PERIYAR UNIVERSITY

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SALEM - 636 011, Tamil Nadu, India.

CENTRE FOR DISTANCE AND ONLINE EDUCATION

(CDOE)

M.Sc. MATHEMATICS

SEMESTER - II

CORE COURSE: REAL ANALYSIS – II

(Candidates admitted from 2024 onwards)

PERIYAR UNIVERSITY

CENTRE FOR DISTANCE AND ONLINE EDUCATION (CDOE)

M.Sc. Mathematics - 2024 admission onwards

CORE – 5
Real Analysis II

Prepared by:

Centre for Distance and Online Education (CDOE)
Periyar University
Salem 636 011

Contents

REAL ANALYSIS - II

OBJECTIVE: This course covers vector and multivariable calculus. This topics include vectors and matrices, parametric curves, partial derivatives, double and triple integrals, and vector calculus in 2 and 3 dimensional spaces, line integrals and integration theorems generalizing the Fundamental theorem of Calculus (Green theorem, Stokes theorem and Gauss's theorem).

UNIT-I: Multivariable Differential Calculus

Introduction - The Directional derivative - Directional derivative and continuity - The total derivative - The total derivative expressed in terms of partial derivatives - The matrix of linear function - The Jacobian matrix - The chain rule - Matrix form of chain rule - The mean - value theorem for differentiable functions - A sufficient condition for differentiability - A sufficient condition for equality of mixed partial derivatives - Taylor's theorem for functions of $\mathbb{R}^n$ to $\mathbb{R}^1$.

UNIT-II: Implicit Functions and Extremum Problems:

Functions with non-zero Jacobian determinants – The inverse function theorem – The Implicit function theorem – Extrema of real valued functions of several variables.

Unit-III: Line Integrals

Introduction – Paths and line integrals – Other notations of line integrals – Basic properties of line integrals – Line integrals with respect to the arc length – Open connected sets & Independence of the path – Second fundamental theorem of calculus for line integrals – The first fundamental theorem of calculus for line integrals.

Unit-IV: Multiple integrals

Introduction – Partitions of rectangle, Step functions – The double integral of a step function – The definition of the double integral of a function defined and bounded on a rectangle – Upper and lower double rectangles – Evaluation of a double integral by repeated one-dimensional integration – Geometric interpretation of the double integral as a volume – Integrability of continuous functions – Integrability of bounded functions with discontinuities.

Unit-V: Green's theorem and Surface integrals Green's theorem in the plane – Change of variables in a double integral – Extensions to higher dimensions – Worked examples. Surface Integrals: Definition of surface integral – Change of parametric representation – Stoke's theorem – The divergence theorem.

REFERENCES:

1. Tom M. Apostol: Mathematical Analysis, 2nd Edition, Addison-Wesley Publishing Company Inc. New York, 1974. (for Units I to II).

2. T.M. Apostol, "Calculus Vol.2, Multi-Variable Calculus and Linear Algebra with Applications to Differential Equations and Probability", Second Edition, John Wiley & Sons, 1969. (Units III to V).

Unit 1

Multivariable Differential Calculus

Objectives

After reading this unit, learners will be able to

- understand the notion of derivative of functions of several variables
- evaluate the partial, directional and total derivatives of functions involving several variables.
- To study the calculus of functions from $\mathbb{R}^n$ to $\mathbb{R}^m$.
- generalize important results in classical calculus in single-variable like chain rule, mean-value theorem, and Taylor's formula to n-variables.

1.1 Partial Derivatives

We shall start by recalling the concept of derivative for a function f of a single variable.

Let $D \subset \mathbb{R}$ and let x_0 be an interior point of D. A function $f : D \to \mathbb{R}$ is said to be differentiable at x_0 if the limit

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

exists and the value of the limit is denoted by $f'(x_0)$.

The term $f(x_0) + (x - x_0)f'(x_0)$ is called the linear approximation of f at x_0.

The geometric interpretation of the derivative $f'(x_0)$ of a function f of a single variable is the slope of the line

$$y = f(x_0) + f'(x_0)(x - x_0)$$

tangent to the graph of the function.

Let $D \subset \mathbb{R}^2$ and let $f : D \to \mathbb{R}$ be any function. Fix $(x_0, y_0) \in D$ and define $D_1,\ D_2 \subset \mathbb{R}$ by $D_1 := \{x \in \mathbb{R} : (x, y_0) \in D\}$ and $D_2 := \{y \in \mathbb{R} : (x_0, y) \in D\}$.

If x_0 is an interior point of D_1, we define the partial derivative of f with respect to x at (x_0, y_0) to be the limit

$$\lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

provided this limit exists. It is denoted by $f_x(x_0, y_0)$.

Similarly, if y_0 is an interior point of D_2, we define the partial derivative of f with respect to y at (x_0, y_0) to be the limit

$$\lim_{k \to 0} \frac{f(x_0, y_0 + k) - f(x_0, y_0)}{k}$$

provided this limit exists. It is denoted by $f_y(x_0, y_0)$.

These partial derivatives are also called the first-order partial derivatives or simply the first partials of f at (x_0, y_0). They are sometimes denoted by

$$\frac{\partial f}{\partial x}(x_0, y_0) \text{ and } \frac{\partial f}{\partial y}(x_0, y_0).$$

Definition 1.1.1. *If the partial derivatives* $\frac{\partial f}{\partial x}(x_0, y_0)$ *and* $\frac{\partial f}{\partial y}(x_0, y_0)$*. exist, then the pair* $(f_x(x_0, y_0), f_y(x_0, y_0))$ *is called the gradient of* f *at* (x_0, y_0) *and is denoted by* $\nabla f(x_0, y_0)$*, i.e.,*

$$\nabla f(x_0, y_0) = \left(f_x(x_0, y_0), f_y(x_0, y_0)\right).$$

The partial derivative $f_x(x_0, y_0)$ gives the rate of change in f at (x_0, y_0) along the x-axis, whereas $f_y(x_0, y_0)$ gives the rate of change in f at (x_0, y_0) along the y-axis.

The above ideas can also be extend to a function of n-variables as follows:

Definition 1.1.2. *Let $U \subset \mathbb{R}^n$ be open and $f : U \to \mathbb{R}$. Let $\{e_1, \ldots, e_n\}$ be the standard basis of $\mathbb{R}^n$. If the following limit exists, we write*

$$\frac{\partial f}{\partial x_j}(x) = \lim_{h \to 0} \frac{f(x_1, \ldots x_{j-1}, x_j + h, x_{j+1}, \ldots, x_n) - f(x)}{h} = \lim_{h \to 0} \frac{f(x + he_j) - f(x)}{h}$$

Note that here

$$\begin{aligned} x + he_j &= (x_1, x_2, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_n) + h(0, 0, \ldots, 1 \ (j^{th}\text{place}), 0, \ldots, 0) \\ &= (x_1, x_2, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_n) + (0, 0, \ldots, h \ (j^{th}\text{place}), 0, \ldots, 0) \\ &= (x_1, \ldots x_{j-1}, x_j + h, x_{j+1}, \ldots, x_n) \end{aligned}$$

Definition 1.1.3. *Let $E \subset \mathbb{R}^n$ be open and $f : E \to \mathbb{R}^m$. Let $\{e_1, \ldots, e_n\}$ and $\{u_1, \ldots, u_m\}$ be the standard basis of $\mathbb{R}^n$ and $\mathbb{R}^m$.*

For $x \in E$,

$$f(x) = (f_1(x), f_2(x), \ldots, f_m(x))$$

where the components of f are the real functions $f_1, \ldots, f_m$ defined by

$$f(x) = \sum_{i=1}^{m} f_i(x) u_i \qquad (x \in E),$$

or by $f_i(x) = f(x) \cdot u_i, \ 1 \leq i \leq m$.

