | V \text{is open in} Y\right\}$$

is a subbasis for the product topology on $X \times Y$.

Proof. Let τ denote the product topology on $X \times Y$ and let τ' be the topology generated by $\mathcal{S}$. Because every element of $\mathcal{S}$ belongs to τ , so does arbitrary unions of finite intersections of elements of $\mathcal{S}$. Thus $\tau \supset \tau'$. For the reverse inclusion, note that every basis element $U \times V$ for the topology τ is a finite intersection of elements of $\mathcal{S}$, since

$$U \times V = \pi_1^{-1}(U) \cap \pi_2^{-1}(V).$$

Therefore, $U \times V \in \tau'$, so that $\tau' \supset \tau$. Hence $\tau = \tau'$. □

Let Us Sum Up:

In this section, we have studied the following concepts:

(1) Product Topology on X $\times$ Y with examples

(2) Projections on product of two topological spaces

(3) Subbasis for the product topology on X $\times$ Y

Check your Progress:

1. Which of the following collections is not a basis for the standard topology on $\mathbb{R}^2$?

   (A) all products of open sets of $\mathbb{R}$

   (B) all products $(a, b)\times(c, d)$ of open intervals in $\mathbb{R}$

   (C) all circular regions in $\mathbb{R}^2$

   (D) None of these

2. Let X and Y be topological spaces. If U is an open subset of X, then $\pi_1^{-1}(U) =$...........?

   (A) $X \times Y$ (B) $U \times Y$ (C) $U \cap X$ (D) $U \times X$

3. The projection map $\pi_2 : X \times Y \to Y$ is

   (A) not a bijection (B) onto (C) an open map (D) All of these

1.5 The Subspace Topology

Let X be a topological space with topology τ. If Y is a subset of X, then the collection

$$\tau_Y = \{Y \cap U | U \in \tau\}$$

is a topology on Y, called the **subspace topology**. With this topology, Y is called a **subspace** of X.

Theorem 1.5.1. *If $\mathscr{B}$ is a basis for the topology of X, then the collection*

$$\mathscr{B}_Y = \{B \cap Y | B \in \mathscr{B}\}$$

is a basis for the subspace topology on Y.

Proof. Consider an open subset $Y \cap U$ of Y where U is open in X and let $y \in Y \cap U$. Since U is open in X and since $y \in U$, there is an element $B \in \mathscr{B}$ such that $y \in B \subset U$.

$$\Rightarrow y \in B \cap Y \subset Y \cap U.$$

Thus, we have a member $B \cap Y \in \mathscr{B}_Y$ such that $y \in B \cap Y \subset Y \cap U$. Therefore, By Lemma 1.2.8, $\mathscr{B}_Y$ is a basis for the subspace topology on Y. □

Definition 1.5.2. *If Y is the subspace of X, we say that a set U is open in Y (or **open relative to Y**) if it belongs to the topology of Y. We say that U is open in X if it belongs to the topology of X.*

Theorem 1.5.3. *Let Y be a subspace of X. If U is open in Y and Y is open in X, then U is open in X.*

Proof. Since U is open in Y, we have $U = Y \cap V$, where V is open in X. Since Y and V are open in X, we have U is open in X. □

Theorem 1.5.4. *If A is a subspace of X and B is a subspace of Y, then the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $X \times Y$.*

Proof. Let the general basis element for the product topology on $X \times Y$ be $U \times V$, where U is open in X and V is open in Y. Therefore, $(A \times B) \cap (U \times V)$ is the general basis element for the subspace topology on $A \times B$.

Now, $(A \times B) \cap (U \times V) = (A \cap U) \times (B \cap V)$.

Since U is open in X, we have $A \cap U$ is open in A. Similarly, $B \cap V$ is open in B. Therefore, $(A \cap U) \times (B \cap V)$ is a general basis element for the product topology on $A \times B$.

Thus, we have proved that the bases for the subspace topology on $A \times B$ and for the product topology on $A \times B$ are the same. Hence the corresponding topologies are same. □

Example 1.5.5. *Consider a subset $Y = [0,1]$ of $\mathbb{R}$ in the subspace topology. Then the basis for the subspace topology on Y is given by*

$$\mathscr{B}_Y = \{Y \cap (a,b) | (a,b) \text{is an open interval in } \mathbb{R}\},$$

where

$$Y \cap (a,b) = \begin{cases} \text{(a,b) if a and b are in } Y, \\ \text{[0,b) if only b is in } Y, \\ \text{(a,1] if only a is in } Y, \\ Y \text{ or } \emptyset \text{ if neither a nor b is in } Y. \end{cases}$$

By the definition, each of the above sets is open in Y. But sets of the second and third types are not open in $\mathbb{R}$.

Remark 1.5.6. *Let X be an ordered set in the order topology and let Y be a subset of X. When the order relation on X is restricted to Y, it makes Y into an ordered set. However,* ***the resulting order topology on Y need not be the same as the topology that Y inherits as a subspace of X****.*

If we consider the previous example, the collection $\mathscr{B}_Y$ forms a basis for the order topology on Y also. Thus in the case of $Y = [0,1]$, its subspace topology as a subspace of $\mathbb{R}$ and its order topology are the same. ***But it is not true always****. For example,*

(i) Consider a subset $Y = [0,1) \cup \{2\}$ of $\mathbb{R}$. Since $\{2\} = Y \cap (1,3)$, we have $\{2\}$ is open in the subspace topology on Y. But in the order topology on Y, any basis element containing 2 is of the form $\{x | x \in Y \text{and } a < x \leq 2\}$ for some $a \in Y$, and hence it cannot be a subset of $\{2\}$. Therefore, $\{2\}$ is not open in the order topology on Y.

(ii) Let us see another example for the above remark. Let $I = [0,1]$ and consider $\mathbb{R} \times \mathbb{R}$ in the dictionary order topology. Then, the dictionary order topology on $I \times I$ is not the same as the subspace topology on $I \times I$ obtained from the dictionary order topology on $\mathbb{R} \times \mathbb{R}$.

For, let $U = \{\frac{1}{2}\} \times (\frac{1}{2}, 1]$. Since $U = (I \times I) \cap V$, where $V = \{\frac{1}{2}\} \times (\frac{1}{2}, 2)$ is open in the dictionary order topology on $\mathbb{R} \times \mathbb{R}$, U is open in the subspace topology on $I \times I$.

But, in the dictionary order topology on $I \times I$, if we consider the point $\frac{1}{2} \times 1$ in U, then any open interval $(a \times b, c \times d)$ containing $\frac{1}{2} \times 1$ will not be a subset of U. Thus U is not open in the dictionary order topology on $I \times I$.

Note: $I \times I$ in the dictionary order topology is called the **ordered square** and is denoted by I_o^2.

Definition 1.5.7. *Let X be an ordered set. Then the subset Y of X is said to be* ***convex*** *in X if for each pair of points $a < b$ of Y, the entire interval (a, b) of points of X lies in Y.*

Note that in an ordered set X, all the intervals and rays are convex in X.

Theorem 1.5.8. *Let X be an ordered set in the order topology and let Y be a subset of X that is convex in X. Then the order topology on Y is the same as the topology Y inherits as a subspace of X.*

Proof. Consider the ray $(a, +\infty)$ in X. We know that open rays are open sets in the order topology. So, $(a, +\infty)$ is open in X.

Now, consider $(a, +\infty) \cap Y$. If $a \in Y$, then

$$(a, +\infty) \cap Y = \{x : x \in Y \text{ and } x > a\},$$

which is an open ray in the ordered set Y.
If $a \notin Y$, then by the convexity of Y, either $x > a \; \forall x \in Y$ or $x < a \; \forall x \in Y$.
That is, either a is a lower bound on Y or an upper bound on Y.
If a is a lower bound on Y, then $(a, +\infty) \cap Y = Y$.
If a is an upper bound on Y, then $(a, +\infty) \cap Y = \emptyset$.
Thus, the set $(a, +\infty) \cap Y$ is either an open ray of Y, or Y itself, or empty.
Similarly, the set $(-\infty, a) \cap Y$ is either an open ray of Y, or Y itself, or empty.
$\therefore$ The collection of all the sets $(a, +\infty) \cap Y$ and $(-\infty, a) \cap Y$ form a subbasis for the subspace topology on Y.
But, we know that each of the above subbasis element is an open set in the order topology on Y.
That is, every subbasis element for the subspace topology on Y belongs to the order topology on Y.
Hence, order topology on $Y \supset$ subspace topology on Y.

Let us prove the reverse inclusion.
We know that, open rays of the ordered set Y form a subbasis for the order topology on Y. Also, we know that any open ray of Y equals the intersection of an open ray of X with Y. So, any open ray of Y is open in the subspace topology on Y.

That is, every subbasis element for the order topology on Y belongs to the subspace topology on Y.

$\therefore$ subspace topology on $Y \supset$ order topology on Y.

Hence the theorem. $\square$

Let Us Sum Up:

In this section, we have studied the following concepts:

(1) Subspace topology with examples

(2) Relation between the subspace topology and the order and product topologies

Check your Progress:

1. In which of the following subspaces of $\mathbb{R}$, the order of topology and the subspace topology are not the same?

   (A) $(-1, 1)$ (B) the set of rationals

   (C) $[0, 2) \cup (1, 3]$ (D) None of these

2. Let Y be a subspace of X. If U is open in X, then

   (A) $U \cap X$ is open in Y (B) $U \cap Y$ is open in Y

   (C) $U \cap X = \phi$ (D) $U \cap Y = \phi$

3. For the subset $Y = [0, 1]$ of $\mathbb{R}$, which of the following is not a basis element for the subspace topology on Y?

   (A) $[0, 1)$ (B) $(0, 1]$ (C) $(-1, 1)$ (D) None of these

1.6 Closed Sets and Limit Points

Definition 1.6.1. *A subset A of a topological space X is said to be* ***closed*** *if $X - A$ is open.*

Example 1.6.2. *1. Consider a subset $[a, b]$ of $\mathbb{R}$. Since $\mathbb{R} - [a, b] = (-\infty, a) \cup (b, \infty)$ is open in $\mathbb{R}$, $[a, b]$ is closed in $\mathbb{R}$.*

2. *Consider a subset* $[a, \infty)$ *of* $\mathbb{R}$. *Since* $\mathbb{R} - [a, \infty) = (-\infty, a)$ *is open in* $\mathbb{R}$, $[a, \infty)$ *is closed in* $\mathbb{R}$.

3. *Consider a subset* $(-\infty, a]$ *of* $\mathbb{R}$. *Since* $\mathbb{R} - (-\infty, a] = (a, \infty)$ *is open in* $\mathbb{R}$, $(-\infty, a]$ *is closed in* $\mathbb{R}$.

4. *Consider a subset* $\{a\}$ *of* $\mathbb{R}$. *Since* $\mathbb{R} - \{a\} = (-\infty, a) \cup (a, \infty)$ *is open in* $\mathbb{R}$, $\{a\}$ *is closed in* $\mathbb{R}$.

5. *Consider* $[a, b)$ *of* $\mathbb{R}$ *with* $a < b$. *Since* $\mathbb{R} - [a, b) = (-\infty, a) \cup [b, \infty)$ *is not open in* $\mathbb{R}$, $[a, b)$ *is not closed in* $\mathbb{R}$.

6. *Consider a subset* $[a, b)$ *of* $\mathbb{R}_l$. *Then* $\mathbb{R} - [a, b) = (-\infty, a) \cup [b, \infty)$. *Since* $(-\infty, a)$ *is open in* $\mathbb{R}$ *and since the lower limit topology on* $\mathbb{R}$ *is (strictly) finer than the standard topology on* $\mathbb{R}$, *the set* $(-\infty, a)$ *is open in* $\mathbb{R}_l$ *also. Also, we know that* $[b, \infty)$ *is open in* $\mathbb{R}_l$. *Hence, their union* $\mathbb{R} - [a, b)$ *is open in* $\mathbb{R}_l$. *Therefore,* $[a, b)$ *is closed in* $\mathbb{R}_l$.

7. *Consider a subset* $A = \{x \times y : x \geq 0$ *and* $y \geq 0\}$ *of* $\mathbb{R}^2$. *Then* $\mathbb{R}^2 - A = \{(-\infty, 0) \times \mathbb{R}\} \cup \{\mathbb{R} \times (-\infty, 0)\}$. *Since* $(-\infty, 0) \times \mathbb{R}$ *and* $\mathbb{R} \times (-\infty, 0)$ *are products of open sets in* $\mathbb{R}$, *they are open in* $\mathbb{R}^2$.
   $\Rightarrow \mathbb{R}^2 - A$ *is open in* $\mathbb{R}^2$.
   $\Rightarrow A$ *is closed in* $\mathbb{R}^2$.

8. *Consider the finite complement topology on a set* X.
   i.e) $\tau_f = \{U \subset X : X - U$ *is finite or* $X - U = X\}$.
   Let $A \subset X$. *Then,* A *is closed in* X *if* $X - A$ *is open in* X.
   That is, A *is closed in* X *if* $X - A \in \tau_f$.
   That is, A *is closed in* X *if* $X - (X - A)$ *is finite (or)* $X - (X - A) = X$. *That is,* A *is closed in* X *if* A *is finite (or)* $A = X$.
   Thus, the closed sets of X *in the finite complement topology are finite subsets of* X, *and* X *itself.*

9. *Consider the discrete topology on a set* X. *We know that, every subset of* X *is open in this topology. Also, if* $A \subset X$, *then* $X - A \subset X$.

$\Rightarrow X - A$ is open $\forall A \subset X$.

That is, complement of every subset of X is open. Hence, every subset of X is closed.

10. *Consider a subset $Y = [1,2] \cup (3,5)$ of $\mathbb{R}$ in the subspace topology.*

    Since $Y - [1,2] = (3,5) = Y \cap (3,5)$ and $Y - (3,5) = [1,2] = Y \cap (0,3)$, both $[1,2]$ and $(3,5)$ are closed in Y.

Theorem 1.6.3. *Let X be a topological space. Then the following conditions hold:*

(1) $\emptyset$ and X are closed.

(2) Arbitrary intersections of closed sets are closed.

(3) Finite unions of closed sets are closed.

Proof. (1) Since $\emptyset$ and X are the complements of the open sets X and $\emptyset$ respectively, $\emptyset$ and X are closed in X.

(2) Consider the collection of closed sets $\{A_\alpha\}_{\alpha \in J}$ in X.

By DeMorgan's law, we have

$$X - \bigcap_{\alpha \in J} A_\alpha = \bigcup_{\alpha \in J} (X - A_\alpha).$$

Since each $X - A_\alpha$ is open, their arbitrary union $\bigcup\limits_{\alpha \in J} (X - A_\alpha)$ is open.

$\Rightarrow X - \bigcap\limits_{\alpha \in J} A_\alpha$ is open.

$\Rightarrow \bigcap\limits_{\alpha \in J} A_\alpha$ is closed.

(3) Let $A_1, A_2, \cdots, A_n$ be closed subsets of X.

By DeMorgan's law, we have

$$X - \bigcup_{i=1}^{n} A_i = \bigcap_{i=1}^{n} (X - A_i).$$

Since each $(X - A_i)$ is open, their finite intersections is also open.

That is, $\bigcap\limits_{i=1}^{n} (X - A_i)$ is open.

$\Rightarrow X - \bigcup\limits_{i=1}^{n} A_i$ is open.

Hence, $\bigcup\limits_{i=1}^{n} A_i$ is closed. $\square$

Definition 1.6.4. *If Y is a subspace of X and $A \subset Y$, then we say that A is* ***closed in*** *Y (that is, A is closed in the subspace topology on Y) if $Y - A$ is open in Y.*

Theorem 1.6.5. *Let Y be a subspace of X. Then a set A is closed in Y if and only if it equals the intersection of a closed set of X with Y.*

Proof. Suppose that A is closed in Y.

$\Rightarrow Y - A$ is open in Y.

$\Rightarrow Y - A = Y \cap U$ for some open set U in X.

Then $X - U$ is closed in X and $Y \cap (X - U) = A$.

i.e) A equals the intersection of a closed set of X with Y.

Conversely, suppose that $A = C \cap Y$, where C is closed in X.

$\Rightarrow X - C$ is open in X.

$\Rightarrow (X - C) \cap Y$ is open in Y.

But $(X - C) \cap Y = Y - A$.

$\therefore Y - A$ is open in Y.

Hence, A is closed in Y. □

Theorem 1.6.6. *Let Y be a subspace of X. If A is closed in Y and Y is closed in X, then A is closed in X.*

Proof. Since A is closed in Y, by previous theorem, we have $A = U \cap Y$, where U is closed in X. Since Y and U are closed in X, their intersection is also closed in X. That is, A is closed in X. □

Definition 1.6.7. *Given a subset A of a topological space X, the **interior of** A is defined as the union of all open sets contained in A, and is denoted by Int A.*

Remark 1.6.8. *The following are true always:*

(i) Int A is an open set.

(ii) Int $A \subset A$.

(iii) If A is open, then Int $A = A$.

(iv) Int A is the largest open set contained in A.

Definition 1.6.9. *Given a subset A of a topological space X, the **closure of** A is defined as the intersection of all closed sets containing A, and is denoted by $\bar{A}$.*

Remark 1.6.10. *The following are true always:*

(i) $\bar{A}$ *is a closed set.*

(ii) If A *is closed, then* $\bar{A} = A$*.*

(iii) $A \subset \bar{A}$*.*

(iv) $\bar{A}$ *is the smallest closed set containing* A*.*

Theorem 1.6.11. *Let* Y *be a subspace of* X *and* A *be a subset of* Y *and let* $\bar{A}$ *denote the closure of* A *in* X*. Then the closure of* A *in* Y *equals* $\bar{A} \cap Y$*.*

Proof. Let B denote the closure of A in Y.

We have to prove that $B = \bar{A} \cap Y$.

Since $\bar{A}$ is closed in X, by Theorem 1.6.5, $\bar{A} \cap Y$ is closed in Y. Also, since $A \subset \bar{A}$ and $A \subset Y$, we have $A \subset \bar{A} \cap Y$. Since B is the intersection of all closed subsets of Y containing A, we must have $B \subset \bar{A} \cap Y$.

Let us prove the other inclusion.

Since B is closed in Y, we have $B = C \cap Y$ for some C closed in X. Then $B \subset C$. Also, we know that $A \subset B$. Therefore, we have $A \subset C$.

i.e) C is a closed subset of X containing A. But, $\bar{A}$ is the intersection of all such closed sets. So, we conclude that $\bar{A} \subset C$.

$\Rightarrow \bar{A} \cap Y \subset C \cap Y = B.$

Hence the theorem. □

Theorem 1.6.12. *Let* A *be a subset of a topological space* X*. Then*

(a) $x \in \bar{A}$ *if and only if every open set* U *containing* x *intersects* A*.*

(b) Suppose the topology of X *is given by a basis. Then* $x \in \bar{A}$ *if and only if every basis element* B *containing* x *intersects* A*.*

Proof. (a) The statement in (a) is equivalent to the following statement.

"$x \notin \bar{A}$ if and only if there exists an open set U containing x that does not intersect A".

Let us prove this statement.

Suppose that $x \notin \bar{A}$. Then $x \in X - \bar{A}$. Since $\bar{A}$ is closed, $X - \bar{A}$ is open. Also, $(X - \bar{A}) \cap A = \emptyset$ because $A \subset \bar{A}$. Thus, we have an open set $X - \bar{A}$ containing x such that $(X - \bar{A}) \cap A = \emptyset$.

Conversely, let U be an open set containing x such that $U \cap A = \emptyset$. Then $X - U$ is a closed set containing A. Therefore, by the definition of $\bar{A}$,
$\bar{A} \subset X - U$. Since $x \notin X - U$, we have $x \notin \bar{A}$ also.

(b) Suppose that $x \in \bar{A}$. Then by (a), every open set containing x intersects A. But, every basis element is an open set. Therefore, every basis element B containing x also intersects A.

Conversely, suppose that every basis element containing x intersects A.
Let U be an open set containing x. Then, by the definition of open set, there exists a basis element B such that $x \in B \subset U$. Therefore, $B \cap A \neq \emptyset$ by our hypothesis. Since $B \cap A \subset U \cap A$, we have $U \cap A \neq \emptyset$. Thus, every open set containing x intersects A. So, by (a), $x \in \bar{A}$. □

Note that from now onwards, the statement "U is an open set containing x" can be rephrased as "U is a neighborhood of x".

Example 1.6.13. *(1) Consider a subset $A = (0, 1]$ of $\mathbb{R}$. Clearly $A \subset \bar{A}$. Since every neighborhood of 0 intersects A, by (a) of Theorem 1.6.12, $0 \in \bar{A}$. But every point outside $[0, 1]$ has a neighborhood disjoint from A. Therefore $\bar{A} = [0, 1]$.*

(2) Consider a subset $B = \{1/n : n \in \mathbb{Z}_+\}$ of $\mathbb{R}$. Clearly, $B \subset \bar{B}$. Since every neighborhood of 0 intersects B, we have $0 \in \bar{B}$. But, every point outside B (except 0) has a neighborhood disjoint from B. Therefore, $\bar{B} = B \cup \{0\}$.

(3) Consider the space $X = \{1, 2, 3\}$ with the topology $\tau = \{\emptyset, X, \{2\}, \{2, 3\}\}$.

(i) Let $A = \{1, 3\} \subset X$.

Clearly, $A \subset \bar{A}$. Consider $2 \in X$. Then neighborhoods of 2 are X, $\{2\}$ and $\{2, 3\}$.

But $\{2\} \cap A = \emptyset$.

$\Rightarrow 2 \notin \bar{A}$.

Therefore $\bar{A} = \{1, 3\}$.

(ii) Let $B = \{2, 3\} \subset X$.

Clearly $B \subset \bar{B}$. Consider $1 \in X$. Then X is the only neighborhood of 1 such that

$X \cap B = X \cap \{2, 3\} \neq \emptyset$.

$\Rightarrow 1 \in \bar{B}$.

Therefore $\bar{B} = \{1, 2, 3\}$.

(4) Consider $\mathbb{Z}_+ = \{1, 2, 3, \cdots\}$ in $\mathbb{R}$. Then, $\bar{\mathbb{Z}}_+ = \mathbb{Z}_+$.

(5) Consider $\mathbb{R}_+ = (0, \infty)$ *in* $\mathbb{R}$. *Then,* $\bar{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{0\}$.

(6) If $\mathbb{Q}$ *is the set of rational numbers in* $\mathbb{R}$, *then* $\bar{\mathbb{Q}} = \mathbb{R}$.

(7) If $C = \{0\} \cup (1, 2)$ *in* $\mathbb{R}$, *then* $\bar{C} = \{0\} \cup [1, 2]$.

(8) Consider the subspace $Y = (0, 1]$ *of* $\mathbb{R}$. *If* $A = (0, \frac{1}{2}) \subset Y$, *then*

$\bar{A}$=*Closure of* A *in* $\mathbb{R} = [0, \frac{1}{2}]$.

$\Rightarrow$ *Closure of* A *in* $Y = \bar{A} \cap Y = [0, \frac{1}{2}] \cap (0, 1] = (0, \frac{1}{2}]$.

Definition 1.6.14. *If* A *is a subset of the topological space* X *and if* $x \in X$, *then we say that* x *is a* ***limit point*** *(or* ***cluster point*** *or a* ***point of accumulation****) of* A *if every neighborhood of* x *intersects* A *in some point other than* x *itself.*

That is, x *is a limit point of* A *if* $x \in \overline{A - \{x\}}$. *The set of all limit points of* A *is denoted by* A'.

Example 1.6.15.

(1) Consider a subset $A = (0, 1] \subset \mathbb{R}$.

Then $A' = [0, 1]$ *because no other point of* $\mathbb{R}$ *is a limit point of* A.

(2) Consider the subset $B = \{1/n : n \in \mathbb{Z}_+\} \subset \mathbb{R}$.

Then $B' = \{0\}$.

(3) Consider the topological space $X = \{1, 2, 3\}$ *with the topology*

$\tau = \{\emptyset, X, \{2\}, \{2, 3\}\}$, *and take* $A = \{1, 3\}$.

Consider $1 \in X$. *Then* X *is the only neighborhood of* 1 *such that*

$$X \cap \big(A - \{1\}\big) = X \cap \{3\} = \{3\} \neq \emptyset.$$

$\Rightarrow 1 \in A'$.

Next, consider $2 \in X$. *Then* $X, \{2\}, \{2, 3\}$ *are the neighborhoods of* 2 *such that*

$$X \cap \big(A - \{2\}\big) = X \cap \{1, 3\} = \{1, 3\} \neq \emptyset,$$

$$\{2, 3\} \cap \big(A - \{2\}\big) = \{2, 3\} \cap \{1, 3\} = \{3\} \neq \emptyset,$$

but

$$\{2\} \cap \big(A - \{2\}\big) = \{2\} \cap \{1, 3\} = \emptyset.$$

$\Rightarrow 2 \notin A'$.

Next, consider $3 \in X$. *Then* X *and* $\{2,3\}$ *are the neighborhoods of 3 such that*

$$X \cap \big(A - \{3\}\big) = X \cap \{1\} = \{1\} \neq \emptyset,$$

but

$$\{2,3\} \cap \big(A - \{3\}\big) = \{2,3\} \cap \{1\} = \emptyset.$$

$\Rightarrow 3 \notin A'$.

Therefore, 1 *is the only limit point of* A*, and hence* $A' = \{1\}$.

(4) Consider $A = [1,2) \subset \mathbb{R}_l$. *Then* $A' = [1,2)$.

Here, 2 *is not a limit point of* A *because the open set* $[2,3)$ *does not intersect* $A - \{2\}$.

Theorem 1.6.16. *Let* A *be a subset of the topological space* X*, and let* A' *be the set of all limit points of* A*. Then* $\bar{A} = A \cup A'$.

Proof. Let $x \in \bar{A}$.

If $x \in A$, then trivially $x \in A \cup A'$. So, suppose that $x \notin A$.

Since $x \in \bar{A}$, every neighborhood U of x must intersect A in a point different from x.

$\Rightarrow x$ is a limit point of A.

i.e) $x \in A'$.

$\Rightarrow x \in A \cup A'$

$$\therefore \bar{A} \subset A \cup A' \tag{1.5}$$

Let us prove the other inclusion.

Let $x \in A'$.

$\Rightarrow$ Every neighborhood of x intersects A (in some point different from x).

$\Rightarrow x \in \bar{A}$.

$\therefore A' \subset \bar{A}$.

But, always $A \subset \bar{A}$.

$$\therefore A \cup A' \subset \bar{A}. \tag{1.6}$$

From (1.5) and (1.6), we have $\bar{A} = A \cup A'$. □

Corollary 1.6.17. *A subset of a topological space is closed if and only if it contains all its limit points.*

Proof. Let A be a subset of X. Then,

$$
\begin{aligned}
A \text{ is closed} &\iff A = \bar{A} \\
&\iff A = A \cup A' \text{ (by previous theorem)} \\
&\iff A \supset A'.
\end{aligned}
$$

□

Definition 1.6.18. *Let X be any arbitrary topological space. Then, we say that the sequence $x_1, x_2, \cdots$ of points of the space X* ***converges*** *to the point x of X provided that, corresponding to each neighborhood U of x, there is a positive integer N such that $x_n \in U$ $\forall \ \ n \geq N$.*

Example 1.6.19. *Let $X = \{a, b, c\}$ and consider the topology $\tau = \{\emptyset, X, \{b\}, \{a, b\}, \{b, c\}\}$ on X. Consider the sequence $\{x_n\}$ defined by $x_n = b \ \ \forall n$. By the definition, this sequence converges not only to the point b, but also to the points a and c.*

Definition 1.6.20. *A topological space X is called a* ***Hausdorff space****, if for each pair x_1, x_2 of distinct points of X, there exist disjoint neighborhoods U_1 and U_2 of x_1 and x_2, respectively.*

Example 1.6.21.

(i) $\mathbb{R}$ is a Hausdorff space.

(ii) Consider $X = \{a, b\}$ with discrete topology. Then X is a Hausdorff space.

Theorem 1.6.22. *Every finite point set in a Hausdorff space X is closed.*

Proof. Given that X is a Hausdorff space.
First, let us prove that every singleton set $\{x_0\} \subset X$ is closed.
Claim: $\overline{\{x_0\}} = \{x_0\}$.
Let $x \in X$ such that $x \neq x_0$.
Since X is Hausdorff, $\exists$ disjoint neighborhoods U and V of x and x_0, respectively.
Thus, we have a neighborhood U of x so that $U \cap \{x_0\} = \emptyset$, because $U \cap V = \emptyset$.
$\Rightarrow x \notin \overline{\{x_0\}}$.

$\therefore \overline{\{x_0\}} = \{x_0\}$.

$\Rightarrow \{x_0\}$ is closed.

Now, consider a finite subset $A = \{x_1, x_2, \cdots, x_n\}$ of X. Then

$A = \{x_1, x_2, \cdots, x_n\} = \{x_1\} \cup \{x_2\} \cup \cdots \cup \{x_n\}$.

Since finite union of closed sets is closed, A is closed. $\square$

Definition 1.6.23. *A topological space X is said to satisfy the T_1-**axiom** if every finite point set of X is closed.*

Theorem 1.6.24. *Let X be a space satisfying the T_1-axiom and let A be a subset of X. Then the point x is a limit point of A if and only if every neighborhood of x contains infinitely many points of A.*

Proof. Suppose that every neighborhood of x contains infinitely many points of A.

$\Rightarrow$ Every neighborhood of x intersects A in some point other than x.

$\Rightarrow x$ is a limit point of A.

Conversely, suppose that x is a limit point of A.

We have to prove that every neighborhood of x contains infinitely many points of A.

Suppose not, then $\exists$ a neighborhood U of x such that U intersects A in only finitely many points. Since x is a limit point of A, U also intersects $A - \{x\}$ in only finitely many points.

So, let $U \cap (A - \{x\}) = \{x_1, x_2, \cdots, x_m\}$.

Since X satisfies the T_1-axiom, $\{x_1, x_2, \cdots, x_m\}$ is closed.

$\Rightarrow X - \{x_1, x_2, \cdots, x_m\}$ is open.

Also, $x \in X - \{x_1, x_2, \cdots, x_m\}$.

i.e) $X - \{x_1, x_2, \cdots, x_m\}$ is a neighborhood of x.

Hence, $U \cap \big(X - \{x_1, x_2, \cdots, x_m\}\big)$ is also a neighborhood of x.

But, $U \cap \big(X - \{x_1, x_2, \cdots, x_m\}\big) \cap (A - \{x\})$

$= U \cap (A - \{x\}) \cap \big(X - \{x_1, x_2, \cdots, x_m\}\big)$

$= \{x_1, x_2, \cdots, x_m\} \cap \big(X - \{x_1, x_2, \cdots, x_m\}\big)$

$= \emptyset$.

This contradicts our hypothesis that x is a limit point of A.

Hence, every neighborhood of x contains infinitely many points of A. $\square$

Theorem 1.6.25. *If X is a Hausdorff space, then a sequence of points of X converges to at most one point of X.*

Proof. Given that X is a Hausdorff space.
Let $\{x_n\}$ be a sequence of points of X that converges to a point x of X.
Let $y \in X$ such that $y \neq x$.
Since X is a Hausdorff space, $\exists$ disjoint neighborhoods U and V of x and y, respectively.
Since $\{x_n\}$ converges to x, U contains x_n for all but finitely many values of n.
But, since $U \cap V = \emptyset$, V can contain only finitely many x_n's.
$\Rightarrow \{x_n\}$ cannot converge to y.
Hence the theorem. □

Definition 1.6.26. *If the sequence $\{x_n\}$ of points of the Hausdorff space X converges to the point x of X, then we can write $x_n \to x$, and we say that x is the **limit** of the sequence $\{x_n\}$.*

Let Us Sum Up:

In this section, we have discussed the following concepts:

(1) Closed sets with examples

(2) Closure of a set with examples

(3) Closure in a subspace

(4) Limit point of a set with examples

(5) Relation between closure and set of limit points

(6) Convergence of a sequence

(7) Hausdorff spaces

(8) T_1- axiom

Check your Progress:

1. If A is a subset of the topological space X, then which of the following is not correct?

   (A) $\bar{A}$ is closed (B) $\bar{A} = A$ (C) $\bar{A} \supset A$ (D) None of these

2. The closure of $\{2\}$ in the discrete topology on $X = \{1\ ,2\ ,3\}$ is

(A) $\{1, 2\}$ (B) $\{2, 3\}$ (C) X (D) $\{2\}$

3. $(-1, 1)$ in $\mathbb{R}$ is

(A) a basis element (B) an open set

(C) not a closed set (D) All of these

Unit Summary:

In this unit, the definition of topological space is introduced with examples. Various concepts like basis for a topology, subbasis for a topology, order topology, product topology, subspace topology, Hausdorff spaces, closed sets and limit points were explained in detail.

Glossary:

- Open set - Member of the topology
- $\mathbb{R}$ - The real line with standard topology
- $\mathbb{R}_l$ - The real line with lower limit topology
- $\mathbb{R}_K$ - The real line with K-topology
- Neighborhood of x - An open set containing x
- $X - A$ - Complement of A in X

Self-Assessment Questions:

(1) Show that the topologies of $\mathbb{R}_l$ and $\mathbb{R}_k$ are not comparable.

(2) Show that if Y is a subspace of X, and A is a subset of Y, then the topology A inherits as a subspace of Y is the same as the topology it inherits as a subspace of X.

(3) Consider the set $Y = [-1, 1]$ as a subspace of $\mathbb{R}$. Which of the following sets are open in Y? Which are open in $\mathbb{R}$?

$A = \{x|1/2 < |x| < 1\}$,

$B = \{x|1/2 < |x| \leq 1\}$,

$C = \{x | 1/2 \leq |x| < 1\}$,

$D = \{x | 1/2 \leq |x| \leq 1\}$,

$E = \{x | 0 < |x| < 1$ and $1/x \notin \mathbb{Z}_+\}$.

(4) Show that if A is closed in X and B is closed in Y, then $A \times B$ is closed in $X \times Y$.

(5) Show that if U is open in X and A is closed in X, then $U - A$ is open in X, and $A - U$ is closed in X.

Exercises:

(1) If $\{\tau_\alpha\}$ is a family of topologies on X, show that $\cap\tau_\alpha$ is a topology on X. Is $\cup\tau_\alpha$ a topology on X?

(2) Let $\{\tau_\alpha\}$ be a family of topologies on X. Show that there is a unique smallest topology on X containing all the collections τ_α, and a unique largest topology contained in all τ_α.

(3) Prove that every simply ordered set is a Hausdorff space in the order topology.

(4) Prove that product of two Hausdorff spaces is a Hausdorff space.

(5) Prove that a subspace of a Hausdorff space is a Hausdorff space.

Answers for check your progress:

Section 1.1	1. (D)	2. (B)	3. (C)
Section 1.2	1. (B)	2. (C)	3. (D)
Section 1.3	1. (A)	2. (D)	3. (C)
Section 1.4	1. (D)	2. (B)	3. (D)
Section 1.5	1. (B)	2. (B)	3. (C)
Section 1.6	1. (B)	2. (D)	3. (D)

Reference:

1. James R. Munkres, Topology (2nd Edition), Prentice Hall of India, New Delhi, 2011.

Suggested Readings:

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2. George F. Simmons, Introduction to Topology and Modern Analysis, McGraw Hill Book Co., 1963.

3. J.L. Kelley, General Topology, Van Nostrand, Reinhold Co., New York, 1955.

4. L. Steen and J. Subhash, Counter Examples in Topology, Holt, Rinehart and Winston, New York, 1970.

5. S. Willard, General Topology, Addison - Wesley, Mass., 1970.

UNIT 2

Unit 2

Continuous Functions

Objectives:

This unit deals with continuous functions defined on a topological space, properties of continuous functions and the concepts of product topology and metric topology.

2.1 Continuous Functions

Definition 2.1.1. *Let X and Y be topological spaces. A function $f : X \to Y$ is said to be* ***continuous*** *if for each open subset V of Y, the set $f^{-1}(V)$ is an open subset of X.*

Here, $f^{-1}(V) = \{x \in X : f(x) \in V\}$ and $f^{-1}(V) = \emptyset$ if V does not intersect the image set $f(X)$.

Continuity of a function not only depends on the the function itself, but also on the topologies given on X and Y.

Remark 2.1.2. *If the topology of the range space Y is given by a basis $\mathscr{B}$, then to prove the continuity of f, it suffices to show that the inverse image of every basis element is open.*

For, let V be open in Y. Then V can be written as a union of basis elements.
i.e) $V = \bigcup\limits_{\alpha \in J} B_\alpha$, where B_α are in $\mathscr{B}$.
Then, $f^{-1}(V) = \bigcup\limits_{\alpha \in J} f^{-1}(B_\alpha)$.
i.e) $f^{-1}(V)$ is open in X if each $f^{-1}(B_\alpha)$ is open in X.

Remark 2.1.3. *If the topology on Y is given by a subbasis $\mathcal{S}$, then to prove the continuity of f it suffices to show that the inverse image of each subbasis element in open.*

For, let B be any basis element for the topology generated by the subbasis $\mathcal{S}$. Then

$$B = S_1 \cap S_2 \cap \cdots \cap S_n, \text{ where } S_1, S_2, \cdots, S_n \in \mathcal{S}.$$

$$\Rightarrow f^{-1}(B) = f^{-1}(S_1) \cap f^{-1}(S_2) \cap \cdots \cap f^{-1}(S_n).$$

$\therefore f^{-1}(B)$ *is open if inverse image of each subbasis element is open.*

Example 2.1.4. *Consider the function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^3$.*
We know that the collection of all open intervals in $\mathbb{R}$ is a basis for the standard topology on $\mathbb{R}$. So, to prove the continuity of f, it is sufficient to prove that inverse image of every open interval is open under f. Consider any open interval (a, b) in the range space $\mathbb{R}$. Then

$$\begin{aligned} f^{-1}\Big((a,b)\Big) &= \{x \in \mathbb{R} : f(x) \in (a,b)\} \\ &= \{x \in \mathbb{R} : x^3 \in (a,b)\} \\ &= \{x \in \mathbb{R} : a < x^3 < b\} \\ &= \{x \in \mathbb{R} : a^{1/3} < x < b^{1/3}\} \\ &= (a^{1/3}, b^{1/3}), \end{aligned}$$

which is open in the domain space $\mathbb{R}$. Therefore, f is continuous.

Example 2.1.5. *Consider the identity function $f : \mathbb{R} \to \mathbb{R}_l$ defined by*
$f(x) = x, \ \forall x \in \mathbb{R}$.
For any basis element $[a, b)$ in $\mathbb{R}_l$,

$$\begin{aligned} f^{-1}\Big([a,b)\Big) &= \{x \in \mathbb{R} : f(x) \in [a,b)\} \\ &= \{x \in \mathbb{R} : x \in [a,b)\} \\ &= \{x \in \mathbb{R} : a \leq x < b\} \\ &= [a,b), \end{aligned}$$

which is not open in $\mathbb{R}$. Therefore, f is not continuous.

Example 2.1.6. *Consider the identity function $g : \mathbb{R}_l \to \mathbb{R}$ defined by*
$g(x) = x, \ \forall x \in \mathbb{R}_l$.

For any basis element (a,b) *in* $\mathbb{R}$*,*

$$\begin{aligned} g^{-1}\Big((a,b)\Big) &= \{x \in \mathbb{R}_l : g(x) \in (a,b)\} \\ &= \{x \in \mathbb{R}_l : x \in (a,b)\} \\ &= \{x \in \mathbb{R}_l : a \leq x < b\} \\ &= (a,b), \end{aligned}$$

which is open in $\mathbb{R}_l$*. Therefore,* g *is continuous.*

Example 2.1.7. *Let* $Y = \{a,b\}$ *with discrete topology and consider the function* $f : \mathbb{R} \to Y$ *by*

$$f(x) = \begin{cases} a & \textit{if } x \leq 0 \\ b & \textit{if } x > 0. \end{cases}$$

We know that $\big\{\{a\},\{b\}\big\}$ *is a basis for the discrete topology on* Y*.*
Now,

$$\begin{aligned} f^{-1}\Big(\{b\}\Big) &= \{x \in \mathbb{R} : f(x) \in \{b\}\} \\ &= \{x \in \mathbb{R} : f(x) = b\} \\ &= \{x \in \mathbb{R} : x > 0\} \\ &= (0,\infty), \end{aligned}$$

which is an open set in $\mathbb{R}$*. But*

$$\begin{aligned} f^{-1}\Big(\{a\}\Big) &= \{x \in \mathbb{R} : f(x) \in \{a\}\} \\ &= \{x \in \mathbb{R} : f(x) = a\} \\ &= \{x \in \mathbb{R} : x \leq 0\} \\ &= (-\infty,0], \end{aligned}$$

which is not open in $\mathbb{R}$*. Therefore,* f *is not continuous.*

Theorem 2.1.8. *Let* X *and* Y *be topological spaces and let* $f : X \to Y$*. Then the following are equivalent:*
(1) f *is continuous.*
(2) For every subset A *of* X*,* $f(\bar{A}) \subset \overline{f(A)}$*.*

(3) For every closed set B of Y, the set $f^{-1}(B)$ is closed in X.

(4) For each $x \in X$ and each neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subset V$.

*If the condition in (4) holds for the point $x \in X$, we say that f is **continuous at the point** x.*

Proof. We show that $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1)$ and that $(1) \Rightarrow (4) \Rightarrow (1)$.

$(1) \Rightarrow (2)$:

Suppose that f is continuous, and let $A \subset C$.

To prove: $f(\bar{A}) \subset \overline{f(A)}$.

Let $x \in \bar{A}$ so that $f(x) \in f(\bar{A})$.

To prove $f(x) \in \overline{f(A)}$, let V be a neighborhood of $f(x)$. Then $x \in f^{-1}(V)$.

Since V is open in Y and f is continuous, we have that $f^{-1}(V)$ is open in X.

i.e) $f^{-1}(V)$ is a neighborhood of x in X. Since $x \in \bar{A}$, $f^{-1}(V)$ must intersect A. So, let $y \in A \cap f^{-1}(V)$.

$\Rightarrow y \in A$ and $y \in f^{-1}(V)$.

$\Rightarrow f(y) \in f(A)$ and $f(y) \in V$.

$\Rightarrow f(y) \in f(A) \cap V$.

i.e) V intersects $f(A)$.

$\therefore$ Every neighborhood of $f(x)$ intersects $f(A)$.

$\Rightarrow f(x) \in \overline{f(A)}$.

Thus, $f(\bar{A}) \subset \overline{f(A)}$.

$(2) \Rightarrow (3)$:

Suppose that (2) holds.

Let B be closed in Y and let $A = f^{-1}(B)$.

To prove: $A = \bar{A}$.

Now, $A = f^{-1}(B) \quad \Rightarrow f(A) = f(f^{-1}(B)) \subset B$.

If $x \in \bar{A}$, then $f(x) \in f(\bar{A}) \subset \overline{f(A)} \subset \bar{B} = B$. [since B is closed]

Thus, $x \in \bar{A} \Rightarrow x \in f^{-1}(B) = A$.

i.e) $\bar{A} \subset A$.

Already, we know that $A \subset \bar{A}$.

Therefore, $A = \bar{A}$.

$(3) \Rightarrow (1)$:

Suppose that (3) holds.

Let V be open in Y. Then $Y - V$ is closed in Y.

$\therefore$ By (3), $f^{-1}(Y - V)$ is closed in X.

But, $f^{-1}(Y - V) = f^{-1}(Y) - f^{-1}(V) = X - f^{-1}(V)$

$\therefore X - f^{-1}(V)$ is closed in X.

$\Rightarrow f^{-1}(V)$ is open in X.

$\therefore$ By the definition, f is continuous.

$(1) \Rightarrow (4)$:

Suppose that f is continuous.

Let $x \in X$ and V be a neighborhood of $f(x)$.

$\therefore$ By (1), $f^{-1}(V)$ is open in X.

Also, $f(x) \in V \Rightarrow x \in f^{-1}(V)$.

$\therefore f^{-1}(V)$ is a neighborhood of x.

Take $U = f^{-1}(V)$.

Then $f(U) = f(f^{-1}(V)) \subset V$.

$\therefore (4)$ holds.

$(4) \Rightarrow (1)$:

Suppose that (4) holds.

Take an open set V in Y.

To prove: $f^{-1}(V)$ is open in X.

Let $x \in f^{-1}(V)$. Then $f(x) \in V$.

So, we have $x \in X$ and a neighborhood V of $f(x)$.

$\therefore$ By (4), there is a neighborhood U_x of x such that $f(U_x) \subset V$.

Then $U_x \subset f^{-1}(V)$.

Thus, $\bigcup_{x \in f^{-1}(V)} U_x = f^{-1}(V)$.

Hence, $f^{-1}(V)$ is open in X. $\square$

Homeomorphism

Definition 2.1.9. *Let X and Y be topological spaces. Let $f : X \to Y$ be a bijection. If both the function f and the inverse function $f^{-1} : Y \to X$ are continuous, then f is called* *a **homeomorphism**.*

Remark 2.1.10. *The condition that f^{-1} be continuous says that for each open set U of X, the inverse image of U under the map $f^{-1} : Y \to X$ is open in Y. But the inverse image of U under the map f^{-1} is $(f^{-1})^{-1}(U) = f(U)$. Therefore, to prove f^{-1} is continuous, it is enough to prove that for every open set U of X, $f(U)$ is open in Y.*

Thus, another way to define a homeomorphism is to say that it is a bijective correspondence $f : X \to Y$ so that U is open in X if and only if $f(U)$ is open in Y.

Note that the above remark shows that a homeomorphism $f : X \to Y$ gives a bijective correspondence not only between X and Y, but between the collection of open sets of X and of Y.

Definition 2.1.11. *Let X be any topological space. A **topological property** of X is a property of the space X which is invariant under homeomorphism.*

*i.e) A property of X is a **toplological property** if whenever the space X possesses that property, then every space homeomorphic to X also possesses that property.*

*In other words, a **toplological property** of X is a property of the space X that can be entirely expressed in terms of the open sets of X.*

Definition 2.1.12. *Suppose that $f : X \to Y$ is an injective continuous map, where X and Y are topological spaces. Let Z be the image set $f(X)$, considered as a subspace of Y. Then the function $f' : X \to Z$, obtained by restricting the range of f, is bijective. If f' happens to be a homeomorphism of X with Z, we say that the map $f : X \to Y$ is a **topological imbedding** or an **imbedding** of X in Y.*

Example 2.1.13. *Consider a function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 3x + 1$.*
Let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(y) = \frac{y-1}{3}$. Then

$$\begin{aligned} f(g(y)) &= f\Big(\frac{y-1}{3}\Big) \\ &= 3\Big(\frac{y-1}{2}\Big) + 1 \\ &= y - 1 + 1 \\ &= y, \forall y \in \mathbb{R}. \end{aligned}$$

Similarly,

$$\begin{aligned}
g(f(x)) &= g\Big(3x+1\Big) \\
&= \frac{(3x+1)-1}{3} \\
&= \frac{3x}{3} \\
&= x, \forall x \in \mathbb{R}.
\end{aligned}$$

$\therefore f(g(x)) = g(f(x)) = x, \forall x \in \mathbb{R}.$

$\Rightarrow f^{-1}$ *exists and* $f^{-1} = g$.

Hence, f *is bijective.*

Now, consider $f : \mathbb{R} \to \mathbb{R}$ *and* $f^{-1} : \mathbb{R} \to \mathbb{R}$.

To prove f *and* f^{-1} *are continuous it is enough to show that inverse image of every open interval is open in the domain.*

First, let us take $f : \mathbb{R} \to \mathbb{R}$ *by* $f(x) = 3x + 1$.

Consider any open interval (a, b) *in the range space* $\mathbb{R}$ *of* f. *Then*

$$\begin{aligned}
f^{-1}((a,b)) &= \{x \in \mathbb{R} : f(x) \in (a,b)\} \\
&= \{x \in \mathbb{R} : 3x+1 \in (a,b)\} \\
&= \{x \in \mathbb{R} : a < 3x+1 < b\} \\
&= \Big\{x \in \mathbb{R} : \frac{a-1}{3} < x < \frac{b-1}{3}\Big\} \\
&= \Big(\frac{a-1}{3}, \frac{b-1}{3}\Big),
\end{aligned}$$

which is open in the domain $\mathbb{R}$.

$\therefore f$ *is continuous.*

Consider any open interval (c, d) *in the range space* $\mathbb{R}$ *of* f^{-1}. *Then*

$$\begin{aligned}
(f^{-1})^{-1}(c,d) &= \{y \in \mathbb{R} : f^{-1}(y) \in (c,d)\} \\
&= \{y \in \mathbb{R} : \frac{y-1}{3} \in (a,b)\} \\
&= \{y \in \mathbb{R} : 3c+1 < y < 3d+1\} \\
&= \Big(3c+1, 3d+1\Big),
\end{aligned}$$

which is open in the domain $\mathbb{R}$.

$\therefore f^{-1}$ *is continuous.*

Hence, f *is a homeomorphism.*

Definition 2.1.14. *If there is a homeomorphism between the spaces X and Y, then we say that X and Y are **homeomorphic**.*

Example 2.1.15. *Consider the identity function $g : \mathbb{R}_l \to \mathbb{R}$.*
Clearly, g is bijective and continuous. But $g^{-1} : \mathbb{R} \to \mathbb{R}_l$ is not continuous.
$\therefore$ g is not a homeomorphism.

Theorem 2.1.16. *(Rules for constructing continuous functions)*
Let X , Y and Z be topological spaces.

(a) (Constant function) If $f : X \to Y$ maps all of X into a single point y_0 of Y, then f is continuous.

(b) (Inclusion) If A is a subspace of X, the inclusion function $j : A \to X$ is continuous.

(c) (Composites) If $f : X \to Y$ and $g : Y \to Z$ are continuous, then the map $g \circ f : X \to Z$ is continuous.

(d) (Restricting the domain) If $f : X \to Y$ is continuous, and if A is a subspace of X, then the restricted function $f \mid A : A \to Y$ is continuous.

(e) (Restricting or expanding the range) Let $f : X \to Y$ be continuous. If Z is a subspace of Y containing the image set $f(X)$, then the function $g : X \to Z$ obtained by restricting the range of f is continuous. If Z is a space having Y as a subspace, then the function $h : X \to Z$ obtained by expanding the range of f is continuous.

(f) (Local formulation of continuity) The map $f : X \to Y$ is continuous if X can be written as the union of open sets U_α such that $f \mid U_\alpha$ is continuous for each α.

Proof. (a) Given that $f(x) = y_0$ for every $x \in X$.
To prove: f is continuous.
Let V be open in Y. Then

$$f^{-1}(V) = \begin{cases} X \text{ if } y_0 \in V, \\ \emptyset \text{ if } y_0 \notin V. \end{cases}$$

Thus, in either case, $f^{-1}(V)$ is open in X.
(b) Given that $A \subset X$ and $j : A \to X$ is an inclusion map.
Then $j(x) = x, \forall x \in A$.

Let V be open in X. Then

$$\begin{aligned} j^{-1}(V) &= \{x \in A : j(x) \in V\} \\ &= \{x \in A : x \in V\} \\ &= A \cap V \end{aligned}$$

$\Rightarrow j^{-1}(V)$ is open in A by the definition of the subspace topology.

(c) Given that $f : X \to Y$ and $g : Y \to Z$ are continuous functions.

To prove: $g \circ f : X \to Z$ is continuous .

Let U be open in Z. Since $g : Y \to Z$ is continuous and U is open in Z, we have $g^{-1}(U)$ is open in Y.

Again, since $f : X \to Y$ is continuous, $f^{-1}(g^{-1}(U))$ is open in X. But $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$.

$\Rightarrow (g \circ f)^{-1}(U)$ is open in X.

$\therefore (g \circ f)$ is continuous.

(d) Given that $f : X \to Y$ is continuous and $A \subset X$.

Consider, the inclusion map $j : A \to X$.

Then $f \circ j$ is a map from A to Y.

i.e) $f \circ j = f|A$.

Since f and j are continuous, by (c), $f|A$ is also continuous.

(e) (i) Given that $f : X \to Y$ is continuous and $f(X) \subset Z \subset Y$.

To prove: $g : X \to Z$ is continuous.

Let B be open in the subspace Z. Then B can be written as $B = Z \cap U$ for some U open in Y.

$$\begin{aligned} \therefore g^{-1}(B) &= g^{-1}(Z \cap U) \\ &= f^{-1}(Z \cap U)\ [because f(X) \subset Z \subset Y)] \\ &= f^{-1}(Z) \cap f^{-1}(U) \\ &= X \cap f^{-1}(U) \\ &= f^{-1}(U). \end{aligned}$$

Since f is continuous, $f^{-1}(U)$ is open in X, and hence $g^{-1}(B)$ is open in X.

$\Rightarrow g$ is continuous.

(ii) Given that $f : X \to Y$ is continuous and Y is the subspace of Z.

To prove: $h : X \to Z$ is continuous .

Consider the inclusion map $j : Y \to Z$.

Then $(j \circ f)$ is a map from X to Z.

i.e) $j \circ f = h$.

$\therefore$ h is continuous, because j and f are continuous.

(f) Given that $X = \cup U_\alpha$, where each U_α is open in X such that $f|U_\alpha : U_\alpha \to Y$ is continuous for each α.

To prove: $f : X \to Y$ is continuous.

Let V be open in Y.

Then $(f|U_\alpha)^{-1}(V)$ is open in U_α, because $f|U_\alpha$ is continuous for each α. But

$$\begin{aligned}(f|U_\alpha)^{-1}(V) &= \{x \in U_\alpha : (f|U_\alpha)(x) \in V\}\\ &= \{x \in U_\alpha : f(x) \in V\}\\ &= U_\alpha \cap f^{-1}(V).\end{aligned}$$

$\Rightarrow U_\alpha \cap f^{-1}(V)$ is open in U_α for each α.

Since U_α is open in X, by Lemma 1.5.3, $U_\alpha \cap f^{-1}(V)$ is open in X, for each α.

$\Rightarrow \underset{\alpha}{\cup}(U_\alpha \cap f^{-1}(V))$ is open in X.

$\Rightarrow (\underset{\alpha}{\cup}U_\alpha) \cap f^{-1}(V)$ is open in X.

$\Rightarrow X \cap f^{-1}(V)$ is open in X.

$\Rightarrow f^{-1}(V)$ is open in X.

Therefore, f is continuous. □

Theorem 2.1.17. *(The pasting lemma).*

Let $X = A \cup B$, where A and B are closed in X. Let $f : A \to Y$ and $g : B \to Y$ be continuous. If $f(x) = g(x)$ for every $x \in A \cap B$, then f and g combine to give a continuous function $h : X \to Y$, defined by

$$h(x) = \begin{cases} f(x) \text{ if } x \in A \\ g(x) \text{ if } x \in B. \end{cases}$$

Proof. Given that $f : A \to Y$ and $g : B \to Y$ are continuous functions and $X = A \cup B$.

To prove: $h : X \to Y$ is continuous.

Let C be closed in Y. Then

$$\begin{aligned} h^{-1}(C) &= \{x \in X : h(x) \in C\} \\ &= \{x \in A \cup B : h(x) \in C\} \\ &= \{x \in A : f(x) \in C\} \cup \{x \in B : g(x) \in C\} \\ &= f^{-1}(C) \cup g^{-1}(C). \end{aligned}$$

Since $f : A \to Y$ is continuous and C is closed in Y, we have $f^{-1}(C)$ is closed in A, and hence $f^{-1}(C)$ is closed X. Similarly, $g^{-1}(C)$ is closed B, and hence in X.
Thus, their union $h^{-1}(C)$ is closed X.
$\Rightarrow h$ is continuous. □

Remark 2.1.18. *The above theorem also holds if A and B are open sets in X. Thus, pasting lemma is just a special case of the local formulation of continuity.*

Example 2.1.19. *Consider the function $h : \mathbb{R} \to \mathbb{R}$ defined by*

$$h(x) = \begin{cases} x \text{ if } x \leq 0, \\ x/2 \text{ if } x \geq 0. \end{cases}$$

Take $A = (-\infty, 0]$ and $B = [0, \infty)$. Clearly, A and B are closed sets in $\mathbb{R}$ such that $A \cup B = \mathbb{R}$. Also, $f : A \to \mathbb{R}$ defined by $f(x) = x$ and
$g : B \to \mathbb{R}$ defined by $g(x) = x/2$ are continuous, and $f(0) = g(0)$.
Thus, by pasting lemma, h is continuous.

Theorem 2.1.20. *(Maps into products).*
Let $f : A \to X \times Y$ be given by the equation

$$f(a) = (f_1(a), f_2(a)).$$

Then f is continuous if and only if the functions

$$f_1 : A \to X \quad \text{and} \quad f_2 : A \to Y$$

are continuous. The maps f_1 and f_2 are called the ***coordinate functions*** *of f.*

Proof. Consider the projections $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$.
Let U be open in X and V be open in Y. Then $\pi_1^{-1}(U) = U \times Y$ is open in $X \times Y$

and $\pi_2^{-1}(V) = X \times V$ is open in $X \times Y$. Therefore, π_1 and π_2 are continuous. Also, $\pi_1 \circ f = f_1$ and $\pi_2 \circ f = f_2$.

Now, suppose that f is continuous. Then both f_1, f_2 are continuous, because they are the composites of continuous functions.

Conversely, suppose that f_1 and f_2 are continuous.

To prove: f is continuous.

We prove this by proving that inverse image of every basis element is open.

Let $U \times V$ be any basis element for the topology on $X \times Y$. Then

$$\begin{aligned} f^{-1}(U \times V) &= \{a \in A : f(a) \in U \times V\} \\ &= \{a \in A : (f_1(a), f_2(a)) \in U \times V\} \\ &= \{a \in A : f_1(a) \in U\} \cap \{a \in A : f_2(a) \in V\} \\ &= f_1^{-1}(U) \cap f_2^{-1}(V) \end{aligned}$$

Since f_1 and f_2 are continuous, $f_1^{-1}(U)$ and $f_2^{-1}(V)$ are open in A. Hence, their intersection $f^{-1}(U \times V)$ is also open in A. Therefore, f is continuous. □

Example 2.1.21. *Consider the function $f : [a, b] \to \mathbb{R}^2$ which is often expressed in the form $f(t) = (x(t), y(t))$, where $x : [a, b] \to \mathbb{R}$ and $y : [a, b] \to \mathbb{R}$. Then f is continuous if both x and y are continuous. This function f is called a* ***parametrized curve*** *in the plane.*

Example 2.1.22. *Consider a vector field in the plane.*
i.e) a function $V : \mathbb{R}^2 \to \mathbb{R}^2$ defined by
$V(x, y) = P(x, y)i + Q(x, y)j = (P(x, y), Q(x, y))$. Then V is continuous if both P and Q are continuous.