For $x \in E, \ 1 \leq i \leq n, \ 1 \leq j \leq m$, we define

$$(D_j f_i)(x) = \frac{\partial f_i}{\partial x_j}(x) = \lim_{t \to 0} \frac{f_i(x + te_j) - f_i(x)}{t}$$

provided the limit exists. Here, $f_i(x)$ means $f_i(x_1, \ldots, x_n)$ and $D_j f_i$ **(called the partial derivative)** *is the derivative of f_i with respect to x_j, keeping the other variables fixed.*

Note: 1. In the case of functions of one variable, the existence of a derivative is sufficient.

2. But in the case of functions of severable variables, continuity or atleast boundedness of the partial derivatives is needed.

Examples

1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x, y) := x^2 + y^2$.

   Then both the partial derivatives of f exist at every point of $\mathbb{R}^2$; in fact,

   $$f_x(x_0, y_0) = 2x_0 \text{ and } f_y(x_0, y_0) = 2y_0 \text{ for any } (x_0, y_0) \in \mathbb{R}^2.$$

2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the norm function given by $f(x, y) := \sqrt{x^2 + y^2}$.

   Then both the partial derivatives of f exist at every point of $\mathbb{R}^2$ except at $(0, 0)$ i.e., the origin

   In fact, for any $(x_0, y_0) \in \mathbb{R}^2$ with $(x_0, y_0) \neq (0, 0)$,

   $$f_x(x_0, y_0) = \frac{x_0}{\sqrt{x_0^2 + y_0^2}} \text{ and } f_y(x_0, y_0) = \frac{y_0}{\sqrt{x_0^2 + y_0^2}}.$$

   To examine whether any of the partial derivatives exist at $(0, 0)$, we have to use the definition.

   This leads to a limit of the quotient $\frac{h}{|h|}$ as $h \to 0$. Clearly, such a limit does not exist.

   It follows that $f_x(0, 0)$ and $f_y(0, 0)$ do not exist.

3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

   $$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & if \quad (x, y) \neq (0, 0) \\ 0 & if \quad (x, y) = (0, 0) \end{cases}$$

   Then for any $h, k \in \mathbb{R}$ with $h \neq 0$ and $k \neq 0$, we have

   $$\frac{f(0 + h, 0) - f(0, 0)}{h} = 0 \text{ and } \frac{f(0, 0 + k) - f(0, 0)}{k} = 0.$$

   Hence $f_x(0, 0)$ and $f_y(0, 0)$ exist and are both equal to 0. However, it is seen already, f is not continuous at $(0, 0)$.

Partial derivative differs from usual derivative for the reason that the existence of partial derivatives $D_1 f, \ldots, D_n f$ at a particular point does not necessarily imply continuity of f at that point.

Check your progress

1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y) = \begin{cases} \dfrac{x^2y}{x^2+y^2} & if \quad (x,y) \neq (0,0) \\ 0 & if \quad (x,y) = (0,0) \end{cases}$$

   Show that f is continuous at $(0,0)$ and find the partial derivatives $f_x(0,0)$ and $f_y(0,0)$.

2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x,y) = |x| + |y|$ for $(x,y) \in \mathbb{R}^2$. Show that $f_x(0,0)$ and $f_y(0,0)$ do not exist.

1.2 The Directional Derivative

In this section, we will introduce the concept of directional derivatives which is a generalization of partial derivatives. Partial derivative specifies the rate of change of a function with respect to each coordinate axis whereas directional derivative examines the rate of change of a function in any specific direction. This is particularly important in multivariable calculus, where functions depend on two or more variables, and understanding how they change in arbitrary directions is key to analyzing surfaces, gradients, and optimizations.

Let $S \subseteq \mathbb{R}^n$. A nearby point of $\mathbf{c} \in S$ is denoted by $\mathbf{c} + \mathbf{u}$, for $\mathbf{u} \neq \mathbf{0}$.

For $h \in \mathbb{R}$, $\mathbf{c} + h\mathbf{u}$ represent points on the line segment joining $\mathbf{c}$ and $\mathbf{c} + \mathbf{u}$.

Here $\mathbf{u}$ is a vector that describes the orientation of the line segment.

We shall assume that $\mathbf{c}$ is an interior point of S. Then there is an n-ball $B(\mathbf{c}; r)$ lying in S, and, if h is small enough, the line segment joining $\mathbf{c}$ to $\mathbf{c} + h\mathbf{u}$ will also lie inside $B(\mathbf{c}; r)$ and hence in S.

Definition 1.2.1. *Let $\mathbf{f} : S \to \mathbb{R}^m$. The directional derivative of $\mathbf{f}$ at $\mathbf{c}$ in the direction $\mathbf{u}$, denoted by $\mathbf{f}'(\mathbf{c}; \mathbf{u})$, is defined by*

$$\mathbf{f}'(\mathbf{c}; \mathbf{u}) = \lim_{h \to 0} \frac{\mathbf{f}(\mathbf{c} + h\mathbf{u}) - \mathbf{f}(\mathbf{c})}{h} \tag{1.1}$$

provided the limit on the right exists.

Geometrically, the directional derivative explains the way in which a function $\mathbf{f}$ changes when we move from a point $\mathbf{c} \in S$ to a nearby point $\mathbf{c} + \mathbf{u}$ $(\mathbf{u} \neq \mathbf{0})$ along a line segment.

1.2.1 Examples

1. If $\mathbf{u} = \mathbf{0}$, then $\mathbf{f}'(\mathbf{c}; \mathbf{0})$ exists and $\mathbf{f}'(\mathbf{c}; \mathbf{0})$ $=\mathbf{0}$ for every $\mathbf{c}$ in S.

2. If $\mathbf{u} = \mathbf{u}_k$, the k-th unit coordinate vector, then $\mathbf{f}'$ $(\mathbf{c}; \mathbf{u}_k)$ is called a partial derivative and is denoted by $D_{\boldsymbol{k}} f(c)$ $\left(= \dfrac{\partial f}{\partial x_k}\right)$.

3. If $\mathbf{f} = (f_1, \ldots, f_m)$, then $\mathbf{f}'(\mathbf{c}; \mathbf{u})$ exists if and only if $f_k'(\mathbf{c}; \mathbf{u})$ exists for each $k = 1, 2, \ldots, m$ and in this case, we write

$$\mathbf{f}'(\mathbf{c}; \mathbf{u}) = (f_1'(\mathbf{c}; \mathbf{u}), \ldots, f_m'(\mathbf{c}; \mathbf{u}))$$

   In particular, when $\mathbf{u} = \mathbf{u}_k$ we have

$$D_k \mathbf{f}(\mathrm{c}) = (D_k f_1(\mathrm{c}), \ldots, D_k f_m(\mathrm{c})) = \left(\frac{\partial f_1}{\partial x_k}(\mathrm{c}), \ldots, \frac{\partial f_m}{\partial x_k}(\mathrm{c})\right)$$

4. If $F(t) = \mathbf{f}(c + tu)$, then $F'(0) = \mathbf{f}'(c; u)$. More generally, $F'(t) = \mathbf{f}'(c + tu; u)$ if either derivative exists.