Let Us Sum Up:

In this section, we have discussed the following concepts:

(1) Continuous functions with examples

(2) Equivalent conditions for continuity

(3) Homeomorphism with examples

(4) Rules for constructing continuous functions

(5) The pasting lemma

(6) Continuity of maps into products

Check your Progress:

1. The identity function $f : \mathbb{R}_l \to \mathbb{R}$ is

   (A) bijective (B) continuous

   (C) not a homeomorphism (D) All of these

2. Which of the following is homeomorphic with $(-1, 1)$?

   (A) $\mathbb{R}$ (B) $(0, \infty)$ (C) $(0, 1)$ (D) All of these

3. The function $f : \mathbb{R} \to \mathbb{R}^\omega$ defined by $f(t) = (t, t, t, ...)$ is continuous when $\mathbb{R}^\omega$ is given the

   (A) box topology (B) product topology (C) order topology (D) None of these

2.2 The Product Topology

Definition 2.2.1. *Let J be an index set. Given a set X, we define a J-**tuple** of elements of X to be a function $x : J \to X$. If $\alpha \in J$, then we denote the value of x at α by x_α instead of $x(\alpha)$, and we call it the α^{th} **coordinate of** x. Also, we denote the function x itself by the symbol $(x_\alpha)_{\alpha \in J}$, and the set of all J-tuples of elements of X by X^J.*

Definition 2.2.2. *Let $\{A_\alpha\}$ be an indexed family of sets and let $X = \bigcup\limits_{\alpha \in J} A_\alpha$. Then, the **cartesian product** of this indexed family is denoted by $\prod\limits_{\alpha \in J} A_\alpha$, and is defined to be the set of all J-tubles $(x_\alpha)_{\alpha \in J}$ of elements of X such that $x_\alpha \in A_\alpha$ for each $\alpha \in J$.*

Note that if all the sets A_α are equal to one set X, then the cartesian product $\prod\limits_{\alpha \in J} A_\alpha$ is the set X^J of all J-tuples of elements of X.

Definition 2.2.3. *Let $\{X_\alpha\}_{\alpha \in J}$ be an indexed family of topological spaces. Consider the collection $\mathscr{C} = \left\{ \prod\limits_{\alpha \in J} U_\alpha : U_\alpha \text{ is open in } X_\alpha \text{ for each } \alpha \right\}$ of subsets of $\prod\limits_{\alpha \in J} X_\alpha$. Then, $\mathscr{C}$ is a basis for the topology on $\prod\limits_{\alpha \in J} X_\alpha$, and the topology generated by this basis is called the **box topology**.*

Definition 2.2.4. *The function* $\pi_\beta : \prod\limits_{\alpha\in J} X_\alpha \to X_\beta$ *defined by*

$$\pi_\beta\Big((x_\alpha)_{\alpha\in J}\Big) = x_\beta$$

is called the ***projection mapping associated with the index*** β.

Definition 2.2.5. *Let* $\mathcal{S}_\beta = \{\pi_\beta^{-1}(U_\beta) : U_\beta \text{ is open in} X_\beta\}$*, and let* $\mathcal{S} = \bigcup\limits_{\beta\in J} \mathcal{S}_\beta$*. Then the topology generated by the subbasis* $\mathcal{S}$ *is called the* ***product topology*** *on* $\prod\limits_{\alpha\in J} X_\alpha$*. In this topology,* $\prod\limits_{\alpha\in J} X_\alpha$ *is called a* ***product space***.

Theorem 2.2.6. *(Comparison of the box and product topologies).*

(i) The box topology on $\prod\limits_{\alpha\in J} X_\alpha$ *has as basis all sets of the form* $\prod U_\alpha$*, where* U_α *is open in* X_α *for each* α.

(ii) The product topology on $\prod\limits_{\alpha\in J} X_\alpha$ *has as basis all sets of the form* $\prod U_\alpha$*, where* U_α *is open in* X_α *for each* α *and* U_α *equals* X_α*, except for finitely many values of* α.

Proof. The statement in (i) is nothing but the definition of box topology.

Let us prove (ii)..

Let $\mathscr{B}$ be a basis obtained from the subbasis $\mathcal{S}$ for the produt topology. on $\prod\limits_{\alpha\in J} X_\alpha$.

Then $\mathscr{B}$ consists of all finite intersections os elements of $\mathcal{S}$.

If $\pi_\beta^{-1}(U_\beta), \pi_\beta^{-1}(V_\beta) \in \mathcal{S}_\beta$, then $\pi_\beta^{-1}(U_\beta) \cap \pi_\beta^{-1}(V_\beta) = \pi_\beta^{-1}(U_\beta \cap V_\beta)$ also belongs to $\mathcal{S}_\beta$. That is, if we intersect finite number of elements belonging to the same one of the sets $\mathcal{S}_\beta$, then their intersection is again an element of $\mathcal{S}_\beta$, and we cannot get any new element in this case. Thus, to generate a new element, we must intersect elements from different sets $\mathcal{S}_\beta$.

Therefore, to get a typical element B of the basis $\mathscr{B}$, choose $\beta_1, \beta_2, \cdots, \beta_n$ as a finite set of distinct indices from the index set J and let U_{β_i} be open in X_{β_i}, for $i = 1, 2, \cdots, n$. Then

$$B = \pi_{\beta_1}^{-1}(U_{\beta_1}) \cap \pi_{\beta_2}^{-1}(U_{\beta_2}) \cap \cdots \cap \pi_{\beta_n}^{-1}(U_{\beta_n}) \tag{2.1}$$

is the typical element of $\mathscr{B}$.

Now,

$x = (x_\alpha) \in B$ if and only if $\quad (x_\alpha) \in \pi_{\beta_1}^{-1}(U_{\beta_1}), (x_\alpha) \in \pi_{\beta_2}^{-1}(U_{\beta_2}), \cdots, (x_\alpha) \in \pi_{\beta_n}^{-1}(U_{\beta_n})$.

i.e) if and only if $\quad \pi_{\beta_1}((x_\alpha)) \in U_{\beta_1}, \cdots \pi_{\beta_n}((x_\alpha)) \in U_{\beta_n}$.

i.e) if and only if $x_{\beta_1} \in U_{\beta_1}, \cdots, x_{\beta_n} \in U_{\beta_n}$.

This means that there is no restriction on the α^{th} coordinate of x if $\alpha \neq \beta_1, \cdots, \beta_n$.

i.e) $x = (x_\alpha) \in B$ if and only if $x_{\beta_1} \in U_{\beta_1}, \cdots, x_{\beta_n} \in U_{\beta_n}$ and $x_\alpha \in X_\alpha$ for $\alpha \neq \beta_1, \cdots, \beta_n$. Therefore, B can be written as $B = \prod\limits_{\alpha \in J} U_\alpha$, where each U_α is open in X_α and $U_\alpha = X_\alpha$ for $\alpha \neq \beta_1, \cdots, \beta_n$. Hence (ii). □

Remark 2.2.7. *For finite products, both box and product topologies are the same.*

For, let $x = (x_1, x_2, \cdots, x_n) \in \prod_{i=1}^n X_i$.

If we consider B as the basis element for the box topology on $\prod_{i=1}^n X_i$, then

$x \in B \Rightarrow (x_1, x_2, \cdots, x_n) \in \prod_{i=1}^n U_i$, where U_i is open in X_i for $i = 1, 2, \cdots, n$.

$\Rightarrow x_i \in U_i$ for each $i = 1, 2, \cdots, n$.

If we consider B as the basis element for the product topology on $\prod_{i=1}^n X_i$, then

$x \in B \Rightarrow x \in \pi_1^{-1}(U_1) \cap \cdots \cap \pi_n^{-1}(U_n)$, where each U_i is open in X_i.

$\Rightarrow x \in \pi_i^{-1}(U_i)$, $\forall\ i = 1, 2, \cdots, n$.

$\Rightarrow \pi_i(x) \in U_i$, $\forall\ i = 1, 2, \cdots, n$.

$\Rightarrow x_i \in U_i$, $\forall\ i = 1, 2, \cdots, n$.

Remark 2.2.8. *The box topology is in general finer than the product topology.*

The following three theorems can be easily proved with the idea of results on $X \times Y$.

Theorem 2.2.9. *Suppose the topology on each space X_α is given by a basis $\mathscr{B}_\alpha$. Then (i)The collection of all sets of the form $\prod_{\alpha \in J} B_\alpha$, where $B_\alpha \in \mathscr{B}_\alpha$ for each α, will serve as a basis for the box topology on $\prod_{\alpha \in J} X_\alpha$.*

(ii)The collection of all sets of the same form, where $B_\alpha \in \mathscr{B}_\alpha$ for finitely many indices α and $B_\alpha = X_\alpha$ for all the remaining indices, will serve as a basis for the product topology $\prod_{\alpha \in J} X_\alpha$.

Example 2.2.10. *Consider the euclidean n-space $\mathbb{R}^n$. A basis for $\mathbb{R}$ consists of all open intervals in $\mathbb{R}$. Then a basis for the topology of $\mathbb{R}^n$ consists of all products of the form*

$$(a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n).$$

Since $\mathbb{R}^n$ is a finite product, the box and product topologies are the same on $\mathbb{R}^n$.

Theorem 2.2.11. *Let A_α be a subspace of X_α, for each $\alpha \in J$. Then $\prod A_\alpha$ is a subspace of $\prod X_\alpha$ if both products are given the box topology, or if both products are given the product topology.*

Theorem 2.2.12. *If each space X_α is a Hausdorff space, then ΠX_α is a Hausdorff space in both the box and product topologies.*

Theorem 2.2.13. *Let $\{X_\alpha\}$ be an indexed family of spaces and let $A_\alpha \subset X_\alpha$ for each α. If $\prod X_\alpha$ is given either the product or the box topology, then*

$$\prod \bar{A}_\alpha = \overline{\prod A_\alpha}.$$

Proof. Let $\mathbf{x} = (x_\alpha)$ be a point of $\prod \bar{A}_\alpha$. We show that $\mathbf{x} \in \overline{\prod A_\alpha}$. Let $U = \prod U_\alpha$ be a basis element for either the box or product topology that contains $\mathbf{x}$. Since $x_\alpha \in \bar{A}_\alpha$, we can choose a point $y_\alpha \in U_\alpha \cap A_\alpha$ for each α. Then $\mathbf{y} = (y_\alpha)$ belongs to both U and $\prod A_\alpha$. Since U is arbitrary, it follows that $\mathbf{x}$ belongs to the closure of $\prod A_\alpha$.

Conversely, suppose $\mathbf{x} = (x_\alpha)$ lies in the closure of $\prod A_\alpha$, in either topology. We show that for any given index β, we have $x_\beta \in \bar{A}_\beta$. Let V_β be an arbitrary open set of X_β containing x_β. Since $\pi_\beta^{-1}(V_\beta)$ is open in $\prod X_\alpha$ in either topology, it contains a point $\mathbf{y} = (y_\alpha)$ of $\prod A_\alpha$. Then y_β belongs to $V_\beta \cap A_\beta$. It follows that $x_\beta \in \bar{A}_\beta$. □

Theorem 2.2.14. *Let $f : A \to \prod_{\alpha \in J} X_\alpha$ be given by the equation*

$$f(a) = (f_\alpha(a))_{\alpha \in J},$$

where $f_\alpha : A \to X_\alpha$ for each α. Let $\prod X_\alpha$ have the product topology. Then the function f is continuous if and only if each function f_α is continuous.

Proof. Let π_β be the projection of the product onto its β^{th} factor. The function π_β is continuous, for if U_β is open in X_β, the set $\pi_\beta^{-1}(U_\beta)$ is a subbasis element for the product topology on $\prod X_\alpha$.

Now, suppose that $f : A \to \prod X_\alpha$ is continuous. The function f_β equals the composite $\pi_\beta \circ f$. Being the composite of two continuous functions, f_β is continuous.

Conversely, suppose that each coordinate function f_α is continuous. To prove that f is continuous, it suffices to prove that the inverse image under f of each subbasis element is open in A.

A typical subbasis element for the product topology on $\prod X_\alpha$ is a set of the form $\pi_\beta^{-1}(U_\beta)$, where β is some index and U_β is open in X_β. Now

$$f^{-1}\left(\pi_\beta^{-1}(U_\beta)\right) = f_\beta^{-1}(U_\beta),$$

because $f_\beta = \pi_\beta \circ f$. Since f_β is continuous, this set is open in A, as desired. □

Example 2.2.15. *Consider $\mathbb{R}^\omega$, the countably infinite product of $\mathbb{R}$ with itself. Recall that*

$$\mathbb{R}^\omega = \prod_{n\in\mathbb{Z}_+} X_n$$

where $X_n = \mathbb{R}$ for each n. Let us define a function $f : \mathbb{R} \to \mathbb{R}^\omega$ by the equation

$$f(t) = (t, t, t, \ldots),$$

where the nth coordinate function of f is the function $f_n(t) = t$. Each of the coordinate functions $f_n : \mathbb{R} \to \mathbb{R}$ is continuous; therefore, the function f is continuous if $\mathbb{R}^\omega$ is given the product topology. But f is not continuous if $\mathbb{R}^\omega$ is given the box topology. Consider, for example, the basis element

$$B = (-1, 1) \times \left(-\frac{1}{2}, \frac{1}{2}\right) \times \left(-\frac{1}{3}, \frac{1}{3}\right) \times \ldots$$

for the box topology. We assert that $f^{-1}(B)$ is not open in $\mathbb{R}$. If $f^{-1}(B)$ were open in $\mathbb{R}$, it would contain some interval $(-\delta, \delta)$ about the point 0 . This would mean that $f((-\delta, \delta)) \subset B$, so that, applying π_n to both sides of the inclusion,

$$f_n((-\delta, \delta)) = (-\delta, \delta) \subset (-1/n, 1/n)$$

for all n, a contradiction.

Let Us Sum Up:

In this section, we have discussed the following concepts:

(1) Box topology

(2) Product topology

(3) Comparison of the box and product topologies

(4) Continuity of maps into product spaces

Check your Progress:

1. If each space X_α is a Hausdorff space, then ΠX_α is a Hausdorff space in ....

   (A) the box topology (B) the product topology

   (C) both the box and product topologies (D) None of these

2. Which of the following is true in general?

   (A) Box topology is finer than the product topology

   (B) Box topology is coarser than the product topology

   (C) Box topology is always not equal to the product topology

   (D) None of these

3. The function $f : \mathbb{R} \to \mathbb{R}^\omega$ defined by $f(t) = (t, t, t, ...)$ is continuous when $\mathbb{R}^\omega$ is given the

   (A) box topology (B) product topology

   (C) order topology (D) None of these

2.3 The Metric Toplogy

Definition 2.3.1. *A **metric** on a set X is a function*

$$d : X \times X \to \mathbb{R}$$

having the following properties:

1. $d(x, y) \geq 0$ *for all* $x, y \in X$ *and* $d(x, y) = 0$ *if and only if* $x = y$.

2. $d(x, y) = d(y, x)$ *for all* $x, y \in X$.

3. *(Triangle inequality):* $d(x, y) + d(y, z) \geq d(x, z)$ *for all* $x, y, z \in X$.

Definition 2.3.2. *Given a metric d on X, the number $d(x, y)$ is called the **distance** between x and y.*

Definition 2.3.3. *Given $\varepsilon > 0$, the set of all points y whose distance from x is less than ε,*

$$i.e., \quad B_d(x, \varepsilon) = \{y \in X \mid d(x, y) < \varepsilon\}$$

is called the ***ε-ball centered at*** *x.*

Definition 2.3.4. *If d is a metric on the set X, then the collection of all ε-balls $B_d(x, \varepsilon)$, for $x \in$ X and $\varepsilon > 0$, is a basis for a topology on X, called the* ***metric topology induced by d****.*

Definition 2.3.5. *If X is a topological space, X is said to be* ***metrizable*** *if there exists a metric d on the set X that induces the topology of X.*

Definition 2.3.6. *A* ***metric space*** *is a metrizable space X together with a specific metric d that gives the topology of X.*

Definition 2.3.7. *A subset A of a metric space (X, d) is said to be* ***bounded*** *if there is some number M such that*

$$d(a_1, a_2) \leq M$$

for every pair a_1, a_2 of points of A.

Definition 2.3.8. *If A is bounded and nonempty, the* ***diameter*** *of A is defined as*

$$\operatorname{diam} A = \sup\{d(a_1, a_2) \mid a_1, a_2 \in A\}.$$

Theorem 2.3.9. *Let X be a metric space with metric d. Define $\bar{d} : X \times X \to \mathbb{R}$ by*

$$\bar{d}(x, y) = \min\{d(x, y), 1\}.$$

Then $\bar{d}$ is a metric that induces the same topology as d.

The metric $\bar{d}$ is called the ***standard bounded metric*** *corresponding to d.*

Proof. Clearly $\bar{d}(x, y) \geq 0$ and $\bar{d}(x, y) = 0$ if and only if $x = y$ is true because d has these properties.

Similarly, $\bar{d}(x, y) = \bar{d}(y, x)$ holds.

As for the triangle inequality, $\bar{d}(x, z) \leq \bar{d}(x, y) + \bar{d}(y, z)$, as the left hand side is at most 1 , if any one of the terms $\bar{d}(x, y) \geq 1$ or $\bar{d}(y, z) \geq 1$, then the inequality clearly hold.

But $\bar{d}(x, y) \geq 1$ if and only if $d(x, y) \geq 1$, and $\bar{d}(y, z) \geq 1$ if and only if $d(y, z) \geq 1$.

$\therefore$ It is enough to consider the case where $d(x, y) < 1$ and $d(y, z) < 1$.

In this case $\bar{d}(x, y) = d(x, y)$ and $\bar{d}(y, z) = d(y, z)$.

So, we have $\bar{d}(x, z) \leq d(x, z) \leq d(x, y) + d(y, z) = \bar{d}(x, y) + \bar{d}(y, z)$.

To show that the metric $\bar{d}$ and d induce the same topology, note that for $\varepsilon < 1$, the $\bar{d}$ and d balls are the same.

Since any metric topology is also generated by the collection of all ε-balls for $\varepsilon < 1$, we conclude that $\bar{d}$ and d induce the same topology. $\square$

Definition 2.3.10. *For $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$, we define*

1. *the **norm of** x by*
$$||x|| = (x_1^2 + x_2^2 + ... + x_n^2)^{1/2},$$

2. *the **euclidean metric** by*
$$d(\mathrm{x}, \mathrm{y}) = ||x - y|| = \left[(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2\right]^{\frac{1}{2}}$$

3. *and the **square metric** by*
$$\rho(x, y) = \max\left\{|x_1 - y_1|, \ldots, |x_n - y_n|\right\}.$$

Lemma 2.3.11. *Let d and d' be two metrics on a set X. Let $\mathcal{T}$ and $\mathcal{T}'$ be the topologies they induce, respectively. Then $\mathcal{T}'$ is finer than $\mathcal{T}$ if and only if for each x in X and each $\varepsilon > 0$, there exists a $\delta > 0$ such that $B_{d'}(x, \delta) \subset B_d(x, \varepsilon)$.*

Proof. Suppose $\mathcal{T}'$ is finer than $\mathcal{T}$.

By Lemma 1.2.9, given a basis element $B_d(x, \varepsilon)$ for $\mathcal{T}$, there exists a basis element B' for the topology $\mathcal{T}'$ such that $x \in B' \subset B_d(x, \varepsilon)$.

But inside B', we can find a ball $B_d'(x, \delta)$ centered at x.

Conversely, if the $\varepsilon - \delta$ condition is satisfied, given a basis B for $\mathcal{T}$ containing x, we can find within B, a ball $B_d(x, \varepsilon)$ centered at x.

Then, there exists a ball $B_{d'}(x, \delta) = B_d(x, \varepsilon) \subset B$.

Again, by Lemma 1.2.9, we conclude that $\mathcal{T}'$ is finer than $\mathcal{T}$. $\square$

Theorem 2.3.12. *The topologies on $\mathbb{R}^n$ induced by the Euclidean metric d and the square metric ρ are the same as the product topology on $\mathbb{R}^n$.*

Proof. Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ be two points in $\mathbb{R}^n$.

We can easily prove that $\rho(x, y) \leq d(x, y) \leq \sqrt{n}\rho(x, y)$.

Let $z \in B_d(x, \in)$. Then $d(x, z) < \varepsilon$

$$\Rightarrow \rho(x, z) < \varepsilon$$

$$\Rightarrow z \in B_\rho(x, \varepsilon) \quad \forall x \in \mathbb{R}^n, \varepsilon > 0$$

Thus $B_d(x, \varepsilon) \subset B_\rho(x, \varepsilon) \quad \forall x \in \mathbb{R}^n, \varepsilon > 0$.

Now, let $t \in B_\rho\left(x, \frac{\varepsilon}{\sqrt{n}}\right)$. Then $\rho(x, t) < \frac{\varepsilon}{\sqrt{n}}$

$$\Rightarrow \sqrt{n}\rho(x, t) < \varepsilon$$

$$\Rightarrow d(x, t) < \varepsilon$$

$$\Rightarrow t \in B_d(x, \varepsilon)$$

Thus $B_\rho\left(x, \frac{\varepsilon}{\sqrt{n}}\right) \subset B_d(x, \varepsilon) \quad \forall x \in \mathbb{R}^n, \varepsilon > 0$.

It follows from the preceding lemma that the topologies induced by the metrics d and ρ are the same.

Now, we shall prove that the product topology and the ρ-metric topology are the same on $\mathbb{R}^n$.

First, let $B = (a_1, b_1) \times \ldots \times (a_n, b_n)$ be a basis element for the product topology , and let $x = (x_1, x_2, \ldots, x_n)$ be an element of B.

For each i, there is an ε_i such that $(x_i - \varepsilon_i, x_i + \varepsilon_i) \subset (a_i, b_i)$;

Choose $\varepsilon = \min\{\varepsilon_1, \ldots, \varepsilon_n\}$.

Then $B_\rho(x, \varepsilon) \subset B$.

As a result the ρ-topology is finer than the product topology.

Conversely, let $B_\rho(x, \varepsilon)$ be a basis element for the ρ-topology.

Given $y \in B_\rho(x, \varepsilon)$, we need to find a basis element B for the product topology such that $y \in B \subset B_\rho(x, \varepsilon)$.

Let $y \in B_\rho(x, \varepsilon)$. Then $\rho(x, y) < \varepsilon$.

$$\Rightarrow \max\{|x_i - y_i| : i = 1, 2, \dots, n\} < \varepsilon$$

$$\Rightarrow |x_i - y_i| < \varepsilon, \forall i = 1, 2, \dots, n$$

$$\Rightarrow y_i \in (x_i - \varepsilon, x_i + \varepsilon), \forall i = 1, 2, \dots, n$$

$$\Rightarrow y \in (x_1 - \varepsilon, x_1 + \varepsilon) \times \dots \times (x_n - \varepsilon, x_n + \varepsilon).$$

That is, $B_\rho(x, \varepsilon)$ itself is a basis element for the product topology on $\mathbb{R}^n$. □

Definition 2.3.13. *Given an index set J, and given points $x = (x_\alpha)_{\alpha \in J}$ and $y = (y_\alpha)_{\alpha \in J}$ of $\mathbb{R}^J$, let us define a metric $\bar{\rho}$ on $\mathbb{R}^J$ by the equation*

$$\bar{\rho}(x, y) = \sup\left\{\bar{d}\,(x_\alpha, y_\alpha) \mid \alpha \in J\right\},$$

*where $\bar{d}$ is the standard bounded metric on $\mathbb{R}$. The metric $\bar{\rho}$ is called the **uniform metric** on $\mathbb{R}^J$, and the topology it induces is called the **uniform topology.***

Theorem 2.3.14. *The uniform topology on $\mathbb{R}^J$ is finer than the product topology but coarser than the box topology; these three topologies are all different if J is infinite.*

Proof. Let $x = (x_\alpha)_{\alpha \in J}$ be a point in $\mathbb{R}^J$ and $\prod_{\alpha \in J} U_\alpha$ be a basis element for the product topology containing the point x.

Then $U_\alpha = \mathbb{R}$ for all but finitely many α 's.

Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be the indices for which $U_{\alpha_i} \neq \mathbb{R}$.

Since U_{α_i} is open in $\mathbb{R}$ containing x_{α_i} and since the metric $\bar{d}$ induces the standard topology on $\mathbb{R}$, for each $i = 1, 2, \dots, n$, there exists $\varepsilon_i > 0$ such that $B_{\bar{d}}\,(x_{\alpha_i}, \varepsilon_i) \subset U_{\alpha_i}$.

Let $\varepsilon = \min\{\varepsilon_1, \dots, \varepsilon_n\}$.

Then for any index $\alpha \in J, B_{\bar{d}}\,(x_\alpha, \varepsilon) \subset U_\alpha$.

If $y \in B_{\bar{\rho}}(x, \varepsilon)$, then $\bar{\rho}(x, y) < \varepsilon$.

This implies for each $\alpha \in J, \bar{d}\,(x_\alpha, y_\alpha) < \varepsilon$.

Therefore, $y \in \prod_{\alpha \in J} U_\alpha$.

Since $x \in B_\rho(x, \varepsilon) \subset \prod_{\alpha \in J} U_\alpha$, it follows that the uniform topology is finer than the product topology.

On the other hand, given a $\bar{\rho}$-ball $B = B_\rho(x, \varepsilon)$, consider the box neighborhood $U = \prod_{\alpha \in J}\left(x_\alpha - \frac{\varepsilon}{2}, x_\alpha + \frac{\varepsilon}{2}\right)$ of x. If $y \in U$ then $|x_\alpha - y_\alpha| < \frac{\varepsilon}{2} \Rightarrow \bar{d}\,(x_\alpha, y_\alpha) < \frac{\varepsilon}{2}$ for each α.

Then, by definition, $\bar{\rho}(x, y) < \varepsilon$, and so $y \in B$.

Thus, $U \subset B$.

This proves that box topology is finer than the uniform topology. $\square$

Theorem 2.3.15. *Let $\bar{d}(a, b) = \min\{|a - b|, 1\}$ be the standard bounded metric on $\mathbb{R}$. If x and y are two points of $\mathbb{R}^\omega$, define*

$$D(x, y) = \sup\left\{\frac{\bar{d}\,(x_i, y_i)}{i}\right\}.$$

Then D is a metric that induces the product topology on $\mathbb{R}^\omega$.

Proof. The first two properties of the metric can be easily verified for D.

Since $\bar{d}$ is a metric, for each i,

$$\frac{\bar{d}\,(x_i, z_i)}{i} \leq \frac{\bar{d}\,(x_i, y_i)}{i} + \frac{\bar{d}\,(y_i, z_i)}{i}$$

$$\implies \frac{\bar{d}\,(x_i, z_i)}{i} \leq D(x, y) + D(y, z).$$

$$\implies \sup\left\{\frac{\bar{d}\,(x_i, z_i)}{i}\right\} = D(x, z) \leq D(x, y) + D(y, z).$$

Let U be open in the metric topology and let x $\in U$.

Choose an ε-ball, $B_D(x, \varepsilon) \subset U$. Let N be such that $\frac{1}{N} < \varepsilon$.

Consider the basis element for the product topology

$$V = (x_1 - \varepsilon, x_1 + \varepsilon) \times \ldots \times (x_N - \varepsilon, x_N + \varepsilon) \times \mathbb{R} \times \mathbb{R} \times \ldots$$

We prove that $V \subset B_D(x, \varepsilon)$.

Given any $y \in \mathbb{R}^\omega$, $\frac{\bar{d}(x_i, y_i)}{i} \leq \frac{1}{N}$ for $i \geq N$.

Therefore,

$$D(x, y) = \sup\left\{\frac{\bar{d}\,(x_i, y_i)}{i}\right\}$$
$$\leq \max\left\{\frac{\bar{d}\,(x_1, y_1)}{1}, \frac{\bar{d}\,(x_2, y_2)}{2}, \ldots, \frac{\bar{d}\,(x_N, y_N)}{N}, \frac{1}{N}\right\}$$

If $y \in V$, then $\dfrac{\bar{d}\,(x_i, y_i)}{i} < \varepsilon$ for $i \leq N$, and hence $y \in B_D(x, \varepsilon)$.

Thus, the product topology is finer than the metric topology.

Conversely, consider a basis element $U = \prod_{i \in \mathbb{Z}_+} U_i$ for the product topology where U_i is an open set in $\mathbb{R}$ for $i = \alpha_1, \alpha_2, \ldots, \alpha_n$ and $U_i = \mathbb{R}$ for all other indices i.

Given $x \in U$, for each $i = \alpha_1, \alpha_2, \ldots, \alpha_n$, choose an interval $(x_i - \varepsilon_i, x_i + \varepsilon_i) \subset U_i$ in $\mathbb{R}$ centered at x_i with $\varepsilon_i \leq 1$.

Define $\varepsilon = \min\left\{ \frac{\varepsilon_i}{i} \middle| \; i = \alpha_1, \alpha_2, \ldots, \alpha_n \right\}$.

We claim that $x \in B_D(x, \varepsilon) \subset U$.

Let $y \in B_D(x, \varepsilon)$.

Then for all i, we have $\frac{\bar{d}(x_i, y_i)}{i} \leq D(x, y) < \varepsilon$.

For any $i \in \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, we have $\varepsilon \leq \frac{\varepsilon_i}{i}$ so that $\bar{d}\,(x_i, y_i) \leq \varepsilon_i \leq 1$.

Then $|x_i - y_i| = \bar{d}\,(x_i, y_i) < \varepsilon_i$.

This proves that $y_i \in U_i$ for all i. Hence, $y \in U$ so that $B_D(x, \varepsilon) \subset U$.

Thus, the metric topology is finer than the product topology. $\square$

Let Us Sum Up:

In this section, we have discussed the following concepts:

(1) Metric topology

(2) Metrizable space

(3) The standard bounded metric

(4) Comparison between the metric topologies and product topology on $\mathbb{R}^n$

(5) Relation between the uniform topology, box topology and the product topology on $\mathbb{R}^J$

(6) Metrizability of $\mathbb{R}^\omega$

Check your Progress:

1. Which of the following does not induces the order topology on $\mathbb{R}$?

   (A) euclidean metric (B) square metric

   (C) $d(x, y) = \min\{1, |x - y|\}$ (D) None of these

2. The uniform topology on $\mathbb{R}^J$, J-an index set, is

   (A) finer than box topology (B) finer than product topology

   (C) the same as the product topology (D) the same as the box topology

3. Which of the following is not true?

(A) Every metric defined on $X(\neq \phi)$ induces a topology on X.

(B) Every topological space is metrizable.

(C) Discrete topological space is metrizable.

(D) None of these.

2.4 The Metric Topology (continued)

Theorem 2.4.1. *Let $f : X \to Y$ and let X and Y be metrizable with metrics d_X and d_Y, respectively. Then continuity of f is equivalent to the requirement that given $x \in X$ and given $\epsilon > 0$, there exists $\delta > 0$ such that*

$$d_X(x, y) < \delta \Longrightarrow dy(f(x), f(y)) < \epsilon.$$

Proof. Suppose that f is continuous. Given x and ϵ, consider the set

$$f^{-1}(B(f(x), \epsilon)),$$

which is open in X and contains the point x. It contains some δ-ball $B(x, \delta)$ centered at x. If y is in this δ-ball, then $f(y)$ is in the ϵ-ball centered at $f(x)$.

Conversely, suppose that the $\epsilon - \delta$ condition is satisfied. Let V be open in Y. We show that $f^{-1}(V)$ is open in X. Let x be a point of the set $f^{-1}(V)$. Since $f(x) \in V$, there is an ϵ-ball $B(f(x), \epsilon)$ centered at $f(x)$ and contained in V. By the $\epsilon - \delta$ condition, there is a δ-ball $B(x, \delta)$ centered at x such that $f(B(x, \delta)) \subset B(f(x), \epsilon)$. Then $B(x, \delta)$ is a neighborhood of x contained in $f^{-1}(V)$, so that $f^{-1}(V)$ is open. Therefore, f is continuous □

Theorem 2.4.2. *(The sequence lemma).*

Let X be a topological space and let $A \subset X$. If there is a sequence of points of A converging to x, then $x \in \bar{A}$. The converse holds if X is metrizable.

Proof. Suppose that $x_n \to x$, where $x_n \in A$. Then, every neighborhood U of x contains a point of A, so $x \in \bar{A}$ by Theorem 1.6.12.

Conversely, suppose that X is metrizable and $x \in \bar{A}$. Let d be a metric for the topology of X. For each positive integer n, take the neighborhood $B_d(x, 1/n)$ of radius

$1/n$ of x, and choose x_n to be a point of its intersection with A. We assert that the sequence x_n converges to x. Any open set U containing x contains an ϵ-ball $B_d(x, \epsilon)$ centered at x. If we choose N so that $1/N < \epsilon$, then U contains x_i for all $i \geq N$. Hence, x_n converges to x. □

Theorem 2.4.3. *Let $f : X \to Y$. If the function f is continuous, then for every convergent sequence $x_n \to x$ in X, the sequence $f(x_n)$ converges to $f(x)$. The converse holds if X is metrizable.*

Proof. Assume that f is continuous. Given $x_n \to x$, we wish to show that $f(x_n) \to f(x)$. Let V be a neighborhood of $f(x)$. Then $f^{-1}(V)$ is a neighborhood of x, and so there is an N such that $x_n \in f^{-1}(V)$ for $n \geq N$. Then $f(x_n) \in V$ for $n \geq N$.

To prove the converse, assume that the convergent sequence condition is satisfied. Let A be a subset of X; we show that $f(\bar{A}) \subset \overline{f(A)}$. If $x \in \bar{A}$, then there is a sequence x_n of points of A converging to x ($\because$ by converse part of sequence lemma). By assumption, the sequence $f(x_n)$ converges to $f(x)$. Since $f(x_n) \in f(A)$, the sequence lemma implies that $f(x) \in \overline{f(A)}$. Therefore, $f(\bar{A}) \subset \overline{f(A)}$. Hence, f is continuous. □

By $\epsilon - \delta$ argument, one can easily prove the following lemma.

Lemma 2.4.4. *The addition, subtraction, and multiplication operations are continuous functions from $\mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$; and the quotient operation is a continuous function from $\mathbb{R} \times (\mathbb{R} - \{0\})$ into $\mathbb{R}$.*

Theorem 2.4.5. *If X is a topological space, and if $f, g : X \to \mathbb{R}$ are continuous functions, then $f + g$, $f - g$, and $f \cdot g$ are continuous. If $g(x) \neq 0$ for all x, then f/g is continuous.*

Proof. The map $h : X \to \mathbb{R} \times \mathbb{R}$ defined by

$$h(x) = f(x) \times g(x),$$

which is continuous, by Theorem 2.1.20. The function $f + g$ equals the composite of h and the addition operation

$$+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}.$$

Therefore $f + g$ is continuous. Similar arguments apply to $f - g$, $f \cdot g$, and f/g. □

Definition 2.4.6. *Let $f_n : X \to Y$ be a sequence of functions from the set X to the metric space Y. Let d be the metric for Y. We say that the sequence (f_n) converges uniformly to the function $f : X \to Y$ if given $\epsilon > 0$, there exists an integer N such that*

$$d\left(f_n(x), f(x)\right) < \epsilon$$

for all $n > N$ and all x in X.

Theorem 2.4.7. *(Uniform limit theorem).*

Let $f_n : X \to Y$ be a sequence of continuous functions from the topological space X to the metric space Y. If (f_n) converges uniformly to f, then f is continuous.

Proof. Let V be open in Y and let x_0 be a point of $f^{-1}(V)$. We wish to find a neighborhood U of x_0 such that $f(U) \subset V$.

Let $y_0 = f\left(x_0\right)$. First choose ϵ so that the ϵ-ball $B\left(y_0, \epsilon\right)$ is contained in V. Then, using uniform convergence, choose N so that for all $n \geq N$ and all $x \in X$,

$$d\left(f_n(x), f(x)\right) < \epsilon/3.$$

Finally, using continuity of f_N, choose a neighborhood U of x_0 such that f_N carries U into the $\epsilon/3$ ball in Y centered at $f_N\left(x_0\right)$.

We claim that f carries U into $B\left(y_0, \epsilon\right)$ and hence into V. For this purpose, note that if $x \in U$, then

$$d\left(f(x), f_N(x)\right) < \epsilon/3 \text{ (by choice of } N \text{)}$$

$$d\left(f_N(x), f_N\left(x_0\right)\right) < \epsilon/3 \text{ (by choice of } U\text{)}$$

$$d\left(f_N\left(x_0\right), f\left(x_0\right)\right) < \epsilon/3 \text{ (by choice of } N\text{)}$$

Adding and using the triangle inequality, we see that $d\left(f(x), f\left(x_0\right)\right) < \epsilon$. Thus, f is continuous. □

Example 2.4.8. $\mathbb{R}^\omega$ *in the box topology is not metrizable.*

Proof. We shall show that the sequence lemma does not hold for $\mathbb{R}^\omega$. Let A be the subset of $\mathbb{R}^\omega$ consisting of those points all of whose coordinates are positive. That is,

$$A = \left\{(x_1, x_2, \ldots) \mid x_i > 0 \text{ for all } i \in \mathbb{Z}_+\right\}.$$

Let **0** be the "origin" in $\mathbb{R}^\omega$, that is, the point $(0, 0, \ldots)$ each of whose coordinates is zero. In the box topology, **0** belongs to $\bar{A}$, for if

$$B = (a_1, b_1) \times (a_2, b_2) \times \cdots$$

is any basis element containing **0** , then B intersects A. For instance, the point

$$\left(\frac{1}{2}b_1, \frac{1}{2}b_2 \ldots\right)$$

belongs to $B \cap A$.

But we assert that there is no sequence of points of A converging to **0**.
For, let (a_n) be a sequence of points of A, where

$$a_n = (x_{1n}, x_{2n}, \ldots, x_{in}, \ldots).$$

Every coordinate x_{in} is positive, so we can construct a basis element B' for the box topology on $\mathbb{R}$ by setting

$$B' = (-x_{11}, x_{11}) \times (-x_{22}, x_{22}) \times \cdots.$$

Then B' contains the origin **0** , but it contains no member of the sequence (a_n). The point a_n cannot belong to B' because its nth coordinate x_{nn} does not belong to the interval $(-x_{nn}, x_{nn})$. Hence, the sequence (a_n) cannot converge to **0** in the box topology. □

Example 2.4.9. *An uncountable product of* $\mathbb{R}$ *with itself is not metrizable.*

Proof. Let J be an uncountable index set. We show that $\mathbb{R}^J$ does not satisfy the sequence lemma (in the product topology).

Let A be the subset of $\mathbb{R}^J$ consisting of all points (x_α) such that $x_\alpha = 1$ for all but finitely many values of α. Let **0** be the origin in $\mathbb{R}^J$, the point each of whose coordinates is 0 .

We assert that **0** belongs to the closure of A. Let $\prod U_\alpha$ be a basis element containing **0** . Then $U_\alpha \neq \mathbb{R}$ for only finitely many values of α, say for $\alpha = \alpha_1, \ldots, \alpha_n$. Let (x_α) be the point of A defined by letting $x_\alpha = 0$ for $\alpha = \alpha_1, \ldots, \alpha_n$ and $x_\alpha = 1$ for all other values of α. Then $(x_\alpha) \in A \cap \prod U_\alpha$, as desired.

But there is no sequence of points of A converging to $\mathbf{0}$.

For, let a_n be a sequence of points of A. Given n, let J_n denote the subset of J consisting of those indices α for which the α th coordinate of a_n is different from 1. The union of all the sets J_n is a countable union of finite sets, and therefore countable. Because J itself is uncountable, there is an index in J, say β, that does not lie in any of the sets J_n. This means that for each of the points a_n, its β th coordinate equals 1.

Now, let U_β be the open interval $(-1, 1)$ in $\mathbb{R}$, and let U be the open set $\pi_\beta^{-1}(U_\beta)$ in $\mathbb{R}^J$. The set U is a neighborhood of $\mathbf{0}$ that contains none of the points a_n. Therefore, the sequence a_n cannot converge to $\mathbf{0}$. □

Let Us Sum Up:

In this section, we have discussed the following concepts:

1. The sequence lemma
2. Uniform limit theorem
3. Metrizability of $\mathbb{R}^\omega$ in the box topology
4. Metrizability of $\mathbb{R}^J$ in the product topology when J is an uncountable index set

Check your Progress:

1. Countable product of $\mathbb{R}$ with itself is metrizable in the

   (A) box topology (B) product topology

   (C) uniform topology (D) standard topology

2. If $f, g : \mathbb{R} \to \mathbb{R}$ are continuous, then which of the following is not true?

   (A) $f + g$ is continuous (B) $f - g$ is continuous

   (C) $f \cdot g$ is continuous (D) f/g is continuous

3. If X is a topological space and $A \subset X$, then the sequence lemma states that

   (A) if there is a sequence of points of A converging to x, then $x \in \bar{A}$

   (B) for every convergent sequence $x_n \to x$ in A, the sequence $f(x_n)$ converges to $f(x)$

(C) (A) is true, and the converse holds if X is metrizable

(D) (B) is true, and the converse holds if X is metrizable

Unit Summary:

In this unit, continuous functions and its properties were discussed. Various rules for constructing continuous functions, the box and product topologies on a product of topological spaces and metric topology were also discussed.

Glossary:

- $f \mid A$ - The domain of f is restricted to A
- $\mathbb{R}^\omega$ - Countably infinite product of $\mathbb{R}$ with itself
- $\mathbb{R}^J$ - Arbitrary product of $\mathbb{R}$ with itself
- π_β - The projection map associated with the index β
- $\operatorname{diam} A$ - diameter of A
- $B_d(x, \varepsilon)$ - ε-ball centered at x
- $\bar{\rho}(x, y)$ - The uniform metric on $\mathbb{R}^J$

Self-Assessment Questions:

(1) Prove that any discrete topological space is metrizable.

(2) Show that the subspace (a, b) of $\mathbb{R}$ is homeomorphic with $(0, 1)$.

(3) Show that the subspace $[a, b]$ of $\mathbb{R}$ is homeomorphic with $[0, 1]$.

(4) Show that $\mathbb{R} \times \mathbb{R}$ in the dictionary order topology is metrizable.

Exercises:

(1) Prove that for functions $f : \mathbb{R} \to \mathbb{R}$, the $\epsilon - \delta$ definition of continuity implies the open set definition.

(2) In $\mathbb{R}^n$, define

$$d'(x, y) = |x_1 - y_1| + \cdots + |x_n - y_n|.$$

Show that d' is a metric that induces the usual topology on $\mathbb{R}^n$.

(3) Given $p \geq 1$, define

$$d'(x, y) = \left[\sum_{i=1}^{n} |x_i - y_i|^p\right]^{1/p},$$

for $x, y \in \mathbb{R}^n$. Assume that d' is a metric. Show that it induces the usual topology on $\mathbb{R}^n$.

(4) If d is a metric on the set X, then prove that the collection of all ϵ-balls $B_d(x, \epsilon)$ for $x \in X$ and $\epsilon > 0$ is a basis for the metric topology on X.

Answers for check your progress:

Section 2.1	1. (D)	2. (D)	3. (B)
Section 2.2	1. (C)	2. (A)	3. (B)
Section 2.3	1. (D)	2. (B)	3. (B)
Section 2.4	1. (B)	2. (D)	3. (C)

Reference:

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3. J.L. Kelley, General Topology, Van Nostrand, Reinhold Co., New York, 1955.

4. L. Steen and J. Subhash, Counter Examples in Topology, Holt, Rinehart and Winston, New York, 1970.

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UNIT 3

Unit 3

Connectedness

Objectives:

This unit deals with connected spaces, connected subspaces of the real line, the concepts of components and local connectedness.

3.1 Connected Spaces

Definition 3.1.1. *Let X be a topological space. A* ***separation*** *of X is a pair U, V of disjoint nonempty open subsets of X whose union is X.*

The space X is said to be ***connected*** *if there does not exist a separation of X.*

Remark 3.1.2. *Connectedness is a topological property, because it is formulated entirely in terms of the collection of open sets of X.*

i.e., ***If X is connected, then any space homeomorphic to X is also connected.***

Remark 3.1.3. ***A space X is connected if and only if the only subsets of X that are both open and closed in X are the empty set and X itself.***

We prove this result by proving that X is not connected if and only if there exists a nonempty proper subset of X, that is both open and closed in X.

Suppose that A is a nonempty proper subset of X, that is both open and closed in X.

Take $U = A$ and $V = X - A$.

Since U is a proper subset of X, V is nonempty.

Thus, U and V are open, disjoint, and nonempty subsets of X such that their union is X.

$\implies$ *U and V constitute a separation of X*

i.e., X is not connected.

Conversely, suppose that U and V form a separation of X.

Then U and V are nonempty open subsets of X such that $U \cup V = X$ and $U \cap V = \emptyset$.

Since V is nonempty, $U \cup V = X$ and $U \cap V = \emptyset$, we have $U \neq X$.

Therefore, U is a proper subset of X.

Also, V is open implies $U = X - V$ is closed.

$\therefore$ We have a nonempty proper subset U of X which is both open and closed in X.

Lemma 3.1.4. ***If Y is a subspace of X, a separation of Y is a pair of disjoint nonempty sets A and B whose union is Y, neither of which contains a limit point of the other. The space Y is connected if there exists no separation of Y.***

Proof. Suppose that A and B form a separation of Y.

Then A is both open and closed in Y.

The closure of A in Y is the set $\overline{A} \cap Y$ (where $\overline{A}$ as usual denotes the closure of A in X).

Since A is closed in Y, $A = \overline{A} \cap Y$; or to say the same thing, $\overline{A} \cap B = \emptyset$.

Since $\overline{A}$ is the union of A and its limit points, B contains no limit points of A.

A similar argument shows that A contains no limit points of B.

Conversely, suppose that A and B are disjoint nonempty sets whose union is Y, neither of which contains a limit point of the other.

Then $A \cap B = \emptyset, A \cap B' = \emptyset$.

$\implies A \cap (B \cup B') = \emptyset$.

$\implies A \cap \overline{B} = \emptyset$.

Similarly, $\overline{A} \cap B = \emptyset$.

Now,

$$\begin{aligned} \overline{A} \cap Y &= \overline{A} \cap (A \cup B) \\ &= (\overline{A} \cap A) \cup (\overline{A} \cap B) \\ &= A \cup \emptyset \\ &= A. \end{aligned}$$

Similarly, $\overline{B} \cap Y = B$.

Thus, both A and B are closed in Y, and since $A = Y - B$ and $B = Y - A$, they are open in Y as well. □

Example 3.1.5. ***Every singleton set is connected*** *because there is no separation for it.*

Example 3.1.6. ***A two-point space in the indiscrete topology is connected*** *because the only open sets in this topology are $\emptyset$ and X.*

Example 3.1.7. ***A two-point space $\{a, b\}$ in the discrete topology is not connected*** *because the open sets $\{a\}$ and $\{b\}$ form a separation of X.*

Example 3.1.8. $\mathbb{R}$ ***is connected.***

Example 3.1.9. ***The subspace $Y = [-1, 0) \cup (0, 1]$ of the real line $\mathbb{R}$ is not connected.***

For,

Each of the sets $[-1, 0)$ and $(0, 1]$ is nonempty and open in Y (although not in $\mathbb{R}$).

Therefore, they form a separation of Y.

Alternatively, we note that neither of these sets contains a limit point of the other.

Example 3.1.10. ***The subspace $X = [-1, 1]$ of the real line $\mathbb{R}$ is connected.***

For,

The sets [-1,0] and (0,1] are disjoint and nonempty, but they do not form a separation of X, because the first set is not open in X.

Alternatively, we note that the first set contains a limit point, 0, of the second.

Indeed, there exists no separation of the space $[-1, 1]$.

Example 3.1.11. ***The rationals $\mathbb{Q}$ are not connected.***

Indeed, the only connected subspaces of $\mathbb{Q}$ are the one-point sets.

For,

If Y is a subspaces of $\mathbb{Q}$ containing two points p and q, one can choose an irrational number a lying between p and q.

Then, we can write Y as the union of open sets $Y \cap (-\infty, a)$ and $Y \cap (a, +\infty)$.

Example 3.1.12. ***The subset $X = \{x \times y | y = 0\} \cup \{x \times y | x > 0$ and $y = 1/x\}$ of the plane $\mathbb{R}^2$ is not connected.***

Lemma 3.1.13. ***If the sets C and D form a separation of X, and if Y is a connected subspace of X, then Y lies entirely within either C or D.***

Proof. Given that C and D form a separation of X.

Then C and D are both open in X and hence $C \cap Y$ and $D \cap Y$ are open in Y. Also,

$$\begin{aligned}(C \cap Y) \cap (D \cap Y) &= (C \cap D) \cap Y \\ &= \emptyset \cap Y \\ &= \emptyset\end{aligned}$$

and

$$\begin{aligned}(C \cap Y) \cup (D \cap Y) &= (C \cup D) \cap Y \\ &= X \cap Y \\ &= Y.\end{aligned}$$

Thus, these two sets are disjoint and their union is Y.

If they were both nonempty, they would constitute a separation of Y. But, given that Y is connected. Therefore, one of them is empty.

Hence, Y must lie entirely in C or in D. □

Theorem 3.1.14. ***The union of a collection of connected subspaces of X that have a point in common is connected.***

Proof. Let $\{A_\alpha\}$ be a collection of connected subspaces which have a point p in common. That is, $p \in \bigcap_\alpha A_\alpha$.

To prove: The space $Y = \bigcup A_\alpha$ is connected.

Suppose that $Y = C \cup D$ is a separation of Y. Then, each A_α, being connected, by Lemma 3.1.13, lies entirely either in C or in D. So, the common point p either belongs to C or D.

Suppose $p \in C$. Then all the A_α's must lie entirely in C, because A_α contains the point p of C.

i.e., $A_\alpha \subset C, \ \ \forall \alpha$.

$\implies \bigcup A_\alpha \subset C.$

$\implies Y \subset C,$

which contradicts the fact that $D \neq \emptyset$. Thus, Y has no separation, and hence it is connected. □

Theorem 3.1.15. ***Let A be a connected subspace of X. If $A \subset B \subset \overline{A}$, then B is also connected.***

(Said differently: If B is formed by adjoining to the connected subspace A some or all of its limit points, then B is connected.)

Proof. Let A be connected and B be such that $A \subset B \subset \overline{A}$.

To prove: B is connected.

Suppose that $B = C \cup D$ is a separation of B.

Since A is a connected subspace of B, by Lemma 3.1.13, the set A must lie entirely in C or in D.

Suppose that $A \subset C$. Then $\overline{A} \subset \overline{C}$.

Also, $B \subset \overline{A}$ and $\overline{A} \subset \overline{C} \implies B \subset \overline{C}$.

Since $\overline{C}$ and D are disjoint, B cannot intersect D.

This implies $B = C$.

This contradicts the fact that D is a nonempty subset of B.

Therefore, B is connected. □

Note: If A is connected, then $\overline{A}$ is also connected.

Theorem 3.1.16. ***The image of a connected space under a continuous map is connected.***

Proof. Let X be connected and let $f : X \to Y$ be a continuous map.

To prove: The image space $Z = f(X)$ is connected.

Since the map obtained from f by restricting its range to the space Z is also continuous, it suffices to consider the case of a continuous surjective map $g : X \to Z$.

Suppose that $Z = A \cup B$ is a separation of Z.

Then A and B are open in Z.

Since g is continuous, $g^{-1}(A)$ and $g^{-1}(B)$ are open in X.

Also,

$$\begin{aligned} g^{-1}(A) \cap g^{-1}(B) &= g^{-1}(A \cap B) \\ &= g^{-1}(\emptyset) \\ &= \emptyset \end{aligned}$$

and, since g is onto, we have

$$\begin{aligned} g^{-1}(A) \cup g^{-1}(B) &= g^{-1}(A \cup B) \\ &= g^{-1}(Z) \\ &= X. \end{aligned}$$

Further, $g^{-1}(A)$ and $g^{-1}(B)$ are non-empty, because g is onto.

Therefore, they form a separation of X, contradicting the assumption that X is connected.

Thus, Z is connected. □

Theorem 3.1.17. ***A finite cartesian product of connected spaces is connected.***

Proof. We prove this by using induction process.

First, let X and Y be connected spaces.

To prove: $X \times Y$ is connected.

Choose a base point $a \times b$ in the product space $X \times Y$.

Then the horizontal slice $X \times b$ is connected, because X is connected and $X \times b$ is homeomorphic with X.

Similarly, each vertical slice $X \times Y$ is connected being homeomorphic with Y.

Put $T_x = (X \times b) \cup (x \times Y)\,, \quad \forall \quad x \in X.$

Clearly, $x \times b \in X \times b$ and $x \times b \in x \times Y$.

Then each T_x is connected, bacause each T_x is the union of two connected spaces, that have the point $x \times b$ in common.

Now, let $T = \bigcup_{x \in X} T_x$.

Then, T is connected, because it is the union of a collection of connected spaces that have the point $a \times b$ in common.

But $T = X \times Y$.

Therefore, $X \times Y$ is connected.

Next, suppose that if $X_1, X_2, ..., X_{n-1}$ are connected spaces, then $X_1 \times X_2 \times ... \times X_{n-1}$

is connected.

Now, let $X_1, X_2, ..., X_n$ be the connected spaces.

Then, by our induction hypothesis, $X_1 \times X_2 \times ... \times X_{n-1}$ is connected.

$\implies (X_1 \times X_2 \times ... \times X_{n-1}) \times X_n$ is connected.

Also, since $X_1 \times X_2 \times ... \times X_n$ is homeomorphic with $(X_1 \times X_2 \times ... \times X_{n-1}) \times X_n$, we have that $X_1 \times X_2 \times ... \times X_n$ is connected.

Hence, by induction, the theorem is proved. □

Example 3.1.18. ***The cartesian product $\mathbb{R}^\omega$ is not connected in the box topology.***

Proof. We know that $\mathbb{R}^\omega = \{(x_1, x_2, ...) | x_i \in \mathbb{R}, i = 1, 2, ...\}$. We can write $\mathbb{R}^\omega$ as the union of the set A consisting of all bounded sequences of real numbers, and the set B of all unbounded sequences. These sets are disjoint and nonempty.

Let $a = (a_1, a_2, ...) \in \mathbb{R}^\omega$.

Consider the open set

$$U = (a_1 - 1, a_1 + 1) \times (a_2 - 1, a_2 + 1) \times \cdots .$$

If a is bounded, then U consists of only bounded sequences so that $a \in U \subset A$, and if a is unbounded, then U consists of only unbounded sequences so that $a \in U \subset B.$

Therefore, A and B are open in the box topology, and so A and B form a separation of $\mathbb{R}^\omega$. Thus, even though $\mathbb{R}$ is connected, $\mathbb{R}^\omega$ is not connected in the box topology. □

Example 3.1.19. ***$\mathbb{R}^\omega$ in the product topology is connected.***

Proof. Assuming that $\mathbb{R}$ is connected, we show that $\mathbb{R}^\omega$ is connected.

Let $\tilde{\mathbb{R}}^n$ denote the subspace of $\mathbb{R}^\omega$ consisting of all sequences $\mathbf{x} = (x_1, x_2, \ldots)$ such that $x_i = 0$ for $i > n$.

That is, $\tilde{\mathbb{R}}^n = \{\mathbf{x} = (x_1, x_2, \ldots) \in \mathbb{R}^\omega : x_i = 0 \quad \text{for} \quad i > n\}$.

The space $\tilde{\mathbb{R}}^n$ is clearly homeomorphic to $\mathbb{R}^n$, so that it is connected, by the preceding theorem. It follows that the space

$\mathbb{R}^\infty = \{(x_1, x_2, \ldots) \in \mathbb{R}^\omega : x_i \neq 0 \quad \text{only for finite number of i's}\}$

that is the union of the spaces $\tilde{\mathbb{R}}^n$ is connected, for these spaces have the point $\mathbf{0} = (0, 0, \ldots)$ in common.

To prove: The closure of $\mathbb{R}^\infty$ equals all of $\mathbb{R}^\omega$.

We know that $\overline{\mathbb{R}^\infty} \subset \mathbb{R}^\omega$.

Let $a = (a_1, a_2, \ldots)$ be a point of $\mathbb{R}^\omega$.

To prove: $a \in \overline{\mathbb{R}^\infty}$.

Let $U = \prod U_i$ be a basis element for the product topology that contains a.

We show that U intersects $\mathbb{R}^\infty$.

Here, $U_i = \mathbb{R}$ for infinitely many values of i.

Therefore, there is an integer N such that $U_i = \mathbb{R}$ for $i > N$.

Since $a_i \in U_i$ for all i, and $0 \in U_i$ for $i > N$, the point

$$\mathbf{x} = (a_1, \ldots, a_n, 0, 0, \ldots)$$

of $\mathbb{R}^\infty$ belongs to U.

$\implies \mathbb{R}^\infty \cap U \neq \emptyset$.

Therefore, $a \in \overline{\mathbb{R}^\infty}$ so that $\mathbb{R}^\omega \subset \overline{\mathbb{R}^\infty}$.

Thus, $\overline{\mathbb{R}^\infty} = \mathbb{R}^\omega$, and hence $\mathbb{R}^\omega$ is connected in the product topology. □

Remark 3.1.20. ***The arbitrary product of connected spaces is connected in the product topology.***

Let Us Sum Up:

In this section, we have discussed the following concepts:

1. Connected spaces with examples
2. Different ways of constructing new connected spaces from the given ones
3. Connectedness of product of connected spaces

Check your Progress:

1. If a set B is formed by adjoining to the connected set A of a topological space X, same or all its limits points, then B is

   (A) not closed (B) connected

   (C) disconnected (D) None of these

2. Which of the following is true?

   (A) Any singleton set is not connected

   (B) $\{a, b, c\}$ with discrete topology is not connected

   (C) $\{a, b\}$ with indiscrete topology is not connected

   (D) The set of rationals is connected

3. If A and B are connected subspaces of X, then which of the following is not true?

   (A) $A \cup B$ is connected (B) $A \times B$ is connected

   (C) $\bar{A} \times \bar{B}$ is connected (D) None of these

3.2 Connected Subspaces of the Real line

Definition 3.2.1. *An ordered set A is said to have the* ***least upper bound property*** *if every nonempty subset A_0 of A that is bounded above has a least upper bound.*

Definition 3.2.2. *A simply ordered set L having more than one element is called a* ***linear continuum*** *if the following hold:*

1. *L has the least upper bound property.*

2. *If $x < y$, there exists z such that $x < z < y$.*

Example 3.2.3. $\mathbb{R}$ ***is a linear continuum.***

Definition 3.2.4. *A subspace Y of L is said to be* **convex** *if for every pair of points a, b of Y with $a < b$, the entire interval $[a, b]$ of points of L lies in Y.*

Theorem 3.2.5. ***If L is a linear continuum in the order topology, then L is connected, and so are intervals and rays in L.***

Proof. We know that the linear continuum L, intervals and rays are all convex.
So, we prove this thoerem by proving that every convex subspace of L is connected.
Let Y be a convex subspace of L.
To prove: Y is connected.