5. If $f(x) = \|x\|^2$, then

$$F(t) = f(\mathbf{c} + t\mathbf{u}) = (\mathbf{c} + t\mathbf{u}) \cdot (\mathbf{c} + t\mathbf{u}) = \|\mathbf{c}\|^2 + 2t\mathbf{c} \cdot \mathbf{u} + t^2 \|\mathbf{u}\|^2$$

   so $F'(t) = 2\mathbf{c} \cdot \mathbf{u} + 2t\|\mathbf{u}\|^2$; hence $F'(0) = f'(\mathbf{c}; \mathbf{u}) = 2\mathbf{c} \cdot \mathbf{u}$.

6. If $\mathbf{f}$ is linear, then

$$\mathbf{f}'(\mathbf{c}; \mathbf{u}) = \lim_{h \to 0} \frac{\mathbf{f}(\mathbf{c} + h\mathbf{u}) - \mathbf{f}(\mathbf{c})}{h} = \lim_{h \to 0} \frac{\mathbf{f}(\mathbf{c}) + h\mathbf{f}(\mathbf{u}) - \mathbf{f}(\mathbf{c})}{h} = \mathbf{f}(\mathbf{u})$$

   for every c and every u.

1.2.2 Remarks

1. If $\mathbf{f}'(\mathbf{c};\mathbf{u})$ exists along every direction $\mathbf{u}$, then in particular all the partial derivatives $D_1\mathbf{f}(\mathrm{c}),\ldots,D_n\mathbf{f}(\mathrm{c})$ exist. However, the converse need not be true. For example, consider the real-valued function $f:\mathbb{R}^2\to\mathbb{R}^1$ given by

$$f(x,y)=\begin{cases} x+y & \text{if } x=0 \text{ or } y=0 \\ 1 & \text{otherwise}\end{cases}$$

Then $D_1 f(0,0)=\lim\limits_{h\to 0}\dfrac{f(h,0)-f(0,0)}{h}=\lim\limits_{h\to 0}\dfrac{h}{h}=1$

Similarly, it can be proved that $D_2 f(0,0)=1$.

However, for any other direction $\mathbf{u}=(a_1,a_2)$, where $a_1\neq 0$ and $a_2\neq 0$, we have

$$\lim_{h\to 0}\frac{f(\mathbf{0}+h\mathbf{u})-f(\mathbf{0})}{h}=\lim_{h\to 0}\frac{f(h\mathbf{u})}{h}=\lim_{h\to 0}\frac{1}{h}$$

and this limit does not exist.

2. A function can have a finite directional derivative $\mathbf{f}'(\mathbf{c};\mathbf{u})$ for every. $\mathbf{u}$ without being continuous at $\mathbf{c}$. For example, let us consider

$$f(x,y)=\begin{cases} \dfrac{xy^2}{x^2+y^4} & \text{if } x\neq 0 \\ 0 & \text{if } x=0\end{cases}$$

Let $\mathbf{u}=(a_1,a_2)$ be any vector in $\mathbb{R}^2$. Then we have

$$\frac{f(0+h\mathbf{u})-f(0)}{h}=\frac{f\left(ha_1,ha_2\right)}{h}=\frac{a_1a_2^2}{a_1^2+h^2a_2^4}$$

and hence

$$f'(\mathbf{0};\mathbf{u})=\begin{cases} a_2^2/a_1 & \text{if } a_1\neq 0 \\ 0 & \text{if } a_1=0\end{cases}$$

Thus, $f'(0;\mathbf{u})$ exists for all $\mathbf{u}$. On the other hand, the function f takes the value $\frac{1}{2}$ at each point of the parabola $x=y^2$ (except at the origin), so f is not continuous at $(0,0)$, since $f(0,0)=0$.

Check your progress

1. Find the directional derivative of the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = x^2 + y^2$.

2. Find the directional derivative of the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = \sqrt{x^2 + y^2}$.

3. Calculate all first-order partial derivatives and the directional derivatives $f'(x; u)$ for each of the real-valued functions defined on $\mathbf{R^n}$ as follows:

   (a) $f(x) = a.x$, where a is a fixed vector in $\mathbf{R^n}$.

   (b) $f(x) = \|x\|^4$.

   (c) $f(x) = x.L(x)$, where $L : \mathbf{R^n} \mapsto \mathbf{R^n}$ is a linear function.

   (d) $f(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j$, where $a_{ij} = a_{ji}$.

1.3 The Total Derivative

The total derivative of a function represents how the function changes in response to changes in all of its input variables, taking into account both direct and indirect dependencies between variables. It generalizes the concept of derivatives to functions of serveral variables and is fundamental in fields like physics, engineering, and economics where systems rely on several interrelated variables.

First we shall motivate this concept by considering a real-valued function f which is differentiable at c. In this case, f can be approximated by a linear polynomial near c . In fact, if $f'(c)$ exists, let $E_c(h)$ denote the difference

$$E_c(h) = \frac{f(c + h) - f(c)}{h} - f'(c) \quad \text{if } h \neq 0 \tag{1.2}$$

with $E_c(0) = 0$. Then we have the equation

$$f(c + h) = f(c) + f'(c)h + hE_c(h) \tag{1.3}$$

which also holds for $h = 0$. This equation is called a **first-order Taylor formula** for approximating $f(c+h) - f(c)$ by $f'(c)h$. The error term is $hE_c(h)$. It follows from (1.2) that $E_c(h) \to 0$ as $h \to 0$. The error term $hE_c(h)$ is said to be of smaller order than h as $h \to 0$.

We must note the following two important properties of the formula (1.3):

1. The quantity $f'(c)h$ is a linear function of h, i.e., if we take $T_c(h) = f'(c)h$, then

$$T_c\left(ah_1 + bh_2\right) = aT_c\left(h_1\right) + bT_c\left(h_2\right)$$

2. The error term $hE_c(h)$ is of smaller order than h as $h \to 0$.

Keeping the above in mind, we define the notion of total derivative of a function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ in such a way that it preserves the above properties.

Definition 1.3.1. *Let* $\mathbf{f} : S \to \mathbb{R}^m$*. Let* $\mathbf{c}$ *be an interior point of* S*, and let* $B(\mathbf{c}; r)$ *be an* n*-ball lying in* S*. Let* $\mathbf{v}$ *be a point in* $\mathbb{R}^n$ *with* $\|\mathbf{v}\| < r$*, so that* $\mathbf{c} + \mathbf{v} \in B(\mathbf{c}; \boldsymbol{r})$*. The function* $\mathbf{f}$ *is said to be differentiable at* $\mathbf{c}$ *if there exists a linear function* $\mathbf{T_c} : \mathbb{R}^n \to \mathbb{R}^m$ *such that*

$$\mathbf{f}(\mathbf{c} + \mathbf{v}) = \mathbf{f}(\mathbf{c}) + \mathbf{T_c}(\mathbf{v}) + \|\mathbf{v}\|\mathbf{E_c}(\mathbf{v}) \tag{1.4}$$

where $\mathbf{E_c}(\mathbf{v}) \to \mathbf{0}$ *as* $\mathbf{v} \to \mathbf{0}$*.*

Equation (1.4) is called a **first-order Taylor formula** and it holds for all $\mathbf{v}$ in $\mathbb{R}^{\boldsymbol{n}}$ with $\|\mathbf{v}\| < r$. The linear function $\mathbf{T_c}$ is called the **total derivative** of $\mathbf{f}$ at $\mathbf{c}$. We can also write (1.4) in the form

$$\mathbf{f}(\mathbf{c} + \mathbf{v}) = \mathbf{f}(\mathbf{c}) + \mathbf{T_c}(\mathbf{v}) + o(\|\mathbf{v}\|) \text{ as } \mathbf{v} \to \mathbf{0}$$

The next results shows that the total derivative is unique, if it exists and gives a relation between total derivative and directional derivatives.

Theorem 1.3.2. *The total derivative of* $\mathbf{f}$ *at* $\mathbf{c}$ *is unique, if it exists*

Proof. Assume $\mathbf{f}$ is differentiable at $\mathbf{c}$ with total derivative $\mathbf{T}_1$ and $\mathbf{T}_2$.

Let $\mathbf{T} = \mathbf{T}_1 - \mathbf{T}_2$. We shall prove that $\mathbf{T}(\mathbf{v}) = 0$ for every $\mathbf{v}$.

Consider

$$\begin{aligned}
\mathbf{T}(\mathbf{v}) &= \mathbf{T}_1(v) - \mathbf{T}_2(v) \\
&= \mathbf{f}(c+v) - \mathbf{f}(c) - \mathbf{T}_1(v) - [\mathbf{f}(c+v) - \mathbf{f}(c) - \mathbf{T}_2(v)]
\end{aligned}$$

and

$$\|\mathbf{T}(\mathbf{v})\| \leq \|\mathbf{f}(c+v) - \mathbf{f}(c) - \mathbf{T}_1(v)\| + \|\mathbf{f}(c+v) - \mathbf{f}(c) - \mathbf{T}_2(v)\|$$

As $\mathbf{v} \to 0$, the two terms on the right tends to 0 and hence $\mathbf{T}(\mathbf{v}) = 0$. This implies that $\mathbf{T}_1 = \mathbf{T_2}$.

Theorem 1.3.3. *Assume* $\mathbf{f}$ *is differentiable at* $\mathbf{c}$ *with total derivative* $\mathbf{T_c}$*. Then the directional derivative* $\mathbf{f}'(\mathbf{c};\mathbf{u})$ *exists for every* $\mathbf{u}$ *in* $\mathbb{R}^n$ *and we have*

$$\mathbf{T_c}(\mathbf{u}) = \mathbf{f}'(\mathbf{c};\mathbf{u})$$

Proof. If $\mathbf{v} = \mathbf{0}$, then $\mathbf{f}'(\mathbf{c};\mathbf{0}) = \mathbf{0}$ and $\mathbf{T_c}(\mathbf{0}) = \mathbf{0}$.

Assume that $\mathbf{v} \neq \mathbf{0}$. Taking $\mathbf{v} = h\mathbf{u}$ in Taylor's formula (1.4),

$$\mathbf{f}(\mathbf{c}+\mathbf{v}) = \mathbf{f}(\mathbf{c}) + \mathbf{T_c}(\mathbf{v}) + \|\mathbf{v}\|\mathbf{E_c}(\mathbf{v})$$

with $h \neq 0$, we have

$$\mathbf{f}(\mathbf{c}+h\mathbf{u}) - \mathbf{f}(\mathbf{c}) = \mathbf{T_c}(h\mathbf{u}) + \|h\mathbf{u}\|\mathbf{E_c}(\mathbf{v}) = h\mathbf{T_c}(\mathbf{u}) + |h|\|\mathbf{u}\|\mathbf{E_c}(\mathbf{v})$$

Dividing the above equation by h on both sides, we have

$$\frac{\mathbf{f}(\mathbf{c}+h\mathbf{u}) - \mathbf{f}(\mathbf{c})}{h} = \mathbf{T_c}(\mathbf{u}) + \frac{|h|}{h}\|\mathbf{u}\|\mathbf{E_c}(\mathbf{v})$$

Allowing $h \to 0$, we note that $\mathbf{v} \to 0$ which implies that $\mathbf{E_c}(\mathbf{v}) \to 0$ and the last term on the right tends to 0. Therefore,

$$\mathbf{T_c}(\mathbf{u}) = \lim_{h\to 0} \frac{\mathbf{f}(\mathbf{c}+h\mathbf{u}) - \mathbf{f}(\mathbf{c})}{h} = \mathbf{f}'(\mathbf{c};\mathbf{u}).$$

Theorem 1.3.4. *If* $\mathbf{f}$ *is differentiable at* $\mathbf{c}$*, then* $\mathbf{f}$ *is continuous at* $\mathbf{c}$*.*

Proof. Note that if $\mathbf{v} \to \mathbf{0}$ in the Taylor formula (1.4),

$$\mathbf{f}(\mathbf{c}+\mathbf{v}) = \mathbf{f}(\mathbf{c}) + \mathbf{T_c}(\mathbf{v}) + \|\mathbf{v}\|\mathbf{E_c}(\mathbf{v}),$$

then the error term $\|\mathbf{v}\|\mathbf{E_c}(\mathbf{v}) \to \mathbf{0}$.

If $\mathbf{v} \in \mathbb{R}^n$ and $\mathbf{v} = v_1\mathbf{u}_1 + \cdots + v_n\mathbf{u}_n$, where $\mathbf{u}_1, \ldots, \mathbf{u}_n$ are the unit coordinate vectors, then by linearity property we have

$$\mathbf{T_c}(\mathbf{v}) = v_1\mathbf{T_c}(\mathbf{u}_1) + \cdots + v_n\mathbf{T_c}(\mathbf{u}_n)$$

As $\mathbf{v} \to 0$, each term on the right tends to 0 and hence $\mathbf{T_c}(\mathbf{v}) \to 0$. Thus, we have

$$\lim_{\mathbf{v}\to 0} \mathbf{f}(\mathbf{c}+\mathbf{v}) = \mathbf{f}(\mathbf{c})$$

which implies that $\mathbf{f}$ is continuous at $\mathbf{c}$.

Note. *Hereafter, we will use the notation* $\mathbf{T_c} = \mathbf{f}'(\mathbf{c})$ *for the total derivative to resemble the notation used in the one-dimensional theory. With this notation, the Taylor formula* (1.4) *takes the form*

$$\mathbf{f}(\mathbf{c}+\mathbf{v}) = \mathbf{f}(\mathbf{c}) + \mathbf{f}'(\mathbf{c})(\mathbf{v}) + \|\mathbf{v}\|\mathbf{E_c}(\mathbf{v})$$

where $\mathbf{E_c}(\mathbf{v}) \to \mathbf{0}$ *as* $\mathbf{v} \to \mathbf{0}$.