Suppose that Y is the union of the disjoint nonempty sets A and B, each of which is open in Y.

Choose $a \in A$ and $b \in B$; suppose for convenience that $a < b$.

The interval $[a, b]$ of points of L is contained in Y.

Let $A_0 = A \cap [a, b]$ and $B_0 = B \cap [a, b]$. Then A_0 and B_0 are open in $[a, b]$ in the subspace topology, which is the same as the order topology.

The sets A_0 and B_0 are nonempty because $a \in A_0$ and $b \in B_0$, and also $A_0 \cap B_0 = \emptyset$. Also,

$$\begin{aligned} A_0 \cup B_0 &= (A \cap [a, b]) \cup (B \cap [a, b]) \\ &= (A \cup B) \cap [a, b] \\ &= Y \cap [a, b] \\ &= [a, b]. \end{aligned}$$

Thus, A_0 and B_0 constitute a separation of $[a, b]$.

Let $c = \sup A_0$. We show that c belongs neither to A_0 nor to B_0, which contradicts the fact that $[a, b]$ is the union of A_0 and B_0.

Case 1: Suppose that $c \in B_0$.

Then $c \neq a$, so either $c = b$ or $a < c < b$.

In either case, it follows from the fact that B_0 is open in $[a, b]$ that there is some interval of the form $(d, c]$ contained in B_0.

If $c = b$, we have a contradiction at once, for d is a smaller upper bound on A_0 than c.

If $c < b$, we note that $(c, b]$ does not intersect A_0 (because c is an upper bound on A_0). Then

$$(d, b] = (d, c] \cup (c, b]$$

does not intersect A_0.

Again, d is a smaller upper bound on A_0 than c, contrary to construction.

Case 2: Suppose that $c \in A_0$.

Then $c \neq b$, so either $c = a$ or $a < c < b$.

Because A_0 is open in $[a, b]$, there must be some interval of the form $[c, e)$ contained in A_0.

Because of order property (2) of the linear continuum L, we can choose a point z of L such that $c < z < e$.

Then $z \in A_0$, contrary to the fact that c is an upper bound for A_0.

Thus, Y is connected. □

Corollary 3.2.6. ***The real line $\mathbb{R}$ is connected and so are intervals and rays in $\mathbb{R}$.***

Proof. Since $\mathbb{R}$ is a linear continuum in the order topology, by previous theorem, it is connected and so are intervals and rays in $\mathbb{R}$. □

Theorem 3.2.7. ***(Intermediate value theorem). Let $f : X \to Y$ be a continuous map, where X is a connected space and Y is an ordered set in the order topology. If a and b are two points of X and if r is a point of Y lying between $f(a)$ and $f(b)$, then there exists a point c of X such that $f(c) = r$.***

Proof. Since $f : X \to Y$ is continuous and X is connected, we have that $f(X)$ is connected. The sets

$$A = f(X) \cap (-\infty, r) \quad \text{and} \quad B = f(X) \cap (r, +\infty)$$

are disjoint, and they are nonempty because one contains $f(a)$ and the other contains $f(b)$.

Both A and B are open in $f(X)$, being the intersection of an open ray in Y with $f(X)$.

Suppose that there is no point c of X such that $f(c) = r$. Then we have

$$\begin{aligned} A \cup B &= [f(X) \cap (-\infty, r)] \cup [f(X) \cap (r, \infty)] \\ &= f(X) \cap [(-\infty, r) \cup (r, \infty)] \\ &= f(X) \cap (Y - \{r\}) \\ &= f(X) \cap Y = f(X). \end{aligned}$$

Thus, A and B would constitute a separation of $f(X)$, contradicting the fact that the image of a connected space under a continuous map is connected. □

Definition 3.2.8. *Given points x and y of the space X, a **path** in X is a continuous map $f : [a, b] \to X$ of some closed interval in the real line into X such that $f(a) = x$ and $f(b) = y$.*

Definition 3.2.9. *A space X is said to be* ***path connected*** *if every pair of points of X can be joined by a path in X.*

Result: Every path connected space is connected.

Proof. Let X be path connected and suppose that $X = A \cup B$ is a separation of X.

Let $f : [a, b] \to X$ be any path in X. Since f is continuous and $[a, b]$ is connected, we have, $f([a, b])$ is a connected subset of X.

Then, by Lemma 3.1.13, it lies entirely within A or B.

Therefore, there is no path in X joining a point of A to the point of B, which is a contradiction to our hypothesis that X is path connected.

Hence, the proof. □

Remark: The converse of the above does not hold.
i.e., a connected space need not be path connected.

For example,

Consider the ordered square I_o^2 in the dictionary order topology. Since I_o^2 is a linear continuum, it is connected.

Let $p = 0 \times 0$ and $q = 1 \times 1$. Suppose that there is a path $f : [a, b] \to I_o^2$ joining p and q.

$\therefore$ By intermediate value theorem, the image set $f([a, b])$ must contain every point $x \times y$ of I_o^2.

Thus, for each $x \in I(= [0, 1])$, the set $U_x = f^{-1}(x \times (0, 1))$ is a nonempty subset of $[a, b]$.

Also, since f is continuous and $x \times (0, 1)$ is open in I_o^2, we have each U_x is open in $[a, b]$.

Now, for each $x \in I$, choose a rational number q_x in U_x.

Since the sets U_x are disjoint, the map $x \to q_x$ is a one-one mapping of I into $\mathbb{Q}$, which is a contradiction to the fact that I is uncountable.

Thus, I_o^2 is not path connected.

Example 3.2.10. ***The unit ball*** $B^n = \{x \in \mathbb{R}^n : ||x|| \leq 1\}$, ***where*** $||x|| = ||(x_1, x_2, ..., x_n)|| = (x_1^2 + x_2^2 + ... + x_n^2)^{1/2}$***, is path connected.***

For, let $x, y \in B^n$. Consider the function $f : [0, 1] \to \mathbb{R}^n$ defined by $f(t) = (1-t)x + ty$.

Clearly, f is continuous and $f(0) = x, f(1) = y$.

Therefore, f is a straight line path in $\mathbb{R}^n$. Also,

$$\begin{aligned} ||f(t)|| &= ||(1-t)x + ty|| \\ &\leq ||(1-t)x|| + ||ty|| \\ &= (1-t)||x|| + t||y|| \\ &\leq 1 - t + t = 1. \end{aligned}$$

$\implies f(t) \in B^n, \ \forall t \ \ \in [0,1]$.

Therefore, the path f between x and y lies in B^n.

$\implies$ *B^n is path connected.*

Example 3.2.11. ***Every open ball $B_d(x, \epsilon)$ and every closed ball $\overline{B}_d(x, \epsilon)$ in $\mathbb{R}^n$ is path connected.***

Example 3.2.12. ***The punctured Euclidean space $\mathbb{R}^n - \{\mathbf{0}\}$ is path connected if $n > 1$.***

For, let $x, y \in \mathbb{R}^n - \{\mathbf{0}\}$. Then we can join x and y by the straight line path between them if that path does not go through the origin.

Otherwise, we can choose a point z not on the straight line joining x and y, and take the broken line path from x to z and then from z to y.

$\therefore \mathbb{R}^n - \{\mathbf{0}\}$ *is path connected if $n > 1$.*

Note: $\mathbb{R} - \{0\}$ **is not path connected**.

Let Us Sum Up:

In this section, we have discussed the following concepts:

1. The least upper bound property
2. Linear continuum and its connectedness
3. Intermediate value theorem
4. Relation between path connectedness and connectedness

Check your Progress:

1. Which of the following is connected?

   (A) $\mathbb{R}_l$ (B) $\mathbb{R}^\omega$ with box topology

(C) $\mathbb{R}^\omega$ with product topology (D) None of these

2. Which of the following is not a linear continuum?

   (A) $\mathbb{R}$ (B) $\mathbb{R} - \{0\}$

   (C) ordered square I_o^2 (D)the set of non-negative reals

3. Which of the following is true?

   (A) Every connected space is path connected

   (B) $\mathbb{R} - \{0\}$ is path connected

   (C) The ordered square I_o^2 is not path connected

   (D) None of these

3.3 Components and Local Connectedness

Definition 3.3.1. *Given X, define an equivalence relation on X by setting $x \sim y$ if there is a connected subspace of X containing both x and y. The equivalence classes are called the **components** (or the "connected components") of X.*

Note : $\sim$ is an equivalence relation on X.

For,

Let $x \in X$. Then $x \sim x$ because $\{x\}$ is a connected subspace of X containing x.

Let $x, y \in X$ such that $x \sim y$.

Then $x \sim y \implies y \sim x$ is obvious by definition.

Let $x, y, z \in X$ such that $x \sim y$ and $y \sim z$.

Now, $x \sim y$ implies that there exists a connected subspace A of X containing x and y .

Next, $y \sim z$ implies that there exists a connected subspace B of X containing y and z .

Therefore, $A \cup B$ is a connected subspace of X containing x and z because $y \in A \cap B$.

$\implies x \sim z$.

Therefore, $\sim$ is an equivalence relation on X.

Theorem 3.3.2. ***The components of X are connected disjoint subspaces of X whose union is X, such that each nonempty connected subspace of X intersects only one of them.***

Proof. Being equivalence classes, the components of X are disjoint and their union is X.

Claim: Each connected subspace A of X intersects only one of them.

Suppose that A intersects the components C_1 and C_2 of X.

Let $x_1 \in A \cap C_1$ and $x_2 \in A \cap C_2$.

$\implies$ A is a connected subspace of X containing x_1 and x_2.

$\implies$ $x_1 \sim x_2$, by definition.

This happens only if $C_1 = C_2$.

i.e., A intersects exactly one component.

Hence our claim.

Next, we prove that component C is connected.

Choose a point x_0 of C. Then $[x_0] = C$.

For each point x of C, we know that $x_0 \sim x$, so there is a connected subspace A_x containing x_0 and x.

By the result just proved, $A_x \subset C$. Therefore,

$$C = \bigcup_{x \in C} A_x.$$

Since the subspaces A_x are connected and have the point x_0 in common, their union is connected.

$\therefore C$ is connected. $\square$

Definition 3.3.3. *Define an equivalence relation on the space X by defining $x \sim y$ if there is a path in X from x to y. The equivalence classes are called the* ***path components*** *of X.*

Note: Let us show this is an equivalence relation.

First we note that if there exists a path $f : [a, b] \to X$ from x to y whose domain is the interval $[a, b]$, then there is also a path g from x to y having the closed interval $[c, d]$ as its domain. (This follows from the fact that any two closed intervals in $\mathbb{R}$ are homeomorphic.)

Now the fact that $x \sim x$ for each x in X follows from the existence of the constant path $f : [a, b] \to X$ defined by the equation $f(t) = x$ for all t.

Symmetry follows from the fact that if $f : [0, 1] \to X$ is a path from x to y, then the "reverse path" $g : [0, 1] \to X$ defined by $g(t) = f(1 - t)$ is a path from y to x.

Finally transitivity is proved as follows: Let $f : [0, 1] \to X$ be a path from x to y, and let $g : [1, 2] \to X$ be a path from y to z. We can "paste f and g together" to get a path $h : [0, 2] \to X$ from x to z; the path h will be continuous by the "pasting lemma."

Theorem 3.3.4. ***The path components of X are path-connected disjoint subspaces of X whose union is X, such that each nonempty path-connected subspace of X intersects only one of them.***

Proof. The proof is similar to that of the preceding theorem. □

Note:

1. Each component of a space X is closed in X, since the closure of a connected subspace of X is connected.
2. If X has only finitely many components, then each component is also open in X, since its complement is a finite union of closed sets.
3. In general, the components of X need not be open in X.

Example 3.3.5. *If $\mathbb{Q}$ is the subspace of $\mathbb{R}$ consisting of the rational numbers, then each component of $\mathbb{Q}$ consists of a single point. None of the components of $\mathbb{Q}$ are open in $\mathbb{Q}$.*

Definition 3.3.6. *A space X is said to be* ***locally connected at x*** *if for every neighborhood U of x, there is a connected neighborhood V of x contained in U.*

If X is locally connected at each of its points, it is said simply to be ***locally connected****. Similarly, a space X is said to be* ***locally path connected at x*** *if for every neighborhood U of x, there is a path-connected neighborhood V of x contained in U.*

If X is locally path connected at each of its points, then it is said to be ***locally path connected.***

Example 3.3.7. *Each interval and each ray in the real line is both connected and locally connected.*

Example 3.3.8. *The subspace $[-1,0) \cup (0,1]$ of $\mathbb{R}$ is not connected, but it is locally connected.*

Example 3.3.9. *The topologist's sine curve is connected but not locally connected.*

Example 3.3.10. *The rationals $\mathbb{Q}$ are neither connected nor locally connected.*

Theorem 3.3.11. ***A space X is locally connected if and only if for every open set U of X, each component of U is open in X.***

Proof. Suppose that X is locally connected.

Let U be an open set in X and C be a component of U.

To prove: C is open in X.

If x is a point of C, then $x \in U$.

Since U is open in X such that $x \in U$, and since X is locally connected, we can choose a connected neighborhood V of x such that $V \subset U$.

Since V is connected, by Lemma 3.1.13, it must lie entirely in the component C of U.

i.e., $x \in V \subset C$.

Therefore, C is open in X.

Conversely, suppose that each component of every open set in X are open in X.

To prove: X is locally connected.

Consider a point x of X and a neighborhood U of x.

Let C be the component of U containing x.

Then by the hypothesis, C is open in X.

Also, C is connected by definition.

That is, we have a connected neighborhood C of x such that $C \subset U$.

$\therefore X$ is locally connected at x and hence X is locally connected. $\square$

A similar proof holds for the following theorem:

Theorem 3.3.12. ***A space X is locally path connected if and only if for every open set U of X, each path component of U is open in X.***

The relation between path components and components is given in the following theorem:

Theorem 3.3.13. ***If X is a topological space, each path component of X lies in a component of X. If X is locally path connected, then the components and the path components of X are the same.***

Proof. Let X be any arbitrary topological space.

Let C be a component of X and x be a point of C.

Let P be the path component of X containing x.

Then P is path connected.

$\implies P$ is connected.

$\implies P \subset C$.

That is, each path component lies in a component of X.

Claim: X is locally path connected $\implies P = C$.

Suppose that $P \subsetneq C$.

Let Q denote the union of all the path components of X that are different from P and intersect C.

Each of them necessarily lies in C, so that $C = P \cup Q$.

Since X is locally path connected, by Theorem 3.3.12, each path component of X is open in X.

Therefore, P (which is a path component) and Q (which is a union of path components) are open in X.

$\implies$ They constitute a separation of C.

This contradicts the fact that C is connected.

Hence, $P = C$. □

Let Us Sum Up:

In this section, we have discussed the following concepts:

1. Components
2. Path components
3. Locally connected space
4. Locally path connected space

Check your Progress:

1. Which of the following is not true in a topological space X?

   (A) Every component of X is connected

   (B) Every path component of X is connected

   (C) Every path component of X is path connected

   (D) Every component of X is open in X

2. Which of the following is locally connected?

   (A) $(0,1) \subset \mathbb{R}$ (B) $(1,\infty) \subset \mathbb{R}$

   (C) $[-1,0) \cup (0,1] \subset \mathbb{R}$ (D) All of these

3. Which of the following is true in a topological space X?

   (A) Every component of X is closed

   (B) Every component of X is open

   (C) Every component of X is both open and closed

   (D) None of these

Unit Summary:

This unit dealt with connected spaces and their properties, ways of constructing connected spaces, connected subspaces of the real line, path connectedness and the concepts of components and local connectedness.

Glossary:

- I_o^2 - Ordered square
- $\overline{B}_d(x,\epsilon)$ - Closed ball
- $\mathbb{R}^n - \{\mathbf{0}\}$ - The punctured Euclidean space

Self-Assessment Questions:

1. Prove that $\mathbb{R}$ is connected.

2. Is the space $\mathbb{R}_l$ connected? Justify your answer.

3. Show that $\mathbb{R}^n$ and $\mathbb{R}$ are not homeomorphic if $n > 1$.

4. Let $Y \subset X$; let X and Y be connected. Show that if A and B form a separation of $X - Y$, then $Y \cup A$ and $Y \cup B$ are connected.

5. Prove that $\mathbb{R}$ is a linear continuum.

Exercises:

1. Let X be an ordered set in the order topology. Show that if X is connected, then X is a linear continuum.

2. Prove that every open ball $B_d(x, \epsilon)$ and every closed ball $\bar{B}_d(x, \epsilon)$ in $\mathbb{R}^n$ is path connected.

3. Let X be locally path connected. Show that every connected open set in X is path connected.

4. Prove that the rationals $\mathbb{Q}$ are neither connected nor locally connected.

5. What are the components and path components of $\mathbb{R}^\omega$ (in the product topology)?

Answers for check your progress:

Section 3.1	1. (B)	2. (B)	3. (A)
Section 3.2	1. (C)	2. (B)	3. (C)
Section 3.3	1. (D)	2. (D)	3. (A)

Reference:

1. James R. Munkres, Topology (2nd Edition), Prentice Hall of India, New Delhi, 2011.

Suggested Readings:

1. J. Dugundji, Topology, Prentice Hall of India, New Delhi, 1975.

2. George F. Simmons, Introduction to Topology and Modern Analysis, McGraw Hill Book Co., 1963.

3. J.L. Kelley, General Topology, Van Nostrand, Reinhold Co., New York, 1955.

4. L. Steen and J. Subhash, Counter Examples in Topology, Holt, Rinehart and Winston, New York, 1970.

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UNIT 4

Unit 4

Compactness

Objectives:

This unit deals with compact spaces, compact subspaces of the real line, limit point compactness and local compactness.

4.1 Compact Spaces

Definition 4.1.1. *A collection $\mathcal{A}$ of subsets of a space X is said to* ***cover*** *X, or to be a* ***covering*** *of X, if the union of the elements of $\mathcal{A}$ is equal to X. It is called an* ***open covering*** *of X if its elements are open subsets of X.*

Definition 4.1.2. *A space X is said to be* ***compact*** *if every open covering $\mathcal{A}$ of X contains a finite subcollection that also covers X.*

Example 4.1.3. ***The real line $\mathbb{R}$ is not compact*** *because the covering of $\mathbb{R}$ by open intervals*

$$\mathcal{A} = \{(n, n+2) \mid n \in \mathbb{Z}\}$$

contains no finite subcollection that covers $\mathbb{R}$.

Example 4.1.4. ***The subspace $X = \{0\} \cup \{1/n \mid n \in \mathbb{Z}_+\}$ is compact.***

For, given an open covering $\mathcal{A}$ of X, there is an element U of $\mathcal{A}$ containing 0. The set U contains all but finitely many of the points $1/n$. For each point of X not in U, choose an element of $\mathcal{A}$ containing it. The collection consisting of these elements of $\mathcal{A}$, along with the element U, is a finite subcollection of $\mathcal{A}$ that covers X.

Example 4.1.5. ***Any finite point space is compact*** *because in this case every open covering of X is finite.*

Example 4.1.6. ***The interval*** $(0,1]$ ***is not compact*** *because the open covering*

$$\mathcal{A} = \{(1/n, 1] \mid n \in \mathbb{Z}_+\}$$

contains no finite subcollection covering $(0,1]$.

Definition 4.1.7. *If Y is a subspace of X, a collection $\mathcal{A}$ of subsets of X is said to* ***cover*** *Y if the union of its elements contains Y.*

Lemma 4.1.8. ***Let Y be a subspace of X. Then Y is compact if and only if every covering of Y by sets open in X contains a finite subcollection covering Y. (or) Let Y be a subspace of X. Then every covering of Y by sets open in Y contains a finite subcollection covering Y if and only if every covering of Y by sets open in X contains a finite subcollection covering*** Y.

Proof. Suppose that Y is compact and let $\mathcal{A} = \{A_\alpha\}_{\alpha \in J}$ be a covering of Y by sets open in X. Then $\{A_\alpha \cap Y : \alpha \in J\}$ is a covering of Y by sets open in Y. Since Y is compact, there exists a finite subcollection

$$\{A_{\alpha_1} \cap Y, A_{\alpha_2} \cap Y, ..., A_{\alpha_n} \cap Y\}$$

that covers Y.

$\implies \{A_{\alpha_1}, A_{\alpha_2}, ..., A_{\alpha_n}\}$ is a finite subcollection of $\mathcal{A}$ that covers Y.

Conversely, suppose that every covering of Y by sets open in X contains a finite subcollection covering Y.We have to prove that Y is compact.

Let $\mathcal{A}' = \{A'_\alpha\}$ be a covering of Y by sets open in Y.

$\implies A'_\alpha = A_\alpha \cap Y$ for some A_α open in X, for each α.

Then the collection $\mathcal{A} = \{A_\alpha\}$ is a covering of Y by sets open in X.

By hypothesis, there exists some finite subcollection $\{A_{\alpha_1}, A_{\alpha_2}, ..., A_{\alpha_n}\}$ that covers Y.

$\implies \{A'_{\alpha_1}, A'_{\alpha_2}, ..., A'_{\alpha_n}\}$ is a finite subcollection of $\mathcal{A}'$ that covers Y. Therefore, Y is compact.

□

Theorem 4.1.9. ***Every closed subspace of a compact space is compact.***

Proof. Let Y be a closed subspace of a compact space X. We have to prove that Y is compact.

Let $\mathcal{A}$ be a covering of Y by sets open in X.

Take $\mathcal{B} = \mathcal{A} \cup \{X - Y\}$.

Since $X - Y$ is open in X, the collection $\mathcal{B}$ is an open covering of X.

Then, there exists some finite subcollection of $\mathcal{B}$ that covers X because X is compact.

If this subcollection contains the set $X - Y$, remove it from the subcollection; otherwise, leave the subcollection as it is.

Thus, the resulting collection is a finite subcollection of $\mathcal{A}$ that covers Y.

Therefore, Y is compact. □

Theorem 4.1.10. ***Every compact subspace of a Hausdorff space is closed.***

Proof. Let Y be a compact subspace of a Hausdorff space. We have to prove that Y is closed.

That is, to prove that $X - Y$ is open.

Let $x_0 \in X - Y$.

If $y \in Y$, then $y \neq x_0$.

Thus, for each point $y \in Y$, we can choose disjoint neighborhoods U_y and V_y of x_0 and y, respectively, because X is Hausdorff.

Then, the collection $\{V_y : y \in Y\}$ is a covering of Y by sets open in X.

Since Y is compact, there exist finitely many of them $V_{y_1}, V_{y_2}, ...V_{y_n}$ so that they cover Y.

Take $V = \{V_{y_1} \cup V_{y_2} \cup ... \cup V_{y_n}\}$ and $U = \{U_{y_1} \cap U_{y_2} \cap ... \cap U_{y_n}\}$.

Clearly, V and U are open sets such that $Y \subset V$, and U is a neighborhood of x_0.

Claim : $U \cap V = \emptyset$.

Let $z \in V$.

$\implies z \in V_{y_i}$, for some i.

$\implies z \notin U_{y_i}$, because $V_{y_i} \cap U_{y_i} = \emptyset$.

$\implies z \notin U$, because $U = \{U_{y_1} \cap U_{y_2} \cap ... \cap U_{y_n}\}$.

Hence our claim.

$\implies U \cap Y = \emptyset$, because $Y \subset V$.

That is, U is a neighborhood of x_0 disjoint from Y.

Therefore, $x_0 \in U \subset X - Y$.

Hence, $X - Y$ is open, and hence Y is closed. □

Lemma 4.1.11. ***If Y is a compact subspace of the Hausdorff space X and x_0 is not in Y, then there exist disjoint open sets U and V of X containing x_0 and Y, respectively.***

Proof. This proof is a part of the previous theorem. □

Theorem 4.1.12. ***The image of a compact space under a continuous map is compact.***

Proof. Let $f : X \to Y$ be continuous and let X be compact.

Let $\mathcal{A}$ be a covering of the set $f(X)$ by sets open in Y.

The collection $\{f^{-1}(A) | A \in \mathcal{A}\}$ is a collection of sets covering X.

These sets are open in X because f is continuous.

Hence, finitely many of them, say $f^{-1}(A_1), ..., f^{-1}(A_n)$, cover X.

Then, the sets $A_1, A_2, ..., A_n$ cover $f(X)$. □

Theorem 4.1.13. ***Let $f : X \to Y$ be a bijective continuous function. If X is compact and Y is Hausdorff, then f is a homeomorphism.***

Proof. Given that $f : X \to Y$ is a bijective continuous function.

To prove: f is a homeomorphism.

It is enough to prove that $f^{-1} : Y \to X$ is continuous.

Let A be a closed set in X.

To prove: $f(A)$ is closed in Y.

Since X is compact and A is closed in X, A is compact.

Since f is continuous, $f(A)$ is compact.

Since Y is Hausdorff and $f(A)$ is a compact subspace of Y, we have $f(A)$ is closed in Y.

□

Theorem 4.1.14. ***The product of finitely many compact spaces is compact.***

Proof. We shall prove that the product of two compact spaces is compact. Then the theorem follows by induction for any finite product.

Step 1 :

Suppose that X and Y are two spaces with Y compact, $x_0 \in X$, and N is an open set of $X \times Y$, containing the slice $x_0 \times Y$ of $X \times Y$.

We prove that "there is a neighborhood W of x_0 in X such that N contains the entire set $W \times Y$" (Here $W \times Y$ is called a tube about $x_0 \times Y$).

First, let us cover the slice $x_0 \times Y$ by basis elements $U \times V$ lying in N.

Since $x_0 \times Y$ is homeomorphic to Y and since Y is compact, we have $x_0 \times Y$ is compact.

Therefore, $x_0 \times Y$ can be covered by finitely many such basis elements
$U_1 \times V_1, U_2 \times V_2, ..., U_n \times V_n$.

$$i.e., \quad x_0 \times Y \subset (U_1 \times V_1) \cup ... \cup (U_n \times V_n). \tag{4.1}$$

Define $W = U_1 \cap U_2 ... \cap U_n$.

Therefore, W is open in X.

Claim: The sets $U_i \times V_i$ covers the tube $W \times Y$.

Let $x \times y \in W \times Y$.

Consider the point $x_0 \times y \in x_0 \times Y$.

By (4.1), $x_0 \times y \in U_i \times V_i$, for some i.

$\implies y \in V_i$, for that i.

But $x \in U_j$, for every $j = 1, 2, ..., n$, because $x \in W$.

$\implies x \times y \in U_i \times V_i$.

Hence our claim.

Since all the sets $U_i \times V_i$ lie in N, and since they cover $W \times Y$, the tube $W \times Y$ lies in N also.

Step 2:

Now, we prove the main theorem.

Let X and Y be compact spaces.

To prove: $X \times Y$ is compact.

Let $\mathcal{A}$ be an open covering of $X \times Y$.

Given $x_0 \in X$, the slice $x_0 \times Y$ is a compact subset of $X \times Y$ and hence $x_0 \times y$ may be covered by finitely many elements $A_1, A_2, \ldots, A_m$ of $\mathcal{A}$.

Take $N = A_1 \cup A_2 U ... U A_m$.

Then, N is an open et in $X \times Y$ such that $x_0 \times y \subset N$.

$\therefore$ By step 1, there exists a neighborhood W of x_0 in X such that N contains the tube $W \times Y$ about $x_0 \times y$.

i.e., $W \times Y$ is covered by finitely many elements $A_1, A_2, \ldots, A_m$ of $\mathcal{A}$.

Thus, for each $x \in X$, we can choose a neighborhood W_x of x such that the tube $W_x \times Y$ can be covered by finitely many elements of $\mathcal{A}$.

Then, the collection $\{W_x : x \in X\}$ is an open covering of X.

$\therefore$ By the compactness of x, there exists a finite subcollection $\{W_1, W_2, \ldots, W_k\}$ covering X.

Then $(W_1 \times Y) \cup \ldots U(W_k \times Y) = X \times Y$ and each $W_i \times Y$ is covered by finitely many elements of $\mathcal{A}$.

i.e., $X \times Y$ can be covered by finitely many elements of $\mathcal{A}$.

$\Rightarrow X \times Y$ is compact. □

Lemma 4.1.15. ***[The tube lemma] Consider the product space $X \times Y$, where Y is compact. If N is an open set of $X \times Y$ containing the slice $x_0 \times Y$ of $X \times Y$, then N contains some tube $W \times Y$ about $x_0 \times Y$, where W is a neighborhood of x_0 in X.***

Proof. Proof of step 1. □

Definition 4.1.16. *The collection $\mathcal{C}$ of subsets of X is said to have the* ***finite intersection property*** *if for every finite subcollection $\{C_1, C_2, ..., C_n\}$ of $\mathcal{C}$, the intersection $C_1 \cap C_2 \cap ... \cap C_n$ is nonempty.*

Theorem 4.1.17. ***Let X be a topological space. Then X is compact if and only if for every collection $\mathcal{C}$ of closed sets in X having the finite intersection property, the intersection $\bigcap_{C \in \mathcal{C}} C$ of all the elements of $\mathcal{C}$ is nonempty***

Proof. Suppose that X is compact.