However, it should be noted that $\mathbf{f}'(\mathbf{c}) : \mathbb{R}^n \to \mathbb{R}^m$ *is a linear function and is not a real number. The vector* $\mathbf{f}'(\mathbf{c})(\mathbf{v})$ *in* $\mathbb{R}^m$ *denotes the value of* $\mathbf{f}'(\mathbf{c})$ *at* $\mathbf{v}$.

Example 1.3.5. *If* $\mathbf{f}$ *is itself a linear function, i.e.,* $\mathbf{f}(\mathbf{c}+\mathbf{v}) = \mathbf{f}(\mathbf{c})+\mathbf{f}(\mathbf{v})$, *then the Taylor's formula takes the form*

$$\mathbf{f}(\mathbf{v}) = \mathbf{f}'(\mathbf{c})(\mathbf{v}) + \|\mathbf{v}\|\mathbf{E_c}(\mathbf{v})$$

for every $\mathbf{c}$. *Therefore, the derivative* $\mathbf{f}'(\mathbf{c}) = \mathbf{f}$. *In other words, the total derivative of a linear function is the function itself.*

1.4 The Total Derivative in terms of Partial Derivatives

Now we are going to prove that the vector $\mathbf{f}'(\mathbf{c})(\mathbf{v})$ can be expressed as a linear combination of the partial derivatives of $\mathbf{f}$.

Theorem 1.4.1. *Let $\mathbf{f} : S \to \mathbb{R}^m$ be differentiable at an interior point $\mathbf{c}$ of S, where $S \subseteq \mathbb{R}^n$. If $\mathbf{v} = v_1\mathbf{u}_1 + \cdots + v_n\mathbf{u}_n$, where $\mathbf{u}_1, \ldots, \mathbf{u}_n$ are the unit coordinate vectors in $\mathbb{R}^{\mathbf{n}}$, then*

$$\mathbf{f}'(\mathbf{c})(\mathbf{v}) = \sum_{k=1}^{n} v_k D_k \mathbf{f}(\mathbf{c})$$

In particular, if f is real-valued $(m = 1)$ we have

$$f'(\mathbf{c})(\mathbf{v}) = \nabla f(\mathbf{c}) \cdot \mathbf{v} \tag{1.5}$$

where $\nabla f(\mathbf{c}) = (D_1 f(\mathbf{c}), \ldots, D_n f(\mathbf{c}))$.

Proof. By the linearity of $\mathbf{f}'(\mathbf{c})$, we write

$$\mathbf{f}'(\mathbf{c})(\mathbf{v}) = \sum_{k=1}^{n} \mathbf{f}'(\mathbf{c})\,(v_k\mathbf{u}_k) = \sum_{k=1}^{n} v_k \mathbf{f}'(\mathbf{c})\,(\mathbf{u}_k) = \sum_{k=1}^{n} v_k \mathbf{f}'\,(\mathbf{c}; \mathbf{u}_k) = \sum_{k=1}^{n} v_k D_k \mathbf{f}(\mathbf{c}).$$

In the case, f is real-valued, we have

$$f'(\mathbf{c})(\mathbf{v}) = \sum_{k=1}^{n} v_k D_k \mathbf{f}(\mathbf{c}) = \nabla f(\mathbf{c}) \cdot \mathbf{v},$$

where $\nabla f(\mathbf{c}) = \left(\dfrac{\partial f}{\partial x_1}(c), \ldots, \dfrac{\partial f}{\partial x_k}(c) \right)$.

Note. *The vector $\nabla f(\mathbf{c})$ in (1.5) is called the* **gradient vector** *of f at c. It is defined at each point where the partials $D_1 f, \ldots, D_n f$ exist. The Taylor formula for a real-valued function f will have the the form*

$$f(\mathbf{c} + \mathbf{v}) = f(\mathbf{c}) + \nabla f(\mathbf{c}) \cdot \mathbf{v} + o(\|\mathbf{v}\|) \quad \textit{as } \mathbf{v} \to \mathbf{0}$$

1.5 The Matrix of a Linear Function

In this section we shall discuss some elementary facts from linear algebra that will be useful in certain calculations with derivatives.

Let $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^m$ be a linear function. (In our applications, $\mathbf{T}$ will be the total derivative of a function $\mathbf{f}$.) We will show that $\mathbf{T}$ determines an $m \times n$ matrix of scalars which is obtained as follows:

Let $\mathbf{u}_1, \ldots, \mathbf{u}_n$ and $\mathbf{e}_1, \ldots, \mathbf{e}_m$ denote the unit coordinate vectors in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. If $\mathbf{x} \in \mathbb{R}^n$, we have $\mathbf{x} = x_1\mathbf{u}_1 + \cdots + x_n\mathbf{u}_n$ so, by linearity,

$$\mathbf{T}(\mathbf{x}) = \sum_{k=1}^{n} x_k \mathbf{T}(\mathbf{u}_k)$$

Therefore $\mathbf{T}$ is completely determined by its action on the coordinate vectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$.

Since $\mathbf{T}(\mathbf{u}_k) \in \mathbb{R}^m$, we can write $\mathbf{T}(\mathbf{u}_k)$ as a linear combination of $e_1, \ldots, e_m$, say

$$\mathbf{T}(\mathbf{u}_k) = \sum_{i=1}^{m} t_{ik}\mathbf{e}_i,$$

where the scalars $t_{1k}, \ldots, t_{mk}$ are the coordinates of $\mathbf{T}(\mathbf{u}_k)$. These scalars can be written vertically as follows:

$$\begin{bmatrix} t_{1k} \\ t_{2k} \\ \vdots \\ t_{mk} \end{bmatrix}$$

This array is called a column vector. We form the column vector for each of $\mathbf{T}(\mathbf{u}_1), \ldots, \mathbf{T}(\mathbf{u}_n)$ and place them side by side to obtain the rectangular array

$$\begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ t_{21} & t_{22} & \cdots & t_{2n} \\ \vdots & \vdots & & \vdots \\ t_{m1} & t_{m2} & \cdots & t_{mn} \end{bmatrix} \tag{1.6}$$

This is called the matrix of $\mathbf{T}$ and is denoted by $m(\mathbf{T})$ which consists of m rows and n columns. We also use the notation $m(\mathbf{T}) = (t_{ik})$ to denote the above matrix.