Let $\mathcal{C}$ be the collection of closed sets in X having the finite intersection property.

Claim: $\bigcap_{C \in \mathcal{C}} C \neq \emptyset$.

Suppose that $\bigcap_{C \in \mathcal{C}} C = \emptyset$.

$\implies X - \bigcap_{C \in \mathcal{C}} C = X$.

$\implies \bigcup_{C \in \mathcal{C}} (X - C) = X$. Then, $\{X - C : C \in \mathcal{C}\}$ is an open covering of X.

Since X is compact, there exists finitely many of them, say $X - C_1, \cdots, X - C_n$ such that they cover X.

That is, $\bigcup_{i=1}^{n} (X - C_i) = X$.

$\implies X - \bigcap_{i=1}^{n} C_i = X$.

$\implies \bigcap_{i=1}^{n} C_i = \emptyset$,

which is a contradiction to the fact that $\mathcal{C}$ has finite intersection property.

Hence our claim.

To prove the converse part, let $\mathcal{A}$ be an open covering of X.

Take $\quad \mathcal{C} = \{X - A : A \in \mathcal{A}\}$.

Then, $\mathcal{C}$ is a collection of closed sets in X.

Now, $\quad \bigcup_{A \in \mathcal{A}} A = X \implies X - \bigcup_{A \in \mathcal{A}} A = \emptyset$.

$\implies \bigcap_{A \in \mathcal{A}} (X - A) = \emptyset$.

$\therefore$ By hypothesis, $\mathcal{C}$ cannot have finite intersection property.

$\implies$ there exists finite number of members, say $X - A_1, X - A_2, ..., X - A_n$ in $\mathcal{C}$ so that $\bigcap_{i=1}^{n} (X - A_i) = \emptyset$.

$\implies X - \bigcup_{i=1}^{n} A_i = \emptyset$.

$\implies \bigcup_{i=1}^{n} A_i = X$.

That is, $\{A_1, A_2, ..., A_n\}$ covers X.

$\therefore X$ is compact. $\square$

Let Us Sum Up:

In this section, we have discussed the following concepts:

1. Compact spaces with examples
2. Constructing new compact spaces from the given ones
3. The tube lemma
4. Finite intersection property

Check your Progress:

1. Which of the following is not true?

   (A) Finite product of compact spaces is compact

   (B) arbitrary product of compact spaces is compact

   (C) homeomorphic image of a compact spaces is compact

   (D) None of these

2. Which of the following is not compact?

   (A) indiscrete topological space

   (B) finite complement topological space

   (C) infinite discrete topological space

   (D) finite topological space

3. If A and B are two compact subsets of a topological space X, then which of the following is not correct?

   (A) $A \times B$ is compact (B) $A \cup B$ is compact

   (C) $A \cap B$ is compact (D) None of these

4.2 Compact subspaces of the real line

Theorem 4.2.1. ***Let X be a simply ordered set having the least upper bound property. In the order topology, each closed interval in X is compact.***

Proof. Let $a, b \in X$ such that $a < b$.

To prove: $[a, b]$ is compact in X.

Let $\mathcal{A}$ be a covering of $[a, b]$ by sets open in $[a, b]$ in the subspace topology.

We shall prove that there exists a finite subcollection of $\mathcal{A}$ covering $[a, b]$.

We prove this by four steps:

Step 1

If $x \in [a, b]$ such that $x \neq b$, then we prove that there is a point $y > x$ in $[a, b]$ such that $[x, y]$ can be covered by atmost two elements of $\mathcal{A}$.

If x has an immediate successor y in X, then $[x, y]$ consists of only two points x and y.

So $[x, y]$ can be covered by at most two elements of $\mathcal{A}$.

If x has no immediate successor in X, choose an element A of $\mathcal{A}$ containing x.

Because $x \neq b$ and A is open, A contains an interval of the form $[x, c)$ for some c in $[a, b]$.

Choose a point y in (x, c).

Then the interval $[x, y]$ is covered by the single element A of $\mathcal{A}$.

Step 2

Let C be the set of all points $y > a$ of $[a, b]$ such that the interval $[a, y]$ can be covered by finitely many elements of $\mathcal{A}$.

Applying Step 1 to the case $x = a$, we see that there exists at least one such y, so C is not empty.

Thus, by hypothesis, C has the least upper bound.

Let c be the least upper bound of the set C.

Then $a < c \leq b$.

Step 3

We show that c belongs to C.

i.e., we show that the interval $[a, c]$ can be covered by finitely many elements of $\mathcal{A}$.

Choose an element A of $\mathscr{A}$ containing c.

Since A is open, it contains an interval of the form $(d, c]$ for some d in $[a, b]$.

If c is not in C, there must be a point z of C lying in the interval (d, c), because otherwise d would be a smaller upper bound on C than c.

Since z is in C, the interval $[a, z]$ can be covered by finitely many, say n, elements of $\mathcal{A}$.

Now $[z, c]$ lies in the single element A of $\mathcal{A}$, hence $[a, c] = [a, z] \cup [z, c]$ can be covered by $n + 1$ elements of $\mathcal{A}$.

Thus c is in C, contrary to the assumption.

Step 4

Finally, we show that $c = b$, and our theorem is proved.

Suppose that $c < b$.

Applying Step 1 to the case $x = c$, we conclude that there exists a point $y > c$ of $[a, b]$ such that the interval $[c, y]$ can be covered by finitely many elements of $\mathcal{A}$.

We proved in Step 3 that c is in C, so $[a, c]$ can be covered by finitely many elements of $\mathcal{A}$. Therefore, the interval

$$[a, y] = [a, c] \cup [c, y]$$

can also be covered by finitely many elements of $\mathcal{A}$.

This means that y is in C, which is a contradiction to the fact that c is an upper bound on C.

$\therefore c = b$.

i.e., $b \in C$.

i.e., $[a, b]$ can be covered by finitely many elements of $mathcal A$.

$\therefore [a, b]$ is compact. □

Corollary 4.2.2. ***Every closed interval in*** $\mathbb{R}$ ***is compact.***

Proof. Since $\mathbb{R}$ satisfies the hypothesis of the previous theorem, every closed interval in $\mathbb{R}$ is compact. □

Theorem 4.2.3. ***A subspace*** A ***of*** $\mathbb{R}^n$ ***is compact if and only if it is closed and is bounded in the euclidean metric*** d ***or the square metric*** ρ***.***

Proof. We know that the inequalities

$$\rho(x, y) \leq d(x, y) \leq \sqrt{n}\rho(x, y)$$

hold.

Then, the subspace A of $\mathbb{R}^n$ is bounded under d if and only if it is bounded under ρ.

So, it will suffice to consider only the metric ρ.

Suppose that A is compact. Then, by Theorem 4.1.10, it is closed.

Consider the collection of open sets

$$\{B_\rho(\mathbf{0}, m) \mid m \in \mathbb{Z}_+\}$$

whose union is all of $\mathbb{R}^n$.

Some finite subcollection covers A. It follows that $A \subset B_\rho(\mathbf{0}, M)$ for some M.

Therefore, for any two points x and y of A, we have $\rho(x, y) \leq 2M$.
Thus, A is bounded under ρ.

Conversely, suppose that A is closed and bounded under ρ.
Suppose that $\rho(x, y) \leq N$ for every pair x, y of points of A.
Choose a point x_0 of A, and let $\rho(x_0, \mathbf{0}) = b$.
The triangle inequality implies that $\rho(x, \mathbf{0}) \leq N + b$ for every x in A.
If $P = N + b$, then A is a subset of the cube $[-P, P]^n$, which is compact.
Being closed, A is also compact. □

Example 4.2.4. *The unit sphere S^{n-1} and the closed unit ball B^n in $\mathbb{R}^n$ are compact because they are closed and bounded.*

Example 4.2.5. *The set*

$$A = \{x \times (1/x) \mid 0 < x \leq 1\}$$

is closed in $\mathbb{R}^2$, but it is not compact because it is not bounded.

Example 4.2.6. *The set*

$$S = \{x \times (\sin(1/x)) \mid 0 < x \leq 1\}$$

is bounded in $\mathbb{R}^2$, but it is not compact because it is not closed.

Theorem 4.2.7. ***(Extreme value theorem). Let $f : X \to Y$ be continuous, where Y is an ordered set in the order topology. If X is compact, then there exist points c and d in X such that $f(c) \leq f(x) \leq f(d)$ for every $x \in X$.***

Proof. Since f is continuous and X is compact, the set $A = f(X)$ is compact.

We show that A has a largest element M and a smallest element m.

Then, since m and M belong to A, we must have $m = f(c)$ and $M = f(d)$ for some points c and d of X.

If A has no largest element, then the collection

$$\{(-\infty, a) \mid a \in A\}$$

forms an open covering of A.

Since A is compact, some finite subcollection

$$\{(-\infty, a_1), \ldots, (-\infty, a_n)\}$$

covers A.

If a_i is the largest of the elements $a_1, \ldots a_n$, then a_i belongs to none of these sets, contrary to the fact that they cover A.

Similarly, we can prove that A has a smallest element m, and hence there exists $c \in X$ such that $m = f(c)$ and $f(c) \leq f(x), \forall x \in X$.

$$\implies f(c) \leq f(x) \leq f(d), \forall x \in X.$$

□

Note: The extreme value theorem of calculus is the special case of this theorem that occurs when we take X to be a closed interval in $\mathbb{R}$ and Y to be $\mathbb{R}$.

Definition 4.2.8. *Let (X, d) be a metric space. Let A be a nonempty subset of X. For each $x \in X$, we define the* ***distance from x to A*** *by the equation*

$$d(x, A) = \inf\{d(x, a) \mid a \in A\}$$

Remark 4.2.9. ***For fixed A, the function $d(x, A)$ is a continuous function of x.***

Proof. Given $x, y \in X$, one has the inequalities

$$d(x, A) \leq d(x, a) \leq d(x, y) + d(y, a)$$

for each $a \in A$.

It follows that

$$d(x, A) - d(x, y) \leq \inf d(y, a) = d(y, A)$$

so that

$$d(x, A) - d(y, A) \leq d(x, y).$$

The same inequality holds with x and y interchanged.

Thus, the continuity of the function $d(x, A)$ follows. □

Definition 4.2.10. *The* ***diameter*** *of a bounded subset A of a metric space (X, d) is the number*

$$\sup \{d(a_1, a_2) \mid a_1, a_2 \in A\}.$$

Lemma 4.2.11. ***(The Lebesgue number lemma). Let $\mathcal{A}$ be an open covering of the metric space (X, d). If X is compact, there is a $\delta > 0$ such that for each subset of X having diameter less than δ, there exists an element of $\mathscr{A}$ containing it.***

The number δ is called a ***Lebesgue number*** *for the covering $\mathcal{A}$.*

Proof. Let $\mathcal{A}$ be an open covering of X.

If X itself is an element of $\mathcal{A}$, then any positive number is a Lebesgue number for $\mathcal{A}$.

So assume X is not an element of $\mathscr{A}$.
Choose a finite subcollection $\{A_1, \ldots, A_n\}$ of $\mathcal{A}$ that covers X.
For each i, set $C_i = X - A_i$, and define $f : X \to \mathbb{R}$ by letting $f(x)$ be the average of the numbers $d(x, C_i)$. i.e.,

$$f(x) = \frac{1}{n} \sum_{i=1}^{n} d(x, C_i).$$

We show that $f(x) > 0$ for all x.
Given $x \in X$, choose i so that $x \in A_i$.
Then, choose ϵ so that the ϵ-neighborhood of x lies in A_i.
Then, $d(x, C_i) \geq \epsilon$, so that $f(x) \geq \epsilon/n$.
Since f is continuous, it has a minimum value δ.
We show that δ is our required Lebesgue number.

Let B be a subset of X of diameter less than δ.
Choose a point x_0 of B.
Then, B lies in the δ-neighborhood of x_0. Now,

$$\delta \leq f(x_0) \leq d(x_0, C_m),$$

where $d(x_0, C_m)$ is the largest of the numbers $d(x_0, C_i)$.
Then, the δ-neighborhood of x_0 is contained in the element $A_m = X - C_m$ of the covering $\mathscr{A}$. □

Definition 4.2.12. *A function f from the metric space (X, d_X) to the metric space (Y, d_Y) is said to be* ***uniformly continuous*** *if given $\epsilon > 0$, there is a $\delta > 0$ such that for every pair of points x_0, x_1 of X,*

$$d_X(x_0, x_1) < \delta \Longrightarrow d_Y(f(x_0), f(x_1)) < \epsilon.$$

Theorem 4.2.13. ***(Uniform continuity theorem). Let $f : X \to Y$ be a continuous map of the compact metric space (X, d_X) to the metric space (Y, d_Y). Then f is uniformly continuous.***

Proof. Given $\epsilon > 0$, take the open covering of Y by balls $B(y, \epsilon/2)$ of radius $\epsilon/2$.
Let $\mathscr{A}$ be the open covering of X by the inverse images of these balls under f.
Choose δ to be a Lebesgue number for the covering $\mathcal{A}$.
Then, if x_1 and x_2 are two points of X such that $d_X(x_1, x_2) < \delta$, the two-point set $\{x_1, x_2\}$ has diameter less than δ, so that its image $\{f(x_1), f(x_2)\}$ lies in some ball $B(y, \epsilon/2)$. Then

$$d_Y(f(x_1), f(x_2)) < \epsilon.$$

□

Definition 4.2.14. *If X is a space, a point x of X is said to be an* ***isolated point*** *of X if the one-point set $\{x\}$ is open in X.*

Theorem 4.2.15. ***Let X be a nonempty compact Hausdorff space. If X has no isolated points, then X is uncountable.***

Proof. Given that X is a nonempty compact Hausdorff space.
Step 1: We show first that given any nonempty open set U of X and any point x of X, there exists a nonempty open set V contained in U such that $x \notin \bar{V}$.

Choose a point y of U different from x. This is possible if x is in U because x is not an isolated point of X.

Also, it is possible if x is not in U simply because U is nonempty.

Now, choose disjoint open sets W_1 and W_2 about x and y, respectively.

Then, the set $V = W_2 \cap U$ is the desired open set.

It is contained in U, it is nonempty because it contains y, and its closure does not contain x.

Step 2: We show that given $f : \mathbb{Z}_+ \to X$, the function f is not surjective. It follows that X is uncountable.

Define $f(n) = x_n, \forall n \in \mathbb{Z}_+$.

Take $U = X$ and $x_1 \in X$.

Then, by Step 1, there exists a nonempty open set $V_1 \subset U$ such that $x_1 \notin \overline{V}$.

In general, given nonempty open set V_{n-1}, we can choose a nonempty open set V_n such that $V_n \subset V_{n-1}$ and $x_n \notin \overline{V}_n$.

Consider the nested sequence

$$\overline{V}_1 \supset \overline{V}_2 \supset \ldots$$

of nonempty closed sets of X.

Clearly, these closed sets have the finite intersection property.

Since X is compact, we have

$$\bigcap_{n \in \mathbb{Z}_+} \overline{V}_n \neq \emptyset,$$

and hence there is a point

$$x \in \bigcap_{n \in \mathbb{Z}_+} \overline{V}_n.$$

$\implies x \in \overline{V}_n, \forall n \in \mathbb{Z}_+$.

$\implies x \neq x_n, \forall n \in \mathbb{Z}_+,$ because $x_n \notin \overline{V}_n, \forall n \in \mathbb{Z}_+$.

$\implies x$ has no pre image under f.

$\implies f$ is not surjective.

This shows that X is uncountable. □

Corollary 4.2.16. ***Every closed interval in $\mathbb{R}$ is uncountable.***

Proof. We know that, every closed interval in $\mathbb{R}$ is compact.

Since $\mathbb{R}$ is Hausdorff, as a subspace of $\mathbb{R}$, any closed interval in $\mathbb{R}$ is Hausdorff, and it has no isolated points.

$\therefore$ By the above theorem, every closed interval in $\mathbb{R}$ is uncountable. □

Let Us Sum Up:

In this section, we have discussed the following concepts:

1. Some compact subspaces of the real line
2. Extreme value theorem
3. The Lebesgue number lemma
4. Uniform continuity theorem

Check your Progress:

1. Which of the following is compact?

   (A) $\mathbb{R}$ with standard topology

   (B) $\mathbb{R}$ with finite complement topology

   (C) the subspace $(1, 3]$ of $\mathbb{R}$

   (D) $\mathbb{R}$ with discrete topology

2. Which of the following is not true for $(0, 1]$ in $\mathbb{R}$ with finite complement topology?

   (A) not closed (B) bounded

   (C) not compact (D) None of these

3. If A is a nonempty compact subset of $\mathbb{R}$, then which of the following is not true always?

   (A) A is a closed set (B) A is a closed interval

   (C) A is not open (D) None of these

4.3 Limit Point Compactness

Definition 4.3.1. *A space X is said to be* ***limit point compact*** *if every infinite subset of X has a limit point.*

Theorem 4.3.2. ***Compactness implies limit point compactness, but not conversely.***

Proof. Let X be a compact space.

Given a subset A of X, we wish to prove that if A is infinite, then A has a limit point.

We prove the contrapositive - if A has no limit point, then A must be finite.
So, suppose that A has no limit point.
Then A contains all its limit points, so that A is closed.

Furthermore, for each $a \in A$, we can choose a neighborhood U_a of a such that U_a intersects A in the point a alone.

The space X is covered by the open set $X - A$ and the open sets U_a; being compact, it can be covered by finitely many of these sets.

Since $X - A$ does not intersect A, and each set U_a contains only one point of A, the set A must be finite. □

Remark: Limit point compactness does not imply compactness.

For,

Let Y consist of two points; give Y the topology consisting of Y and the empty set.

Then the space $X = \mathbb{Z}_+ \times Y$ is limit point compact, for every nonempty subset of X has a limit point.

It is not compact, for the covering of X by the open sets $U_n = \{n\} \times Y$ has no finite subcollection covering X.

Definition 4.3.3. *Let X be a topological space. If (x_n) is a sequence of points of X, and if*

$$n_1 < n_2 < \cdots < n_i < \cdots$$

is an increasing sequence of positive integers, then the sequence (y_i) defined by setting $y_i = x_{n_i}$ is called a ***subsequence*** *of the sequence (x_n). The space X is said to be* ***sequentially compact*** *if every sequence of points of X has a convergent subsequence.*

Theorem 4.3.4. ***Let X be a metrizable space. Then the following are equivalent:***

1. ***X is compact.***

2. ***X is limit point compact.***

3. ***X is sequentially compact.***

Proof. We have already proved that (1) ⇒ (2).

To show that (2) ⇒ (3), assume that X is limit point compact.

Given a sequence (x_n) of points of X, consider the set $A = \{x_n \mid n \in \mathbb{Z}_+\}$.

If the set A is finite, then there is a point x such that $x = x_n$ for infinitely many values of n.

In this case, the sequence (x_n) has a subsequence that is constant, and therefore converges trivially.

On the other hand, if A is infinite, then A has a limit point x.

We define a subsequence of (x_n) converging to x as follows:

First choose n_1 so that

$$x_{n_1} \in B(x, 1).$$

Then suppose that the positive integer n_{i-1} is given.

Because the ball $B(x, 1/i)$ intersects A in infinitely many points, we can choose an index $n_i > n_{i-1}$ such that

$$x_{n_i} \in B(x, 1/i)$$

Then the subsequence $x_{n_1}, x_{n_2}, \ldots$ converges to x.

Finally, we show that (3) ⇒ (1). This proof consists of 3 parts.

First, we show that if X is sequentially compact, then the Lebesgue number lemma holds for X.

Let $\mathscr{A}$ be an open covering of X.

We assume that there is no $\delta > 0$ such that each set of diameter less than δ has an element of $\mathscr{A}$ containing it, and derive a contradiction.

Our assumption implies in particular that for each positive integer n, there exists a set of diameter less than $1/n$ that is not contained in any element of $\mathcal{A}$; let C_n be such a set.

Choose a point $x_n \in C_n$, for each n. By hypothesis, some subsequence (x_{n_i}) of the sequence (x_n) converges, say to the point a.

Now a belongs to some element A of the collection $\mathcal{A}$; because A is open, we may choose an $\epsilon > 0$ such that $B(a, \epsilon) \subset A$.

If i is large enough that $1/n_i < \epsilon/2$, then the set C_{n_i} lies in the $\epsilon/2$-neighborhood of x_{n_i}; if i is also chosen large enough that $d(x_{n_i}, a) < \epsilon/2$, then C_{n_i} lies in the ϵ-neighborhood of a.

But this means that $C_{n_i} \subset A$, contrary to hypothesis.

Second, we show that if X is sequentially compact, then given $\epsilon > 0$, there exists a finite covering of X by open ϵ-balls.

Once again, we proceed by contradiction. Assume that there exists an $\epsilon > 0$ such that X cannot be covered by finitely many ϵ-balls.

Construct a sequence of points x_n of X as follows:

First, choose x_1 to be any point of X. Noting that the ball $B(x_1, \epsilon)$ is not all of X (otherwise X could be covered by a single ϵ-ball), choose x_2 to be a point of X not in $B(x_1, \epsilon)$.

In general, given $x_1, \ldots, x_n$, choose x_{n+1} to be a point not in the union

$$B(x_1, \epsilon) \cup \cdots \cup B(x_n, \epsilon)$$

using the fact that these balls do not cover X.

Note that by construction $d(x_{n+1}, x_i) \geq \epsilon$ for $i = 1, \ldots, n$.

Therefore, the sequence (x_n) can have no convergent subsequence; in fact, any ball of radius $\epsilon/2$ can contain x_n for at most one value of n.

Finally, we show that if X is sequentially compact, then X is compact.

Let $\mathscr{A}$ be an open covering of X.

Because X is sequentially compact, the open covering $\mathscr{A}$ has a Lebesgue number δ.

Let $\epsilon = \delta/3$; use sequential compactness of X to find a finite covering of X by open $\epsilon-$balls.

Each of these balls has diameter of at most $2\delta/3$, so it lies in an element of $\mathscr{A}$.

Choosing one such element of $\mathscr{A}$ for each of these $\epsilon-$ balls, we obtain a finite subcollection of $\mathscr{A}$ that covers X. □

Let Us Sum Up:

In this section, we have discussed the following concepts:

1. Limit point compactness

2. Sequentially compactness

3. Relation between compactness, limit point compactness and sequentially compactness in metrizable spaces

Check your Progress:

1. If X is a metrizable space, then

   (A) X is compact (B) X is limit point compact

   (C) X is sequentially compact (D) None of these

2. If X is a compact space, then

   (A) X is limit point compact, and conversely

   (B) X is sequentially compact, and conversely

   (C) both (a) and (b)

   (D) None of these

3. A topological space X is limit point compact if

   (A) every closed subset of X has a limit point

   (B) every infinte subset of X has a limit point

   (C) every sequence has a limit

   (D) All of these

4.4 Local Compactness

Definition 4.4.1. *The space X is said to be* ***locally compact at*** x *if there is some compact subspace C of X that contains a neighbourhood of x.*

If X is locally compact at each of its points, then X is said to be ***locally compact****.*

Note: Every compact space is locally compact.

For,

If X is compact, then every open cover of X has a finite subcollection, say $\{U_1, U_2, ..., U_n\}$, that covers X.

Let $x \in X$. Then $x \in U_j$ for some j.

$\therefore X$ is a compact space such that it contains the neighborhood U_j of x.

i.e., X is locally compact.

Example 4.4.2. ***The real line $\mathbb{R}$ is locally compact.***

For,

Let $x \in \mathbb{R}$. Then x belongs to some open interval (a, b) in $\mathbb{R}$.

$\therefore [a, b]$ is the compact subspace of $\mathbb{R}$ containing the neighborhood (a, b) of x.

$\implies \mathbb{R}$ is locally compact.

Remark: The above example shows that a locally compact space need not be compact.

Example 4.4.3. ***The space $\mathbb{R}^n$ is locally compact.***

For,

Let $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$.

Let $(a_1, b_1) \times (a_2, b_2) \times ... \times (a_n, b_n)$ be any basis element in $mathbbR^n$ containing x.

Then $[a_1, b_1] \times [a_2, b_2] \times ... \times [a_n, b_n]$ is the compact subspace of $\mathbb{R}^n$ containing the neighborhood $(a_1, b_1) \times (a_2, b_2) \times ... \times (a_n, b_n)$ of x.

$\therefore \mathbb{R}^n$ is locally compact.

Example 4.4.4. ***The space $\mathbb{R}^\omega$ is not locally compact.***

For,

Let $x = (x_\alpha)_{\alpha \in \omega} \in \mathbb{R}^\omega$.

Let $B = (a_1, b_1) \times (a_2, b_2) \times ... \times (a_n, b_n) \times \mathbb{R} \times ... \times \mathbb{R} \times ...$ be any basis element containing x.

If there exists a compact subspace A such that $B \subset A$, then A is closed, because $\mathbb{R}^\omega$ is Hausdorff.

But $\overline{B}$ is the smallest closed set containing B.

$\implies \overline{B} \subset A$.

$\implies \overline{B}$ must be compact, but $\overline{B} = [a_1, b_1] \times [a_2, b_2] \times ... \times [a_n, b_n] \times \mathbb{R} \times ... \times \mathbb{R} \times ...$ is not compact.

$\therefore B$ is not contained in any compact subspace of $\mathbb{R}^\omega$.

i.e., $\mathbb{R}^\omega$ is not locally compact.

Example 4.4.5. ***Every simply ordered set X having the least upper bound property is locally compact****, because given a basis element for X, it is contained in a closed interval in X, which is compact.*

Theorem 4.4.6. ***Let X be a space. Then X is locally compact Hausdorff if and only if there exists a space Y satisfying the following conditions:***

1. ***X is a subspace of Y.***
2. ***The set $Y - X$ consists of a single point.***
3. ***Y is a compact Hausdorff space.***

If Y and Y' are two spaces satisfying these conditions, then there is a homeomorphism of Y with Y' that equals the identity map on X.

Proof. **Step 1.**

We first verify uniqueness.

Let Y and Y' be two spaces satisfying these conditions.

Define $h : Y \to Y'$ by letting h map the single point p of $Y - X$ to the point q of $Y' - X$, and letting h equal the identity on X.

We show that if U is open in Y, then $h(U)$ is open in Y'.

Symmetry then implies that h is a homeomorphism.

First, consider the case where U does not contain p.

Then $h(U) = U$.

Since U is open in Y and is contained in X, it is open in X.

Because X is open in Y', the set U is also open in Y', as desired.

Second, suppose that U contains p.

Since $C = Y - U$ is closed in Y, it is compact as a subspace of Y.

Because C is contained in X, it is a compact subspace of X.

Then, because X is a subspace of Y', the space C is also a compact subspace of Y'.

Because Y' is Hausdorff, C is closed in Y', so that $h(U) = Y' - C$ is open in Y', as desired.

Step 2 :

Now we suppose X is locally compact Hausdorff and construct the space Y.

Step 1 gives us an idea how to proceed.

Let us take some object that is not a point of X, denote it by the symbol ∞ for convenience, and adjoin it to X, forming the set $Y = X \cup \{\infty\}$.

Topologize Y by defining the collection of open sets of Y to consist of (1) all sets U that are open in X, and (2) all sets of the form $Y - C$, where C is a compact subspace of X.

We need to check that this collection is, in fact, a topology on Y.

The empty set is a set of type (1), and the space Y is a set of type (2).

Checking that the intersection of two open sets is open involves three cases:

$$U_1 \cap U_2 \text{ is of type } (1).$$

$$(Y - C_1) \cap (Y - C_2) = Y - (C_1 \cup C_2) \text{ is of type (2).}$$

$$U_1 \cap (Y - C_1) = U_1 \cap (X - C_1) \text{ is of type (1)}$$

because C_1 is closed in X.

Similarly, one checks that the union of any collection of open sets is open:

$$\begin{aligned} &\bigcup U_\alpha = U \quad \text{ is of type (1).} \\ &U\,(Y - C_\beta) = Y - \left(\bigcap C_\beta\right) = Y - C \quad \text{ is of type (2).} \\ &\left(\bigcup U_\alpha\right) \cup \left(\bigcup (Y - C_\beta)\right) = U \cup (Y - C) = Y - (C - U), \end{aligned} \tag{2}$$

which is of type (2) because $C - U$ is a closed subspace of C and therefore compact.

Now we show that X is a subspace of Y.

Given any open set of Y, we show its intersection with X is open in X.

If U is of type (1), then $U \cap X = U$.

If $Y - C$ is of type (2), then $(Y - C) \cap X = X - C$.

Both of these sets are open in X.

Conversely, any set open in X is a set of type (1) and therefore open in Y by definition.

To show that Y is compact, let $\mathscr{A}$ be an open covering of Y.

The collection $\mathscr{A}$ must contain an open set of type (2), say $Y - C$, since none of the open sets of type (1) contain the point ∞.

Take all the members of $\mathcal{A}$ different from $Y - C$ and intersect them with X.

They form a collection of open sets of X covering C.

Because C is compact, finitely many of them cover C.

The corresponding finite collection of elements of $\mathcal{A}$ will, along with the element $Y - C$, cover all of Y.

To show that Y is Hausdorff, let x and y be two points of Y.

If both of them lie in X, there are disjoint sets U and V open in X containing them, respectively.

On the other hand, if $x \in X$ and $y = \infty$, we can choose a compact set C in X containing a neighborhood U of x.