Theorem 1.5.1. *let $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^m$ and $\mathbf{S} : \mathbb{R}^m \to \mathbb{R}^p$ be two linear functions, with the domain of $\mathbf{S}$ containing the range of $\mathbf{T}$. Then we can form the composition $\mathbf{S} \circ \mathbf{T}$ defined by*

$$(\mathbf{S} \circ \mathbf{T})(\mathbf{x}) = \mathbf{S}[\mathbf{T}(\mathbf{x})] \quad \textit{for all } \mathbf{x} \textit{ in } \mathbb{R}^n$$

The composition $\mathbf{S} \circ \mathbf{T}$ is also linear and it maps $\mathbb{R}^n$ into $\mathbb{R}^p$. Further,

$$m(\mathbf{S} \circ \mathbf{T}) = m(\mathbf{S} \circ \mathbf{T}) = \left[\sum_{k=1}^{m} s_{ik}t_{kj}\right]_{i,j=1}^{p,n}$$

Proof. We shall first prove the linearity of $\mathbf{S} \circ \mathbf{T}$. For $x_1, x_2 \in \mathbb{R}^n$ and scalars α, β, we have

$$\begin{aligned}
(\mathbf{S} \circ \mathbf{T})(\alpha x_1 + \beta x_2) &= \mathbf{S}(\mathbf{T}(\alpha x_1 + \beta x_2)) \\
&= \mathbf{S}(\alpha \mathbf{T} x_1 + \beta \mathbf{T} x_2)) \\
&= \alpha \mathbf{S}(\mathbf{T} x_1) + \beta \mathbf{S}(\mathbf{T} x_2)) \\
&= \alpha(\mathbf{S} \circ \mathbf{T})(x_1) + \beta(\mathbf{S} \circ \mathbf{T})(x_2)
\end{aligned}$$

which shows that $\mathbf{S} \circ \mathbf{T}$ is linear map from $\mathbb{R}^n$ into $\mathbb{R}^p$.

Denote the unit coordinate vectors in $\mathbb{R}^n, \mathbb{R}^m$, and $\mathbb{R}^p$, respectively, by $\mathbf{u}_1, \ldots, \mathbf{u}_n,\ \mathbf{e}_1, \ldots, \mathbf{e}_m$, and $\mathbf{w}_1, \ldots, \mathbf{w}_p$.

Suppose that $\mathbf{S}$ and $\mathbf{T}$ have matrices (s_{ij}) and (t_{ij}), respectively. This means that

$$\mathbf{S}\left(\mathbf{e}_k\right) = \sum_{i=1}^{p} s_{ik} \mathbf{w}_i \text{ and } \mathbf{T}\left(\mathbf{u}_j\right) = \sum_{k=1}^{m} t_{kj} \mathbf{e}_k$$

for $k = 1, 2, \ldots, m$, and $j = 1, 2, \ldots, n$. Then

$$(\mathbf{S} \circ \mathbf{T})\left(\mathbf{u}_j\right) = \mathbf{S}\left[\mathbf{T}\left(\mathbf{u}_j\right)\right] = \sum_{k=1}^{m} t_{kj} \mathbf{S}\left(\mathbf{e}_k\right) = \sum_{k=1}^{m} t_{kj} \sum_{i=1}^{p} s_{ik} \mathbf{w}_i = \sum_{i=1}^{p} \left(\sum_{k=1}^{m} s_{ik} t_{kj} \right) \mathbf{w}_i$$

so that

$$m(\mathbf{S} \circ \mathbf{T}) = \left[\sum_{k=1}^{m} s_{ik} t_{kj} \right]_{i,j=1}^{p,n}$$

In other words, $m(\mathbf{S} \circ \mathbf{T})$ is a $p \times n$ matrix whose entry in the ith row and jth column is $\sum_{k=1}^{m} s_{ik} t_{kj}$, the dot product of the ith row of $m(\mathbf{S})$ with the jth column of $m(\mathbf{T})$. This matrix is also called the product $m(\mathbf{S})m(\mathbf{T})$. Thus, $m(\mathbf{S} \circ \mathbf{T}) = m(\mathbf{S})m(\mathbf{T})$.

1.6 The Jacobian Matrix

Next we shall show find matrices associated with total derivatives.

Suppose $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at a point $\mathbf{c} \in \mathbb{R}^n$, and let $\mathbf{T} = \mathbf{f}'(\mathbf{c})$ be the total derivative of $\mathbf{f}$ at $\mathbf{c}$. To find the matrix associated with $\mathbf{T}$ we consider its action on the unit coordinate vectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$. We know that

$$\mathbf{T}\left(\mathbf{u}_k\right) = \mathbf{f}'\left(\mathbf{c}; \mathbf{u}_k\right) = D_k \mathbf{f}(\mathbf{c})$$

If we take $\mathbf{f} = (f_1, \ldots, f_m)$ and $\mathbf{e}_1, \ldots, \mathbf{e}_m$ are the unit coordinate vectors of $\mathbb{R}^m$, then $D_k\mathbf{f} = (D_k f_1, \ldots, D_k f_m)$ and

$$\mathbf{T}(\mathbf{u}_k) = D_k\mathbf{f}(\mathbf{c}) = \sum_{i=1}^{m} D_k f_i(\mathbf{c})\mathbf{e}_i$$

Therefore the matrix of $\mathbf{T}$ is $m(\mathbf{T}) = (D_k f_i(\mathbf{c}))$ and will be denoted by $\mathbf{Df}(\mathbf{c})$, i.e.,

$$\mathbf{Df}(\mathbf{c}) = \begin{bmatrix} D_1 f_1(\mathbf{c}) & D_2 f_1(\mathbf{c}) & \cdots & D_n f_1(\mathbf{c}) \\ D_1 f_2(\mathbf{c}) & D_2 f_2(\mathbf{c}) & \cdots & D_n f_2(\mathbf{c}) \\ \vdots & \vdots & & \vdots \\ D_1 f_m(\mathbf{c}) & D_2 f_m(\mathbf{c}) & \cdots & D_n f_m(\mathbf{c}) \end{bmatrix}. \tag{1.7}$$

This matrix is called the **Jacobian matrix** of $\mathbf{f}$ at $\mathbf{c}$. The Jacobian matrix $\mathbf{Df}(\mathbf{c})$ is defined at each point $\mathbf{c}$ in $\mathbb{R}^n$ where all the partial derivatives $D_k f_i(\mathbf{c})$ exist.

The k-th row of the Jacobian matrix (1.7) is a vector in $\mathbb{R}^n$ called the gradient vector of f_k, denoted by $\nabla f_k(\mathbf{c})$, i.e.,

$$\nabla f_k(\mathbf{c}) = (D_1 f_k(\mathbf{c}), \ldots, D_n f_k(\mathbf{c}))$$

In the special case when f is real-valued $(m = 1)$, the Jacobian matrix consists of only one row. In this case $\mathbf{D}f(\mathbf{c}) = \nabla f(\mathbf{c})$, and Equation (1.5) of Theorem 1.5 shows that the directional derivative $f'(\mathbf{c}; \mathbf{v})$ is the dot product of the gradient vector $\nabla f(\mathbf{c})$ with the direction $\mathbf{v}$.

For a vector-valued function $\mathbf{f} = (f_1, \ldots, f_m)$, we have

$$\mathbf{f}'(\mathbf{c})(\mathbf{v}) = \mathbf{f}'(\mathbf{c}; \mathbf{v}) = \sum_{k=1}^{m} f_k'(\mathbf{c}; \mathbf{v})\mathbf{e}_k = \sum_{k=1}^{m} \{\nabla f_k(\mathbf{c}) \cdot \mathbf{v}\}\, \mathbf{e}_k \tag{1.8}$$

so the vector $\mathbf{f}'(\mathbf{c})(\mathbf{v})$ has components

$$(\nabla f_1(\mathbf{c}) \cdot \mathbf{v}, \ldots, \nabla f_m(\mathbf{c}) \cdot \mathbf{v})$$

Thus, the components of $\mathbf{f}'(\mathbf{c})(\mathbf{v})$ are obtained by taking the dot product of the successive rows of the Jacobian matrix with the vector $\mathbf{v}$. If we regard $\mathbf{f}'(\mathbf{c})(\mathbf{v})$ as an $m \times 1$ matrix, or column vector, then $\mathbf{f}'(\mathbf{c})(\mathbf{v})$ is equal to the matrix product $\mathbf{Df}(\mathbf{c})\mathbf{v}$, where $\mathrm{Df}(\mathbf{c})$ is the $m \times n$ Jacobian matrix and $\mathbf{v}$ is regarded as an $n \times 1$ matrix, or column vector.