Then U and $Y - C$ are disjoint neighborhoods of x and ∞, respectively, in Y.

Step 3:

Finally, we prove the converse.

Suppose a space Y satisfying conditions (1)-(3) exists.

Then X is Hausdorff because it is a subspace of the Hausdorff space Y.

Given $x \in X$, we show X is locally compact at x.

Choose disjoint open sets U and V of Y containing x and the single point of $Y - X$, respectively.

Then the set $C = Y - V$ is closed in Y, so it is a compact subspace of Y.

Since C lies in X, it is also compact as a subspace of X.

It contains the neighborhood U of x.

$\implies X$ is locally compact at x.

Since $x \in X$ is arbitrary, X is locally compact. $\square$

Definition 4.4.7. *If Y is a compact Hausdorff space and X is a proper subspace of Y whose closure equals Y, then Y is said to be a **compactification** of X.*

*If $Y - X$ equals a single point, then Y is called the **one point compactification of** X.*

Note: In Theorem 4.4.6, we have shown that X has the one point compactification $Y \iff X$ is locally compact, Hausdorff space, that is, not itself compact.

Example 4.4.8. *We know that $X = (0, 1)$ with the usual topology is not compact. Take $Y = [0, 1]$, Clearly, $\overline{X} = Y$, and Y is the compact Hausdorff space.*

$\implies Y$ is the compactification of X.

Example 4.4.9. *We know that $X = [0, 1)$ with the usual topology is not compact. Take $Y = [0, 1]$, Clearly, $\overline{X} = Y$, and $Y - X = \{1\}$.*
$\implies$ Y is the one point compactification of X.

Theorem 4.4.10. ***Let X be a Hausdorff space. Then X is locally compact $\iff$ given x in X and given a neighborhood U of x, there is a neighborhood V of x such that $\overline{V}$ is compact and $\overline{V} \subset U$.***

Proof. Suppose that X is locally compact. Let $x \in X$ and U be any neighborhood of x.

Then, X has a one point compactification Y.

Take $C = Y - U$.

Since U is open in X and X is open in Y, we have U is open in Y.

$\therefore$ C is closed in Y.

Thus, C is a compact subspace of Y, because Y is compact.

Also, $x \notin C$.

$\therefore$ There exist disjoint open sets V and W of x and C, respectively. Thus, $\overline{V}$ in Y is compact.

Claim: $\overline{V} \cap C = \emptyset$.

Suppose that $y \in \overline{V} \cap C$.

$\implies$ $y \in \overline{V}$ and $y \in C$.

$\therefore$ $y \in \overline{V}$ and $y \in W$, because $C \subset W$.

Since $y \in \overline{V}$, the neighborhood W of y must intersect V which is impossible, because $V \cap W = \emptyset$.

Hence our claim.

$\therefore$ $\overline{V} \subset U$.

Conversely, given $x \in X$ and given a neighborhood U of x, there is a neighborhood V of x such that $\overline{V}$ is compact and $\overline{V} \subset U$.

Since $V \subset \overline{V}$, $\overline{V}$ is the required compact set containing a neighborhood V of x.

$\implies$ X is locally compact. □

Corollary 4.4.11. ***Let X be a locally compact Hausdorff and A be a subspace of X. If A is closed in X (or) open in X, then A is locally compact.***

Proof. Given that X is locally compact and Hausdorff.

Case 1:

Suppose that A is closed in X.

Let $x \in A$. Then $x \in X$.

Since X is locally compact, there exists a compact subspace C of X containing the neighborhood U of x in X.

Since A is closed in X, $A \cap C$ is closed in C.

$\implies A \cap C$ is compact.

Now, $U \subset C \implies A \cap U \subset A \cap C$.

Also, $A \cap U$ is a neighborhood of x in A.

$\therefore$ We have a compact subspace $A \cap C$ of A containing the neighborhood $A \cap U$ of x in A.

$\implies A$ is locally compact.

Case 2:

Suppose that A is ope in X. Let $x \in A$.

Then $x \in X$ and A is a neighborhood of x in X.

Therefore, by previous theorem, there exists a neighborhood V of x in X such that $\overline{V}$ is compact and $\overline{V} \subset A$.

Thus, $\overline{V}$ is the required compact subspace of A containing the neighborhood V of x in A, because $V \cap A = V$.

$\implies A$ is locally compact. □

Corollary 4.4.12. ***A space X is homeomorphic to an open subspace of a compact Hausdorff space if and only if X is locally compact Hausdorff.***

Proof. Suppose that X is locally compact and Hausdorff.

Then by Theorem 4.4.6, there exists a one point compactification Y of X.

Thus, Y is compact Hausdorff space such that X is homeomorphic to the open subspace $Y - \{p\}$, where $p \notin X$.

Conversely, suppose that X is homeomorphic to an open subspace A of a compact Hausdorff space Y.

$\implies Y$ is locally compact also.

Then by previous corollary, A is locally compact.

Also, since Y is Hausdorff, A is Hausdorff.

Since X is homeomorphic to A, X is locally compact, Hausdorff. □

Let Us Sum Up:

In this section, we have discussed the following concepts:

1. Locally compact space with examples
2. Compactification
3. Equivalent condition for local compactness

Check your Progress:

1. Which of the following is not locally compact?

   (A) $\mathbb{R}$ (B) $\mathbb{R}^2$

   (C) $\mathbb{R}^\omega$ (D) None of these

2. If X is a locally compact Hausdorff space, then which of the following is true?

   (A) There exists a space Y such that X is a subspace of Y

   (B) There exists a space Y such that the set $Y - X$ consists of a single point

   (C) There exists a compact Hausdorff space Y

   (D) All of these

3. Which of the following is true?

   (A) Every compact space is locally compact

   (B) Every locally compact space is compact

   (C) $\mathbb{R}^\omega$ is locally compact

   (D) All of these

Unit Summary:

This unit dealt with compact spaces with examples, ways of constructing new compact spaces out of existing ones, identifying compact subspaces of the real line, characterization of compact subspaces of $\mathbb{R}^n$, relation between compactness, limit point compactness and sequentially compactness, and local compactness.

Glossary:

- Open cover of X - A cover of X with open sets in X
- $d(x, A)$ - Distance from x to A
- x is an isolated point of X - The one-point set $\{x\}$ is open in X

Self-Assessment Questions:

1. Show that the rationals $\mathbb{Q}$ are not locally compact.
2. Show that a finite union of compact subspaces of X is compact.
3. Show that if $f : X \to Y$ is continuous, where X is compact and Y is Hausdorff, then f is a closed map.
4. Show by an example that limit point compactness needn't imply compactness.

Exercises:

1. Prove that every subset of $\mathbb{R}$ is compact in the finite complement topology.
2. Prove that if X is an ordered set in which every closed interval is compact, then X has the least upper bound property.
3. Show that $[0, 1]$ is not limit point compact as a subspace of $\mathbb{R}_l$.
4. Let X be limit point compact. If $f : X \to Y$ is continuous, does it follow that $f(X)$ is limit point compact?
5. Let X be a locally compact space. If $f : X \to Y$ is continuous, does it follow that $f(X)$ is locally compact?

Answers for check your progress:

Section 4.1	1. (D)	2. (C)	3. (B)
Section 4.2	1. (B)	2. (D)	3. (B)
Section 4.3	1. (D)	2. (D)	3. (B)
Section 4.4	1. (C)	2. (D)	3. (A)

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UNIT 5

Unit 5

Countability and Separation Axioms

Objectives:

This unit provides introduction to some countability and separation axioms together with their consequences. The main aim of this unit is to prove the Urysohn Metrization Theorem and the Tietze Extension Theorem.

5.1 Countability Axioms

Definition 5.1.1. *The space X is said to have a* ***countable basis at*** x *if there is a countable collection $\mathscr{B} = \{U_n\}_{n\in\mathbb{Z}_+}$ of neighborhoods of x such that each neighborhood U of x contains atleast one of the elements U_n of $\mathscr{B}$.*

Definition 5.1.2. *A space that has a countable basis at each of its points is said to satisfy the* ***first countability axiom (or) to be first countable.***

Example 5.1.3. ***Every metrizable space is first countable.***

Proof. Let X be metrizable and let $x \in X$.

Let $\mathscr{B} = \{B_d(x, 1/n) : n \in \mathbb{Z}_+\}$.

Clearly, $\mathscr{B}$ is a countable collection of neighborhoods of x.

Now, consider any neighborhood $B_d(x, r)$ of x.

If $r > 1$, then every element of $\mathscr{B}$ is contained in $B_d(x, r)$.

If $r \leq 1$, then clearly atleast one of the member of $\mathscr{B}$ is contained in $B_d(x, r)$.

$\implies$ $\mathscr{B}$ is a countable basis at x.

Since $x \in X$ is arbitrary, X is first countable. $\square$

Theorem 5.1.4. ***Let X be a topological space.***

(a) ***Let A be a subset of X. If there is a sequence of points of A converging to x, then $x \in \bar{A}$. The converse holds if X is first countable.***

(b) ***If $f : X \to Y$ is continuous, then for every convergent sequence $x_n \to x$ in X, the sequence $(f(x_n))$ converges to $f(x)$. The converse holds if X is first countable.***

Proof. The proof is a direct generalization of the proof given under the hypothesis of metrizability. □

Definition 5.1.5. *If a space X has a countable basis for its topology, then X is said to satisfy the* ***second countability axiom****, or to be* ***second countable.***

Example 5.1.6. $\mathbb{R}$ *is second countable, since $\mathscr{B} = \{(a, b) : a, b \in \mathbb{Q}\}$ is a countable basis for* $\mathbb{R}$.

Example 5.1.7. $\mathbb{R}^n$ *is second countable because it has countable basis consisting of all products of open intervals having rational end points.*

Example 5.1.8. $\mathbb{R}^\omega$ *is second countable because the collection of all products $\prod_{n \in \mathbb{Z}_+} U_n$, where U_n is an open interval with rational end points for finitely many values of n and $U_n = \mathbb{R}$ for all other values of n, is a countable basis for $\mathbb{R}^\omega$.*

Remark 5.1.9. ***Second countability implies first countability.***

Proof. Suppose that X is second countable.

Then there exists a countable basis $\mathscr{B}$ for the topology of X.

Let $x \in X$.

Then the subcollection of $\mathscr{B}$ consisting of those basis elements containing the point x is a countable basis of X.

$\implies$ X is first countable. □

Theorem 5.1.10.

(a) ***A subspace of a first-countable space is first-countable, and a countable product of first-countable spaces is first-countable.***

(b) A subspace of a second countable space is second-countable, and a countable product of second-countable spaces is second-countable.

Proof. (a) Let X be a first countable space, and let Y be a subspace of X.

To prove: Y is a first countable space.

Let $y \in Y$.

Since $Y \subset X$, $y \in X$.

Since X is first countable, there exists a countable basis at y.

i.e., there exists a collection $\mathscr{B} = \{U_n\}_{n\in\mathbb{Z}_+}$ of neighborhoods of y in X such that each neighborhood U of y in X contains atleast one of the member of $\mathscr{B}$.

Then $\mathscr{B}' = \{U_n \cap Y : U_n \in \mathscr{B}\}$ is a countable collection of neighborhoods of y in Y.

Let $V \cap Y$ be a neighborhood of y in Y.

Since V is a neighborhood of y in X, there exists a member U_n in $\mathscr{B}$ such that $U_n \subset V$.

$\implies U_n \cap Y \subset V \cap Y$.

i.e., we have a member $U_n \cap Y \in \mathscr{B}'$ such that $U_n \cap Y \subset V \cap Y$.

$\implies Y$ has a countable basis $\mathscr{B}'$ at y.

Since $y \in Y$ is arbitrary, each point of Y has a countable basis.

$\implies Y$ is first countable.

Let each X_n, $n \in \mathbb{Z}_+$ be first countable.

To prove: $\prod\limits_{n\in\mathbb{Z}_+} X_n$ is first countable.

Let $x = (x_1, x_2, ...) \in \prod\limits_{n\in\mathbb{Z}_+} X_n$.

Then $x_i \in X_i, \ \forall\, i \in \mathbb{Z}_+$.

Since each X_n is first countable, there exists a countable basis $\mathscr{B}_n$ at x_n, for each n.

Then $\mathscr{C} = \{\prod U_n | U_n \in \mathscr{B}_n,$ for finitely many values of n and $U_n = X_n$ for all other values of $n\}$ is a countable collection of neighborhoods of x in $\prod X_n$.

Let $\prod V_n$ be a neighborhood of x.

Since $x \in \prod V_n$, we have $x_n \in V_n$ for each n.

Since V_n is a neighborhood of x_n in X_n, and X_n is first countable, there exists an element $U_n \in \mathscr{B}_n$ such that $U_n \subset V_n$ for each n.

Thus, we can choose $\prod U_n$ in $\mathscr{C}$ such that $\prod U_n \subset \prod V_n$.

$\implies \mathscr{C}$ is a countable basis at x.

Hence, $\prod_{n\in\mathbb{Z}_+} X_n$ is first countable.

(b) Let X be a topological space and let A be its subspace.

Suppose that X is second countable.

Then, by the definition of second countable space, X has a countable basis $\mathscr{B}$ for its topology.

To prove: A is a second countable space.

It is enough to prove that A has a countable basis for its subspace topology.

Take $\mathscr{B}' = \{B \cap A | B \in \mathscr{B}\}$.

Then $\mathscr{B}'$ is a countable collection of open subsets of A.

Let U be open in A.

$\implies U = V \cap A$, where V is open in X.

Since $\mathscr{B}$ is a basis for the topology of X, there exists a member B in $\mathscr{B}$ such that $B \subseteq V$.

$\implies B \cap A \subseteq V \cap A = U$.

i.e., we have a member $B \cap A$ in $\mathscr{B}'$ such that $B \cap A \subset U$.

$\implies \mathscr{B}'$ is a basis for the topology of A.

$\therefore \mathscr{B}'$ is a countable basis for the topology of A.

i.e., A is second countable.

The proof for the countable product case is similar as in (a). □

Definition 5.1.11. *A subset D of X is said to be* ***dense*** *in X if $\bar{D} = X$.*

Theorem 5.1.12. ***Suppose that X has a countable basis. Then***

(a) Every open covering of X contains a countable subcollection covering X.

(b) There exists a countable subset of X that is dense in X.

Proof. Let $\mathcal{B} = \{B_n\}_{n\in\mathbb{Z}_+}$ be a countable basis for X.

(a) Let $\mathcal{A}$ be an open covering of X.

For each positive integer n for which it is possible, choose an element A_n of $\mathcal{A}$ containing the basis element B_n.

Let $\mathcal{A}' = \{A_n\}$.

$\therefore$ The collection $\mathcal{A}'$ is countable.

Thus, we get a countable subcollection $\mathcal{A}'$ of $\mathcal{A}$.

To prove: $\mathcal{A}'$ covers X.

Let $x \in X$.

Since $\mathcal{A}$ is an open covering of X, we can choose an element A of $\mathcal{A}$ containing x.

Since A is open, there is a basis element B_n such that $x \in B_n \subset A$.

$\therefore$ There exists $A_n = A \in \mathscr{A}'$ such that $x \in A_n$.

Thus $\mathscr{A}'$ covers X.

(b) Let $\mathcal{B} = \{B_n\}_{n\in\mathbb{Z}_+}$ be a countable basis for X.

Choose a point x_n from each nonempty basis element B_n.

Let D be the set consisting of the points x_n.

Clearly, D is a countable subset of X.

To prove: D is dense in X.

i.e., to prove $\overline{D} = X$.

Let $x \in X$ and let U be an open set containing x.

Then, there exists a basis element $B_i \in \mathcal{B}$ such that $x \in B_i \subset U$.

We know that $x_i \in B_i$.

$\implies x_i \in U$.

But, already we know that $x_i \in D$.

$\therefore U \cap D$ is nonempty.

Thus $x \in \overline{D}$.

Hence, $X \subset \overline{D}$. But $\overline{D} \subset X$.

This implies $\overline{D} = X$.

Hence, D is dense in X. $\square$

Definition 5.1.13. *A space for which every open covering contains a countable subcovering is called a* ***Lindelof space.***

Definition 5.1.14. *A space having a countable dense subset is called a* ***separable space.***

Example 5.1.15. ***The space $\mathbb{R}_l$ satisfies all the countability axioms, but the second.***

Proof. Given $x \in \mathbb{R}_\ell$, the set of all basis elements of the form $[x, x+1/n)$ is a countable basis at x.

So, $\mathbb{R}_l$ is first countable.

It is easy to see that the rational numbers are dense in $\mathbb{R}_\ell$.

Hence, $\mathbb{R}_l$ is separable.

To see that $\mathbb{R}_\ell$ has no countable basis, let $\mathscr{B}$ be a basis for $\mathbb{R}_\ell$.

Choose for each x, an element B_x of $\mathscr{B}$ such that $x \in B_x \subset [x, x+1)$.

If $x \neq y$, then $B_x \neq B_y$, since $x = \inf B_x$ and $y = \inf B_y$.

Therefore, $\mathscr{B}$ must be uncountable, and so $\mathbb{R}_l$ is not second countable.

To show that $\mathbb{R}_\ell$ is Lindelöf, it will suffice to show that every open covering of $\mathbb{R}_\ell$ by basis elements contains a countable subcollection covering $\mathbb{R}_\ell$.

So, let

$$\mathcal{A} = \{[a_\alpha, b_\alpha)\}_{\alpha \in J}$$

be a covering of $\mathbb{R}$ by basis elements for the lower limit topology.

We wish to find a countable subcollection that covers $\mathbb{R}$.

Let C be the set

$$C = \bigcup_{\alpha \in J} (a_\alpha, b_\alpha)$$

which is a subset of $\mathbb{R}$.

We show the set $\mathbb{R} - C$ is countable.

Let x be a point of $\mathbb{R} - C$.

We know that x belongs to no open interval (a_α, b_α).

Therefore, $x = a_\beta$ for some index β.

Choose such a β and then choose q_x to be a rational number belonging to the interval (a_β, b_β).

Because (a_β, b_β) is contained in C, so is the interval $(a_\beta, q_x) = (x, q_x)$.

It follows that if x and y are two points of $\mathbb{R} - C$ with $x < y$, then $q_x < q_y$.

(For otherwise, we would have $x < y < q_y \leq q_x$, so that y would lie in the interval (x, q_x) and hence in C.)

Therefore, the map $x \to q_x$ of $\mathbb{R} - C$ into $\mathbb{Q}$ is injective, so that $\mathbb{R} - C$ is countable.

Now, we show that some countable subcollection of $\mathcal{A}$ covers $\mathbb{R}$.

To begin, choose for each element of $\mathbb{R} - C$ an element of $\mathcal{A}$ containing it.

One obtains a countable subcollection $\mathcal{A}'$ of $\mathcal{A}$ that covers $\mathbb{R} - C$.

Now, take the set C and topologize it as a subspace of $\mathbb{R}$.

In this topology, C satisfies the second countability axiom.

Now, C is covered by the sets (a_α, b_α), which are open in $\mathbb{R}$ and hence open in C.

Then, some countable subcollection covers C.

Suppose that this subcollection consists of the elements (a_α, b_α) for $\alpha = \alpha_1, \alpha_2, ...$

Then $\mathcal{A}'' = \{[a_\alpha, b_\alpha) : \alpha = \alpha_1, \alpha_2, ...\}$ is a countable subcollection of $\mathcal{A}$ that covers C.

Thus $\mathcal{A}' \cup \mathcal{A}''$ is a countable subcollection of $\mathcal{A}$ that covers $\mathbb{R}_\ell$.

$\implies$ $\mathbb{R}_\ell$ is Lindelof.

□

Example 5.1.16. ***The product of two Lindelof spaces need not be Lindelof.***

Example 5.1.17. ***The subspace of a Lindelof space need not be Lindelof.***

Let Us Sum Up:

In this section, we have discussed the following concepts:

1. First countable space
2. Second countable space
3. Lindelof space
4. Separable space
5. Relation between countability axioms

Check your Progress:

1. The real line with lower limit topology is not

   (A) first countable (B) second countable

   (C) a separable space (D) None of these

2. Which of the following is not true?

   (A) subspace of a first countable space is first countable

   (B) subspace of a second countable space is second countable

   (C) subspace of a Lindelof space is Lindelof

(D) None of these

3. The real line $\mathbb{R}$ is

(A) first countable (B) a Lindelof space

(C) a separable space (D) All of these

5.2 The Separation Axioms

Definition 5.2.1. *A space X is said to be **Hausdorff** if for each pair x, y of distinct points of X, there exist disjoint open sets containing x and y, respectively.*

Definition 5.2.2. *Suppose that one-point sets are closed in X. Then X is said to be **regular** if for each pair consisting of a point x and a closed set B disjoint from x, there exist disjoint open sets containing x and B, respectively.*

Definition 5.2.3. *The space X is said to be **normal** if for each pair A, B of disjoint closed sets of X, there exist disjoint open sets containing A and B, respectively.*

Note: The above three axioms are known as separation axioms.

Remark 5.2.4. ***Every regular space is Hausdorff.***

Proof. Let X be a regular space.

Let $x, y \in X$ such that $x \neq y$.

Since X is regular, $\{y\}$ is closed in X.

Therefore, there exists disjoint open sets U and V containing x and $\{y\}$, respectively.

i.e., U and V are disjoint open sets in X such that $x \in U$ and $y \in V$.

$\implies$ X is Hausdorff. □

Remark 5.2.5. ***Every normal space is regular.***

Proof. Let X be a normal space.

Let $x \in X$ and B be any closed set in X such that $x \notin B$.

Then $\{x\}$ and B are disjoint closed sets in X.

Since X is normal, there exists disjoint open sets U and V containing the closed sets $\{x\}$ and B, respectively.

i.e., U and V are disjoint open sets in X containing x and B, respectively.

$\implies X$ is regular. □

Example 5.2.6. $\mathbb{R}_K$ ***is Hausdorff, but not regular.***

Example 5.2.7. $\mathbb{R}_l$ ***is normal.***

Example 5.2.8. $\mathbb{R}_l$ ***is normal*** $\implies$ $\mathbb{R}_l$ ***is regular*** $\implies$ $\mathbb{R}_l$ ***is Hausdorff.***

Lemma 5.2.9. ***Let*** X ***be a topological space. Let one-point sets in*** X ***be closed.***

(a) X ***is regular if and only if given a point*** x ***of*** X ***and a neighborhood*** U ***of*** x***, there is a neighborhood*** V ***of*** x ***such that*** $\bar{V} \subset U$.

(b) X ***is normal if and only if given a closed set*** A ***and an open set*** U ***containing*** A***, there is an open set*** V ***containing*** A ***such that*** $\bar{V} \subset U$.

Proof. (a) Suppose that X is regular.

Suppose that the point x and the neighborhood U of x are given.

Let $B = X - U$.

Then B is a closed set.

By hypothesis, there exist disjoint open sets V and W containing x and B, respectively.

The set $\bar{V}$ is disjoint from B, since if $y \in B$, the set W is a neighborhood of y disjoint from V.

Therefore, $\bar{V} \subset U$.

To prove the converse, suppose the point x and the closed set B not containing x are given.

Let $U = X - B$.

By hypothesis, there is a neighborhood V of x such that $\bar{V} \subset U$.

The open sets V and $X - \bar{V}$ are disjoint open sets containing x and B, respectively.

Thus, X is regular.

(b) Suppose that X is normal.

Suppose that the closed set A and the open set U containing A are given.

Let $B = X - U$.

Then, B is a closed set disjoint from A.

By hypothesis, there exist disjoint open sets V and W containing A and B, respectively.

The set $\bar{V}$ is disjoint from B, since if $y \in B$, the set W is a neighborhood of y disjoint from V.

Therefore, $\bar{V} \subset U$.

To prove the converse, suppose that A and B are disjoint closed sets in X.

Let $U = X - B$.

By hypothesis, there is an open set V of A such that $\bar{V} \subset U$.

The open sets V and $X - \bar{V}$ are disjoint open sets containing A and B, respectively.

Thus, X is normal. □

Theorem 5.2.10.

(a) A subspace of a Hausdorff space is Hausdorff; a product of Hausdorff spaces is Hausdorff.

(b) A subspace of a regular space is regular; a product of regular spaces is regular.

Proof. (a) Let X be Hausdorff. Let Y be a subspace of X.

To prove: Y is Hausdorff.

Let x and y be two distinct points of the subspace Y of X.

If U and V are disjoint neighborhoods in X of x and y, respectively, then $U \cap Y$ and $V \cap Y$ are disjoint neighborhoods of x and y in Y.

$\implies$ Y is Hausdorff.

Let $\{X_\alpha\}$ be a family of Hausdorff spaces.

To prove: $\prod X_\alpha$ is a Hausdorff space.

Let $\mathbf{x} = (x_\alpha)$ and $\mathbf{y} = (y_\alpha)$ be distinct points of the product space $\prod X_\alpha$.

Because $\mathbf{x} \neq \mathbf{y}$, there is some index β such that $x_\beta \neq y_\beta$.

Choose disjoint open sets U and V in X_β containing x_β and y_β, respectively.

Then, the sets $\pi_\beta^{-1}(U)$ and $\pi_\beta^{-1}(V)$ are disjoint open sets in $\prod X_\alpha$ containing $\mathbf{x}$ and $\mathbf{y}$, respectively.

$\implies \prod X_\alpha$ is a Hausdorff space.

(b) Let X be a regular space. Let Y be a subspace of X.

Then, all the one-point sets are closed in Y.

To prove: Y is regular.

Let x be a point of Y, and B be a closed subset of Y disjoint from x.

Now $\bar{B} \cap Y = B$, where $\bar{B}$ denotes the closure of B in X.

Therefore, $x \notin \bar{B}$, and so, using the regularity of X, we can choose disjoint open sets U and V of X containing x and $\bar{B}$, respectively.

Then $U \cap Y$ and $V \cap Y$ are disjoint open sets in Y containing x and B, respectively.

$\implies Y$ is regular.

Let $\{X_\alpha\}$ be a family of regular spaces.

Let $X = \prod X_\alpha$.

To prove: X is regular.

By (a), X is Hausdorff, so that one-point sets are closed in X.

We use the preceding lemma to prove regularity of X.

Let $\mathbf{x} = (x_\alpha)$ be a point of X, and let U be a neighborhood of $\mathbf{x}$ in X.

Choose a basis element $\prod U_\alpha$ about $\mathbf{x}$ contained in U.

Choose, for each α, a neighborhood V_α of x_α in X_α such that $\bar{V}_\alpha \subset U_\alpha$.

If it happens that $U_\alpha = X_\alpha$, choose $V_\alpha = X_\alpha$.

Then $V = \prod V_\alpha$ is a neighborhood of $\mathbf{x}$ in X.

Since $\bar{V} = \prod \bar{V}_\alpha$, then we have $\bar{V} \subset \prod U_\alpha \subset U$, so that X is regular. □

Example 5.2.11. $\mathbb{R}_l$ ***is regular*** $\implies$ $\mathbb{R}_l^2$ ***is regular. But*** $\mathbb{R}_l^2$ ***is not normal.***

Let Us Sum Up:

In this section, we have discussed the following concepts:

1. Hausdorff space
2. Regular space
3. Normal space
4. Relation between separation axioms
5. Equivalent conditions for regularity and normality

Check your Progress:

1. $\mathbb{R}_K$ is

   (A) Hausdorff (B) regular

   (C) Normal (D) All of these

2. Which of the following is not true?

   (A) Product of Hausdorff spaces is Hausdorff

   (B) Product of regular spaces is regular

   (C) Product of normal spaces is normal

   (D) None of these

3. Which of the following is not a separation axiom?

   (A) Hausdorff space (B) separable space

   (C) regular space (D) Normal space

5.3 Normal spaces

Theorem 5.3.1. ***Every regular space with countable basis is normal.***

Proof. Let X be a regular space with a countable basis $\mathscr{B}$.

To prove : X is normal.

Clearly, one point sets are closed in X, by the regularity of X.

Let A and B be disjoint closed sets in X.

Since X is regular, for each $x \in A$, there exists a neighborhood U_x not intersecting B.

Then $\exists$ a neighborhood V_x such that $\overline{V_x} \subset U_x$

Since V_x is an open set containing x, we can choose an element of $\mathscr{B}$ containing x and contained in V_x.

By choosing such a basis element, for each $x \in A$, we construct a countable covering of A by open sets, those closure do not intersect B.

Let us denote this cover by $\{U_n\}_{n\in\mathbb{Z}_+}$.

Similarly, choose a countable collection $\{V_n\}$ of open sets covering B such that each $\bar{V}_n$ is disjoint from A.

Take $U = \bigcup U_n$ and $V = \bigcup V_n$.

Then U and V are open sets containing A and B, respectively, but they need not be disjoint.

Let us construct the required disjoint open sets in the following way:

Given n, define

$$U'_n = U_n - \bigcup_{i=1}^{n} \overline{V_i} \quad \text{and} \quad V'_n = V_n - \bigcup_{i=1}^{n} \bar{U}_i.$$

Then each U'_n is open, being the difference of an open set U_n and a closed set $\bigcup \overline{V}_i$.

Similarly, each V'_n is open.

Since every x in A belongs to U_n for some n and x belongs to none of the sets $\bar{V}_i$, the collection $\{U'_n\}$ also covers A.

Similarly, the collection $\{V'_n\}$ covers B.

Now, take

$$U' = \bigcup_{n \in \mathbb{Z}_+} U'_n, \qquad V' = \bigcup_{n \in \mathbb{Z}_+} V'_n.$$

Clearly, U' and V' are open sets containing A and B, respectively.