Note. *Equation (1.8), used in conjunction with the triangle inequality and the Cauchy-Schwarz inequality, gives us*

$$\|\mathbf{f}'(\mathbf{c})(\mathbf{v})\| = \left\| \sum_{k=1}^{m} \{\nabla f_k(\mathbf{c}) \cdot \mathbf{v}\} \, \mathbf{e}_k \right\| \leq \sum_{k=1}^{m} |\nabla f_k(\mathbf{c}) \cdot \mathbf{v}| \leq \|\mathbf{v}\| \sum_{k=1}^{m} \|\nabla f_k(\mathbf{c})\|$$

Therefore we have

$$\|\mathbf{f}'(\mathbf{c})(\mathbf{v})\| \leq M\|\mathbf{v}\| \tag{1.9}$$

where $M = \sum_{k=1}^{m} \|\nabla f_k(\mathbf{c})\|$. *This inequality will be used in the proof of the chain rule. It also shows that* $\mathbf{f}'(\mathbf{c})(\mathbf{v}) \to \mathbf{0}$ *as* $\mathbf{v} \to \mathbf{0}$.

1.7 The Chain Rule

The chain rule is a fundamental rule in calculus for differentiating compositions of functions. Let us first recall the chain rule in one-dimension without proving it.

Theorem 1.7.1. *Let f be defined on an open interval S, let g be defined on $f(S)$, and consider the composite function $g \circ f$ defined on S by the equation*

$$(g \circ f)(x) = g[f(x)].$$

Assume that there is a point c in S such that $f(c)$ is an interior point of $f(S)$. If f is differentiable at c and if g is differentiable at $f(c)$, then $g \circ f$ is differentiable at c and we have

$$(g \circ f)'(c) = g'[f(c)]f'(c).$$

For functions of several variables, the chain rule becomes more involved but follows the same principle.

Let $\mathbf{f}$ and $\mathbf{g}$ be functions such that the composition $\mathbf{h} = \mathbf{f} \circ \mathbf{g}$ is defined in a neighborhood of a point $\mathbf{a}$. The chain rule helps to evaluate the total derivative of $\mathbf{h}$ in terms of total derivatives of $\mathbf{f}$ and $\mathbf{g}$.

Theorem 1.7.2. *Assume that $\mathbf{g}$ is differentiable at $\mathbf{a}$, with total derivative $\mathbf{g}'(\mathbf{a})$. Let $\mathbf{b} = \mathbf{g}(\mathbf{a})$ and assume that $\mathbf{f}$ is differentiable at $\mathbf{b}$, with total derivative $\mathbf{f}'(\mathbf{b})$. Then the* *composite function* $\mathbf{h} = \mathbf{f} \circ \mathbf{g}$ *is differentiable at* $\mathbf{a}$*, and the total derivative* $\mathbf{h}'(\mathbf{a})$ *is given by*

$$\mathbf{h}'(\mathbf{a}) = \mathbf{f}'(\mathbf{b}) \circ \mathbf{g}'(\mathbf{a})$$

the composition of the linear functions $\mathbf{f}'(\mathbf{b})$ *and* $\mathbf{g}'(\mathbf{a})$.

Proof. We shall prove this interesting theorem step-by-step.

1. Since g is differentiable at $\mathbf{a}$ with total derivative $\mathbf{g}'(\mathbf{a})$, we have

$$\mathbf{g}(\mathbf{a}+\mathbf{y}) - \mathbf{g}(\mathbf{a}) = \mathbf{g}'(\mathbf{a})(\mathbf{y}) + \|\mathbf{y}\|\mathbf{E}_{\mathbf{a}}(\mathbf{y})$$

   where $\mathbf{E}_{\mathbf{a}}(\mathbf{y}) \to 0$ as $\mathbf{y} \to 0$.

   Taking $\mathbf{b} = \mathbf{g}(\mathbf{a})$ and $\mathbf{v} = \mathbf{g}(\mathbf{a}+\mathbf{y}) - \mathbf{g}(\mathbf{a})$, the above equation becomes

$$\mathbf{v} = \mathbf{g}'(\mathbf{a})(\mathbf{y}) + \|\mathbf{y}\|\mathbf{E}_{\mathbf{a}}(\mathbf{y})$$

2. Since $\mathbf{f}$ is differentiable at $\mathbf{b}$ with total derivative $\mathbf{f}'(\mathbf{b})$, we have

$$\begin{aligned} \mathbf{f}(\mathbf{b}+\mathbf{v}) - \mathbf{f}(\mathbf{b}) &= \mathbf{f}'(\mathbf{b})(\mathbf{v}) + \|\mathbf{v}\|\mathbf{E}_{\mathbf{b}}(\mathbf{v}) \\ &= \mathbf{f}'(\mathbf{b})[\mathbf{g}'(\mathbf{a})(\mathbf{y}) + \|\mathbf{y}\|\mathbf{E}_{\mathbf{a}}(\mathbf{y})] + \|\mathbf{v}\|\mathbf{E}_{\mathbf{b}}(\mathbf{v}) \end{aligned}$$

   where $\mathbf{E}_{\mathbf{b}}(\mathbf{v}) \to 0$ as $\mathbf{v} \to 0$.

3. Consider the difference $\mathbf{h}(\mathbf{a}+\mathbf{y}) - \mathbf{h}(\mathbf{a})$ for small $\|\mathbf{y}\|$, and show that we have a first-order Taylor formula. We have

$$\mathbf{h}(\mathbf{a}+\mathbf{y}) - \mathbf{h}(\mathbf{a}) = \mathbf{f}[\mathbf{g}(\mathbf{a}+\mathbf{y})] - \mathbf{f}[\mathbf{g}(\mathbf{a})] = \mathbf{f}(\mathbf{b}+\mathbf{v}) - \mathbf{f}(\mathbf{b}) \tag{1.10}$$

   where $\mathbf{b} = \mathbf{g}(\mathbf{a})$ and $\mathbf{v} = \mathbf{g}(\mathbf{a}+\mathbf{y}) - \mathbf{b}$. The Taylor formula for $\mathbf{g}(\mathbf{a}+\mathbf{y})$ implies

$$\mathbf{v} = \mathbf{g}'(\mathbf{a})(\mathbf{y}) + \|\mathbf{y}\|\mathbf{E}_{\mathbf{a}}(\mathbf{y}), \quad \text{where } \mathbf{E}_{\mathbf{a}}(\mathbf{y}) \to \mathbf{0} \text{ as } \mathbf{y} \to \mathbf{0} \tag{1.11}$$