Claim: $U' \bigcap V' = \emptyset$.

Suppose that $x \in U' \cap V'$.

$\therefore x \in U'_j$ and $x \in V'_k$ for some j, k.

Suppose that $j \leq k$.

Then $x \in U'_j \implies x \in U_j$

But $x \in V'_k \Rightarrow x \notin \bigcup_{i=1}^{k} \bar{U}_i \quad \Rightarrow x \notin \bar{U}_j$, because $j \leq k$, a contradiction to $x \in U_j$.

Similar contradiction arises if $j > k$.

$\therefore U'$ and V' are disjoint.

Hence the proof. □

Theorem 5.3.2. ***Every metrizable space is normal.***

Proof. Let X be a metrizable space with metric d.

Since X is Hausdorff, every one point set is closed in X.

Let A and B be disjoint closed subsets of X.

For each $a \in A$, choose ϵ_a so that the ball $B(a, \epsilon_a)$ does not intersect B.

Similarly, for each b in B, choose ϵ_b so that the ball $B(b, \epsilon_b)$ does not intersect A.

Define

$$U = \bigcup_{a \in A} B(a, \epsilon_a/2) \quad \text{and} \quad V = \bigcup_{b \in B} B(b, \epsilon_b/2).$$

Then U and V are open sets containing A and B, respectively.

We claim that they are disjoint.

For if $z \in U \cap V$, then

$$z \in B(a, \epsilon_a/2) \cap B(b, \epsilon_b/2)$$

for some $a \in A$ and some $b \in B$.

By triangle inequality, $d(a, b) < (\epsilon_a + \epsilon_b)/2$.

If $\epsilon_a \leq \epsilon_b$, then $d(a, b) < \epsilon_b$, so that the ball $B(b, \epsilon_b)$ contains the point a.

If $\epsilon_b \leq \epsilon_a$, then $d(a, b) < \epsilon_a$, so that the ball $B(a, \epsilon_a)$ contains the point b. Neither situation is possible. Hence our claim. □

Theorem 5.3.3. ***Every compact Hausdorff space is normal.***

Proof. Let X be a compact Hausdorff space.

First let us prove that X is regular.

Clearly, one point sets are closed in X.

Next, if x is a point of X and B is a closed set in X not containing x, then B is compact.

Then by Lemma 4.1.11, there exist disjoint open sets about x and B, respectively.

Essentially, the same argument as given in that lemma can be used to show that X is normal:

Given disjoint closed sets A and B in X, choose, for each point a of A, disjoint open sets U_a and V_a containing a and B, respectively. (Here we use regularity of X.)

The collection $\{U_a\}$ covers A; because A is compact, A may be covered by finitely many sets $U_{a_1}, \ldots, U_{a_m}$. Then

$$U = U_{a_1} \cup \cdots \cup U_{a_m} \quad \text{and} \quad V = V_{a_1} \cap \cdots \cap V_{a_m}$$

are disjoint open sets containing A and B, respectively. □

Theorem 5.3.4. ***Every well-ordered set X is normal in the order topology.***

Proof. Let X be a well-ordered set.

We assert that every interval of the form $(x, y]$ is open in X:

If X has a largest element and y is that element, $(x, y]$ is just a basis element about y.

If y is not the largest element of X, then $(x, y]$ equals the open set (x, y'), where y' is the immediate successor of y.

Now let A and B be disjoint closed sets in X.

Assume for the moment that neither A nor B contains the smallest element a_0 of X.

For each $a \in A$, there exists a basis element about a disjoint from B.

It contains some interval of the form $(x, a]$. (Here is where we use the fact that a is not the smallest element of X.)

Choose, for each $a \in A$, such an interval $(x_a, a]$ disjoint from B.

Similarly, for each $b \in B$, choose an interval $(y_b, b]$ disjoint from A.

The sets

$$U = \bigcup_{a \in A} (x_a, a] \quad \text{and} \quad V = \bigcup_{b \in B} (y_b, b]$$

are open sets containing A and B, respectively.

We assert they are disjoint.

For suppose that $z \in U \cap V$.

Then $z \in (x_a, a] \cap (y_b, b]$ for some $a \in A$ and some $b \in B$.

Assume that $a < b$.

Then if $a \leq y_b$, the two intervals are disjoint, while if $a > y_b$, we have $a \in (y_b, b]$, contrary to the fact that $(y_b, b]$ is disjoint from A.

A similar contradiction occurs if $b < a$.

Finally, assume that A and B are disjoint closed sets in X, and A contains the smallest element a_0 of X.

The set $\{a_0\}$ is both open and closed in X.

Then as we discussed earlier, there exist disjoint open sets U and V containing the closed sets $A - \{a_0\}$ and B, respectively.

Then $U \cup \{a_0\}$ and V are disjoint open sets containing A and B, respectively. $\square$

Example 5.3.5. ***If J is uncountable, the product space $\mathbb{R}^J$ is not normal.***

Let Us Sum Up:

In this section, we have discussed the following concepts:

1. Condition for a regular space to be normal
2. Relation between metrizable space and a nomal space
3. Normal space in an ordered set

Check your Progress:

1. Which of the following is not normal?

   (A) Regular space (B) compact Hausdorff space

   (C) Metrizable space (D) Well ordered set with order topology

2. Every metrizable space is

   (A) normal (B) regular

   (C) Hausdorff (D) All of these

3. Which of the following is not normal?

   (A) $\mathbb{R}$ (B) $\mathbb{R}_l$ (C) $\mathbb{R}_k$ (D) $\mathbb{R}^3$

5.4 The Urysohn Lemma

Theorem 5.4.1. ***(The Urysohn Lemma).***

Let X be a normal space and let A and B be disjoint closed subsets of X.

Let $[a, b]$ be a closed interval in the real line. Then there exists a continuous map

$$f : X \to [a, b]$$

such that $f(x) = a$ for every x in A, and $f(x) = b$ for every x in B.

Definition 5.4.2. *If A and B are two subsets of the topological space X, and if there is a continuous function $f : X \to [0, 1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$, we say that A and B* ***can be separated by a continuous function****.*

Definition 5.4.3. *A space X is said to be* ***completely regular*** *if one point sets are closed in X and if for each point x_0 and each closed set A not containing x_0, there is a continuous function $f : X \to [0, 1]$ such that $f(x_0) = 1$ and $f(A) = \{0\}$.*

Theorem 5.4.4. ***A subspace a completely regular spaces is completely regular.***

Proof. Let X be a completely regular space and let $Y \subset X$.

To prove: Y is completely regular.

Consider a subset $\{p\}$ in Y.

Since $\{p\} \subset X$, it is closed in X.

$\implies \{p\} \cap Y$ is closed in Y.

$\implies \{p\}$ is closed in Y.

$\therefore$ All one point sets in Y are closed in Y.

Let $x_0 \in Y$ and let A be a closed set of Y such that $x_0 \notin A$.

We know that, Closure of A in Y $= \bar{A} \cap Y$, where $\bar{A}$ is the closure of A in X.

Since A is closed in Y, we have $A = \bar{A} \cap Y$.

Then $x_0 \notin \bar{A}$, because $x_0 \in Y$ and $x_0 \notin A$.

Now, we have a point x_0 of X and a closed set $\bar{A}$ of X such that $x_0 \notin \bar{A}$.

Since X is completely regular, we can choose a continuous function $f : X \to [0, 1]$ such that $f(x_0) = 1$ and $f(\bar{A}) = \{0\}$.

Then the restricted function $f_{|_Y} : Y \to [0, 1]$ is also continuous and $f(x_0) = 1, f(A) = \{0\}$.

Thus, Y is completely regular. □

Theorem 5.4.5. ***A product of a completely regular spaces is completely regular.***

Proof. Let $\{X_\alpha\}$ be a family of completely regular spaces.

To prove: $X = \prod X_\alpha$ is completely regular.

First, we prove that one-point sets are closed in X.

Since each X_α is completely regular, each X_α is regular.

Then $X = \prod X_\alpha$ is regular.

$\implies$ each one point set is closed.

Now, let $b = (b_\alpha)$ be a point of X and let A be a closed set of X disjoint from b.

Choose a basis element $\prod U_\alpha$ containing b that doesn't intersect A, where $U_\alpha = X_\alpha$, except for finitely many α say $\alpha_1, \alpha_2, ..., \alpha_n$.

Now, we have $b_{\alpha_i} \in U_{\alpha_i} \subset X_{\alpha_i}$ and $X_{\alpha_i} - U_{\alpha_i}$ is a closed set in X_{α_i} disjoint from b_{α_i} for $i = 1, 2, ...n$.

Since each X_{α_i} are completely regular, given $i = 1, 2, ..., n$, choose a continuous function $f_i : X_{\alpha_i} \to [0, 1]$ such that $f_i(b_{\alpha_i}) = 1$ and $f_i(X_{\alpha_i} - U_{\alpha_i}) = \{0\}$.

We know that $\pi_{\alpha_i} : \prod X_\alpha \to X_{\alpha_i}$ is a continuous map.

Then $\phi_i = f_i \circ \pi_{\alpha_i} : \prod X_\alpha \to [0, 1]$ is a continuous function because the composition of two continuous functions is continuous.

$$\begin{aligned}
\therefore \phi_i(X - \prod U_\alpha)) &= f_i \circ \pi_{\alpha_i}(X - \prod U_\alpha) \\
&= f_i(X_{\alpha_i} - U_{\alpha_i}) \\
&= \{0\}.
\end{aligned}$$

Thus, $\quad \phi_i(X - \prod U_\alpha)) = \{0\} \quad$ for $i = 1, 2, ..., n$.

Also,

$$\begin{aligned}
\phi_i(b) &= f_i \circ \pi_{\alpha_i}(b) \\
&= f_i(\pi_{\alpha_i}(b)) \\
&= f_i(b_{\alpha_i}) \\
&= 1.
\end{aligned}$$

Consider the product $f = \phi_1.\phi_2...\phi_n$.

Then $f : \prod X_\alpha \to [0, 1]$ defined by $f(x) = \phi_1(x).\phi_2(x)...\phi_n(x)$ is a continuous function.

$\therefore f(b) = \phi_1(b).\phi_2(b)...\phi_n(b) = 1$.

Since $A \cap \prod U_\alpha = \emptyset$,

$$\begin{aligned}
f(X - \prod U_\alpha) &= \phi_1(X - \prod U_\alpha)\phi_2(X - \prod U_\alpha)...\phi_n(X - \prod U_\alpha) \\
&= \{0\}.
\end{aligned}$$

$\implies f(A) = \{0\}$.

$\therefore \prod X_\alpha$ is completely regular. □

Let Us Sum Up:

In this section, we have discussed the following concepts:

1. The Urysohn Lemma
2. Completely regular space

Check your Progress:

1. The Urysohn Lemma is true in a ....

   (A) Regular space (B) Hausdorff space

   (C) Normal space (D) None of these

2. Which of the following is not true?

   (A) Product of Hausdorff spaces is Hausdorff

   (B) Product of regular spaces is regular

   (C) Product of normal spaces is normal

   (D) None of these

3. Which of the following is not true?

   (A) Arbitrary product of completely regular spaces is completely regular

   (B) A subspace a completely regular spaces is completely regular

   (C) Finite product of completely regular spaces is completely regular

   (D) None of these

5.5 The Urysohn Metrization Theorem

Theorem 5.5.1. ***(Urysohn metrization theorem).***

Every regular space X with a countable basis is metrizable.

Proof. We shall prove that X is metrizable by imbedding X in a metrizable space Y.

That is, by showing X homeomorphic with a subspace of Y .

Step 1: We prove that there exists a countable collection of continuous functions $f_n : X \to [0,1]$ having the property that given any point x_0 of X and any n neighborhood U of x_0, there exists an index n such that f_n is positive at x_0 and vanishes outside U.

Let $\{B_n\}$ be a countable basis for X.

Then, given $x_0 \in X$ and given a neighborhood U of x_0, we can choose a basis element B_m such that $x_0 \in B_m \subset U$.

Since B_m is an open set containing x_0, by the regularity of X, $\exists$ a neighborhood V of x_0 such that $\bar{V} \subset B_m$.

$\therefore$ We can choose a basis element B_n so that $x_0 \in B_n \subset V$.

$\Rightarrow x_0 \in B_n$ and $\bar{B}_n \subset \bar{V} \subset B_m \Rightarrow \bar{B}_n \subset B_m$.

Also, $\bar{B}_n$ and $X - B_m$ are disjoint closed subsets of the normal space X.

$\therefore$ By Urysohn lemma, for each pair n, m of indices for which $\overline{B_n} \subset B_m$, we can choose a continuous function $g_{n,m} : x \to [0,1]$ such that

$g_{n,m}\left(\bar{B}_n\right) = \{1\}$ and $g_{n,m}\left(x - B_m\right) = \{0\}$.

$\Rightarrow g_{n,m}\left(x_0\right) = 1 > 0$ and $g_{n,m}(x - U) = \{0\}$.

$\therefore \quad \{g_{n,m}\}$ is a countable collection of continuous functions from X to $[0,1]$.

So, this collection can be reindexed with the positive integers as $\{f_n\}$.

Step 2 :

Consider the metrizable space $\mathbb{R}^\omega$ in the product topology.

Define $F : X \to \mathbb{R}^\omega$ by $F(x) = (f_1(x), f_2(x), \ldots)$.

To prove: F is an imbedding.

Since $\mathbb{R}^\omega$ has the product topology and each f_n is continuous, we have F is continuous.

Next, let $x, y \in X$ such that $x \neq y$.

Choose a neighborhood U of x disjoint from y, because X is Hausdorff.

Then by Step.1, there exists some index n such that $f_n(x) > 0$ and

$$f_n(y) = 0.$$

$\Rightarrow f_n(x) \neq f_n(y)$ for the above n.

$\Rightarrow F(x) \neq F(y)$.

$\therefore F$ is injective.

Finally, we prove that F is a homeomorphism of X onto its image $z = F(x)$ of $\mathbb{R}^\omega$.

(i.e) We need to show that F is open.

Let U be open in X.

To prove: $F(U)$ is open in Z.

Let $z_0 \in F(U)$.

Then, there exists $x_0 \in U$ such that $F(x_0) = z_0$.

$\therefore$ By Step.1, there exists an index N such that $f_N(x_0) > 0$ and $f_N(x - U) = \{0\}$.

Consider the open ray $(0, +\infty)$ in $\mathbb{R}_+$, and the projection map $\pi_N : \mathbb{R}^\omega \to \mathbb{R}$.

Since π_N is continuous, $\pi_N^{-1}((0, +\infty)) = V(\text{say})$ is open in $\mathbb{R}^\omega$.

Let $W = V \cap Z$. Then W is open in Z.

Claim: $z_0 \in W \subset F(U)$.

We know that $z_0 \in F(U) \subset Z \subset \mathbb{R}^\omega$.

$$\therefore \pi_N(z_0) = \pi_N(F(x_0)) = f_N(x_0) > 0$$

$$\Rightarrow \pi_N(z_0) \in (0, +\infty)$$

$$\Rightarrow z_0 \in \pi_N^{-1}((0, +\infty)) = V.$$

Thus $z_0 \in W$.

Next, let $z \in W$.

$$\Rightarrow \quad z \in \pi_N^{-1}((0, \infty)) \text{ and } z \in Z$$

$$\Rightarrow \pi_N(z) \in (0, \infty) \text{ and } z = F(x) \text{ for some } x \in X.$$

$$\Rightarrow \pi_N(F(x)) = f_N(x) \in (0, \infty)$$

(i.e) $f_N(x) > 0.$

Since f_N vanishes outside U, the point x must be in U.

$$\Rightarrow \quad F(x) \in F(u)$$

That is, $z \in F(U)$.

Hence, $W \subset F(U)$

$\therefore W$ is an open set of Z such that $z_0 \in W \subset F(U)$.

$\Rightarrow F(U)$ is open in Z.

Thus, F is an imbedding of X in $\mathbb{R}^\omega$. $\square$

Theorem 5.5.2. ***(Imbedding theorem).***

Let X be a space in which one-point sets are closed.

Suppose that $\{f_\alpha\}_{\alpha \in J}$ is an indexed family of continuous functions $f_\alpha : X \to \mathbb{R}$ satisfying the requirement that for each point x_0 of X and each neighborhood U of x_0, there is an index α such that f_α is positive at x_0 and vanishes outside U.

Then the function $F : X \to \mathbb{R}^J$ defined by

$$F(X) = (f_\alpha(x))_{\alpha \in J}$$

is an imbedding of X in $\mathbb{R}^J$.

If f_α maps X into $[0, 1]$ for each α, then F imbeds X in $[0, 1]^J$.

Proof. The proof is almost a copy of Step 2 of the preceding proof. One merely replaces n by α, and $\mathbb{R}^\omega$ by $\mathbb{R}^J$, throughout. One needs one-point sets in X to be closed in order to be sure that given $x \neq y$, there is an index α such that $f_\alpha(x) \neq f_\alpha(y)$. □

A family of continuous functions that satisfies the hypotheses of this theorem is said to separate points from closed sets in X. The existence of such a family is readily seen to be equivalent, for a space X in which one-point sets are closed, to the requirement that X be completely regular. Therefore, one has the following immediate corollary:

Theorem 5.5.3. ***A space X is completely regular if and only if it is homeomorphic to a subspace of $[0, 1]^J$ for some J.***

Let Us Sum Up:

In this section, we have discussed the following concepts:

1. The Urysohn metrization theorem
2. Imbedding theorem

Check your Progress:

1. Every regular space with a countable basis is

   (A) normal only (B) metrizable only

   (C) both normal and metrizable (D) None of these

2. Which of the following is not true?

   (A) Every regular space with a countable basis is normal

   (B) Every regular space with a countable basis is metrizable

   (C) Every regular space with a countable basis is second countable

   (D) None of these

3. Which of the following is not a countable basis for $\mathbb{R}$?

   (A) $\{(a, b) : a, b \in \mathbb{Q}\}$ (B) $\{(a, b) : a, b \in \mathbb{Z}\}$

   (C) $\{(a, b) : a, b \in \mathbb{R}\}$ (D) None of these

5.6 The Tietze Extension Theorem

Theorem 5.6.1. ***(The Tietze Extension Theorem).***

Let X be a normal space and let A be a closed subspace of X.

(a) Any continuous map of A into the closed interal $[a, b]$ of $\mathbb{R}$ may be extended to a continuous map of all of X into $[a, b]$.

(b) Any continuous map of A into $\mathbb{R}$ may be extended to a continuous map of all of X into $\mathbb{R}$.

Proof. **Step 1:**

Let $f : A \to [-r, r]$ be continuous.

Let us construct a continuous function $g : x \to \mathbb{R}$ such that

$$|g(x)| \leqslant \frac{1}{3}r \quad \forall x \in X \text{and}$$
$$|g(a) - f(a)| \leq \frac{2}{3}r \quad \forall a \in A.$$

Divide the interval $[-r, r]$ into three equal intervals I_1, I_2, I_3 of length $\frac{2}{3}r$.

$$\therefore I_1 = \left[-r, \frac{-1}{3}r\right], I_2 = \left[\frac{-1}{3}r, \frac{1}{3}r\right], I_3 = \left[\frac{1}{3}r, r\right]$$

Take $B = f^{-1}(I_1)$ and $C = f^{-1}(I_3)$.

Since I_1 and I_3 are disjoint closed subsets of $[-r, r]$ and since f is continuous, we have B and C are disjoint closed subsets of A.

Since A is closed in X, B and C are closed in X.

$\therefore$ By Urysohn lemma, there exists a continuous function $g : X \to I_2$ such that $g(x) = \frac{-1}{3}r$ for each x in B, and $g(x) = \frac{1}{3}r$ for each x in C.

Then $|g(x)| \leq \frac{1}{3}r \quad \forall x \in X$.

Claim: $|g(a) - f(a)| \leq \frac{2}{3}r \quad \forall a \in A$.

Let $a \in A$.

Then there are 3 cases.

If $a \in B$ then $f(a) \in I_1$ and $g(a) = \frac{-1}{3}r \in I_1$, and hence $|f(a) - g(a)| \leq \frac{2}{3}r$.

If $a \in C$, then $f(a) \in I_3$ and $g(a) = \frac{1}{3}r \in I_3$, and hence $|g(a) - f(a)| \leq \frac{2}{3}r$.

If $a \notin B \cup C$ then $f(a) \in I_2$ and $g(a) \in I_2$, and hence $|g(a) - f(a)| \leq \frac{2}{3}r$.

Hence our claim.

Step.2:

We now prove part (a).

Without loss of generality, we can replace the arbitrary closed interval $[a, b]$ of $\mathbb{R}$ by the interval $[-1, 1]$.

Let $f : A \to [-1, 1]$ be a continuous map.

Then by Step 1, with $r = 1$, there exists a continuous function $g_1 : X \to \mathbb{R}$ such that

$$|g_1(x)| \leq \frac{1}{3} \quad \forall x \in X$$
$$\text{and} \quad |f(a) - g(a)| \leq \frac{2}{3} \quad \forall a \in A.$$

Now, consider the function $f - g_1$.

It maps A into $\left[-\frac{2}{3}, \frac{2}{3}\right]$ and is continuous.

So, applying Step.1 with $r = \frac{2}{3}$, we get a continuous function $g_2 : X \to \mathbb{R}$ such that

$$|g_2(x)| \leq \frac{1}{3}\left(\frac{2}{3}\right) \quad \forall x \in X$$
$$\text{and} \quad |f(a) - g_1(a) - g_2(a)| \leq \frac{2}{3}\left(\frac{2}{3}\right) \quad \forall a \in A.$$

Then, we apply Step.1 to the function $f - g_1 - g_2$ and so on.

At the general step, we have real-valued functions $g_1, g_2, \ldots, g_n$ defined on all of X such that

$$|f(a) - g_1(a) - \cdots - g_n(a)| \leq \left(\frac{2}{3}\right)^n \quad \forall a \in A \qquad (1)$$

and $|g_n(x)| \leq \frac{1}{3}\left(\frac{2}{3}\right)^{n-1} \quad \forall x \in X$.

Applying Step.1 so $f - g_1 - \cdots - g_n$ with $r = \left(\frac{2}{3}\right)^n$, we obtain a continuous function $g_{n+1} : X \to \mathbb{R}$ such that

$$|g_{n+1}(x)| \leq \frac{1}{3}\left(\frac{2}{3}\right)^n \quad \forall x \in X$$

and $|f(a) - g_1(a) - \cdots - g_{n+1}(a)| \leq \left(\frac{2}{3}\right)^{n+1} \quad \forall a \in A$.

Thus by induction, the functions g_n are defined for all n.

Since $|g_n(x)| \leqslant \frac{1}{3}\left(\frac{2}{3}\right)^{n-1}$ and since the geometric series $\frac{1}{3}\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^{n-1}$ converges, by comparison test $\sum_{n=1}^{\infty} g_n(x)$ also converges.

So, define $g(x) = \sum_{n=1}^{\infty} g_n(x), \ \forall x \in X$.

To prove that g is continuous, it is enough to show that $\sum g_n(x)$ converges to $g(x)$ uniformly, because all g_n's are continuous.

Let (s_n) be the sequence of partial sum of $\sum g_n$.

Then, for $k > n$,

$$\begin{aligned}
|s_k(x) - s_n(x)| &= \left|\sum_{i=1}^{k} g_i(x) - \sum_{i=1}^{n} g_i(x)\right| \\
&= \left|\sum_{i=n+1}^{k} g_i(x)\right| \\
&\leq \sum_{i=n+1}^{k} |g_i(x)| \\
&\leq \frac{1}{3}\sum_{i=n+1}^{k}\left(\frac{2}{3}\right)^{i-1} \\
&< \frac{1}{3}\sum_{i=n+1}^{\infty}\left(\frac{2}{3}\right)^{i-1} \\
&= \frac{1}{3}\left[\left(\frac{2}{3}\right)^n + \left(\frac{2}{3}\right)^{n+1} + \cdots\right] \\
&= \frac{1}{3}\left[\frac{(2/3)^n}{1 - \frac{2}{3}}\right] \\
&= \left(\frac{2}{3}\right)^n.
\end{aligned}$$

Fixing n and letting $k \to \infty$, we get

$$|g(x) - s_n(x)| \leq \left(\frac{2}{3}\right)^n \quad \forall x \in X.$$

$\therefore (s_n) \to g$ uniformly.

$\Rightarrow g$ is continuous.

Next, we show that $g(a) = f(a) \quad \forall a \in A$.

By (1), we have

$$\left| f(a) - \sum_{i=1}^{n} g_i(a) \right| = |f(a) - s_n(a)| \leq \left(\frac{2}{3}\right)^n \quad \forall a \in A.$$
$$\Rightarrow \lim_{n\to\infty} s_n(a) = f(a) \quad \forall a \in A$$
$$\Rightarrow \quad g(a) = f(a) \quad \forall a \in A, \text{ because } \lim_{n\to\infty} s_n(x) = g(x).$$

Finally, we show that g maps X into the interval $[-1, 1]$.

We have $g : X \to \mathbb{R}$ is continuous.

Since

$$|g(x)| = \left| \sum_{n=1}^{\infty} g_n(x) \right| \leq \sum_{n=1}^{\infty} |g_n(x)|$$
$$\leq \sum_{n=1}^{\infty} \frac{1}{3} \left(\frac{2}{3}\right)^{n-1} = 1$$

for all $x \in X$, we have g is a continuous function from X into $[-1, 1]$.

Step 3:

We now prove part (b).

Suppose that $f : A \to \mathbb{R}$ is continuous.

Since $\mathbb{R}$ is homeomorphic to $(-1, 1)$, we can replace $\mathbb{R}$ by $(-1, 1)$.

(i.e) f is a continuous function from A to $(-1, 1)$.

Then by (a), f can be extended to a continuous map $g : X \to [-1, 1]$.

But, we need a continuous function $h : X \to (-1, 1)$.

Given g, let us define a subset D of X by $D = g^{-1}(\{-1\}) \cup g^{-1}(\{1\})$.

Since g is continuous, D is closed in X.

Because $g(A) = f(A) \subset (-1, 1)$, the closed sets A and D are disjoint.

$\therefore$ By the Urysohn lemma, there is a continuous function $\phi : X \to [0, 1]$ such that $\phi(D) = \{0\}$ and $\phi(A) = \{1\}$.

Define $h(x) = \phi(x)g(x) \quad \forall x \in X$.

Clearly, h is continuous, and h is an extension of f, because for $a \in A$,

$$h(a) = \phi(a)g(a) = 1 \cdot g(a) = f(a).$$

Finally, we prove that h maps X into $(-1, 1)$.

Let $x \in X$.

If $x \in D$, then $h(x) = 0.g(x) = 0$.

If $x \notin D$, then $|g(x)| < 1$, and hence $|h(x)| \leq 1.|g(x)| < 1$.

$\Rightarrow$ In either case, $|h(x)| < 1$.

$\Rightarrow \quad h(x) \in (-1, 1)$.

$\therefore \quad h : X \to (-1, 1)$ is the required continuous function. □

Let Us Sum Up:

This section dealt with the problem, of extending a continuous real-valued function that is defined on a subspace of a space X to a continuous function defined on all of X, known as the Tietze extension theorem.

Check your Progress:

1. If A is a closed subspace of the normal space X, then any continuous map of A into the closed interal $[a, b]$ of $\mathbb{R}$ may be extended to a continuous map of all of X into $[a, b]$. This is known as ....

   (A) Urysohn Metrization Theorem

   (B) Tietze Extension Theorem

   (C) Imbedding theorem

   (D) None of these

2. The Tietze extension theorem is true in a ....

   (A) Regular space (B) Hausdorff space

   (C) Normal space (D) None of these

3. The Tietze extension theorem implies ....

   (A) Urysohn lemma

   (B) Urysohn Metrization Theorem

   (C) Imbedding theorem

   (D) None of these

Unit Summary:

This unit dealt with some countability and separation axioms with their consequences. Further, major results, like, Urysohn Metrization Theorem and the Tietze Extension Theorem were proved as applications of Urysohn lemma.

Glossary:

- $\{U_n\}_{n\in\mathbb{Z}_+}$ - Countable collection of neighborhoods U_n
- $\{B_n\}_{n\in\mathbb{Z}_+}$ - Countable basis
- $[0,1]^J$ - Arbitrary product of $[0,1]$ with itself, where J is some index set

Self-Assessment Questions:

1. Prove or disprove: A subspace of a Lindelof space is Lindelof.
2. Give a counter example to prove that the product of two Lindelof spaces is not Lindelof.
3. Is every Hausdorff space regular? Justify your answer.
4. Prove that a closed subspace of a normal space is normal.

Exercises:

1. Show that if X is Lindelof and Y is compact, then $X \times Y$ is Lindelof.
2. Show that every order topology is regular.
3. Let $f, g : X \to Y$ be continuous. Assume that Y is Hausdorff. Show that $\{x | f(x) = g(x)\}$ is closed in X.
4. Is $\mathbb{R}_l$ second countable? Justify your answer.
5. Give an example showing that a Hausdorff space with countable basis need not be metrizable.

Answers for check your progress:

Section 5.1	1. (B)	2. (C)	3. (D)
Section 5.2	1. (A)	2. (C)	3. (B)
Section 5.3	1. (A)	2. (D)	3. (C)
Section 5.4	1. (C)	2. (C)	3. (D)
Section 5.5	1. (C)	2. (D)	3. (C)
Section 5.6	1. (B)	2. (C)	3. (A)

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