   The Taylor formula for $\mathbf{f}(\mathbf{b}+\mathbf{v})$ implies

$$\mathbf{f}(\mathbf{b}+\mathbf{v}) - \mathbf{f}(\mathbf{b}) = \mathbf{f}'(\mathbf{b})(\mathbf{v}) + \|\mathbf{v}\|\mathbf{E}_{\mathbf{b}}(\mathbf{v}), \quad \text{where } E_{\mathbf{b}}(\mathbf{v}) \to \mathbf{0} \text{ as } \mathbf{v} \to \mathbf{0} \tag{1.12}$$

Using (1.11) in (1.12) we find

$$\begin{aligned}\mathbf{f}(\mathbf{b}+\mathbf{v})-\mathbf{f}(\mathbf{b}) &= \mathbf{f}'(\mathbf{b})\left[\mathbf{g}'(\mathbf{a})(\mathbf{y})\right]+\mathbf{f}'(\mathbf{b})\left[\|\mathbf{y}\|\mathbf{E}_{\mathbf{a}}(\mathbf{y})\right]+\|\mathbf{v}\|\mathbf{E}_{\mathbf{b}}(\mathbf{v}) \\ &= \mathbf{f}'(\mathbf{b})\left[\mathbf{g}'(\mathbf{a})(\mathbf{y})\right]+\|\mathbf{y}\|\mathbf{E}(\mathbf{y}) \qquad (1.13)\end{aligned}$$

where $\mathbf{E}(\mathbf{0})=\mathbf{0}$ and

$$\mathbf{E}(\mathbf{y})=\mathbf{f}'(\mathbf{b})\left[\mathbf{E}_{\mathbf{a}}(\mathbf{y})\right]+\frac{\|\mathbf{v}\|}{\|\mathbf{y}\|}\mathbf{E}_{\mathbf{b}}(\mathbf{v}) \quad \text{ if } \mathbf{y}\neq\mathbf{0} \qquad (1.14)$$

To complete the proof we need to show that $\mathbf{E}(\mathbf{y})\to\mathbf{0}$ as $\mathbf{y}\to\mathbf{0}$.

The first term on the right of (1.14) tends to 0 as $y\to 0$ because $E_a(\mathbf{y})\to 0$. In the second term, the factor $\mathbf{E}_b(\mathbf{v})\to 0$ because $\mathbf{v}\to 0$ as $\mathbf{y}\to 0$. Now we show that the quotient $\|\mathbf{v}\|/\|\mathbf{y}\|$ remains bounded as $\mathbf{y}\to\mathbf{0}$. Using (1.11) and using the fact that $\|\mathbf{f}'(\mathbf{c})(\mathbf{v})\|\le M\|\mathbf{v}\|$ to estimate the numerator we find

$$\|\mathbf{v}\|\le\|\mathbf{g}'(\mathbf{a})(\mathbf{y})\|+\|\mathbf{y}\|\,\|\mathbf{E}_{\mathbf{a}}(\mathbf{y})\|\le\|\mathbf{y}\|\left\{M+\|\mathbf{E}_{\mathbf{a}}(\mathbf{y})\|\right\}$$

where $M=\sum_{k=1}^{m}\|\nabla g_k(\mathbf{a})\|$. Hence

$$\frac{\|\mathbf{v}\|}{\|\mathbf{y}\|}\le M+\|\mathbf{E}_{\mathbf{a}}(\mathbf{y})\|\,,$$

so $\|\mathbf{v}\|/\|\mathbf{y}\|$ remains bounded as $\mathbf{y}\to\mathbf{0}$. Using (1.10) and (1.13) we obtain the Taylor formula

$$\mathbf{h}(\mathbf{a}+\mathbf{y})-\mathbf{h}(\mathbf{a})=\mathbf{f}'(\mathbf{b})\left[\mathbf{g}'(\mathbf{a})(\mathbf{y})\right]+\|\mathbf{y}\|\mathbf{E}(\mathbf{y})$$

where $\mathbf{E}(\mathbf{y})\to\mathbf{0}$ as $\mathbf{y}\to\mathbf{0}$. This proves that $\mathbf{h}$ is differentiable at a and that its total derivative at $\mathbf{a}$ is the composition $\mathbf{f}'(\mathbf{b})\circ\mathbf{g}'(\mathbf{a})$.

1.8 The Mean-value Theorem for Differentiable Functions

The Mean-Value Theorem for functions from $\mathbb{R}^1$ to $\mathbb{R}^1$ states that

$$f(y)-f(x)=f'(z)(y-x)$$

where $z \in (x, y)$. We will see that this equation is false for vector-valued functions from $\mathbb{R}^n$ to $\mathbb{R}^m$, when $m > 1$. In this section, we will derive a generalized Mean-Value Theorem for vector-valued functions.

Let us use the notation $L(\mathbf{x}, \mathbf{y})$ to denote the line segment joining two points $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$. That is,

$$L(\mathbf{x}, \mathbf{y}) = \{t\mathbf{x} + (1-t)\mathbf{y} : 0 \leq t \leq 1\}.$$

Theorem 1.8.1. (Mean-Value Theorem) *Let $S \subset \mathbb{R}^n$ be open and assume that $\mathbf{f} : S \to \mathbb{R}^m$ is differentiable at each point of S. Let $\mathbf{x}, \mathbf{y} \in S$ such that $L(\mathbf{x}, \mathbf{y}) \subseteq S$. Then for every vector $\mathbf{a} \in \mathbb{R}^m$, there is a point $\mathbf{z} \in L(\mathbf{x}, \mathbf{y})$ such that*

$$\mathbf{a} \cdot \{\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x})\} = \mathbf{a} \cdot \{\mathbf{f}'(\mathbf{z})(\mathbf{y} - \mathbf{x})\} \tag{1.15}$$

Proof. Let us take $\mathbf{u} = \mathbf{y} - \mathbf{x}$. Since S is open and $L(\mathbf{x}, \mathbf{y}) \subseteq S$, there is a $\delta > 0$ such that $\mathbf{x} + t\mathbf{u} \in S$ for all real $t \in (-\delta, 1+\delta)$. Let $\mathbf{a} \in \mathbb{R}^m$ be a fixed vector and let us define $F : (-\delta, 1+\delta) \to \mathbb{R}$ by

$$F(t) = \mathbf{a} \cdot \mathbf{f}(\mathbf{x} + t\mathbf{u})$$

Then F is differentiable on $(-\delta, 1+\delta)$ and its derivative is given by

$$F'(t) = \mathbf{a} \cdot \mathbf{f}'(\mathbf{x} + t\mathbf{u}; \mathbf{u}) = \mathbf{a} \cdot \{\mathbf{f}'(\mathbf{x} + t\mathbf{u})(\mathbf{u})\}$$

By the usual Mean-Value Theorem, we have

$$F(1) - F(0) = F'(\theta), \quad \text{where } 0 < \theta < 1$$

Note that

$$F(1) - F(0) = \mathbf{a} \cdot \{\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x})\}$$

and

$$F'(\theta) = \mathbf{a} \cdot \{\mathbf{f}'(\mathbf{x} + \theta\mathbf{u})(\mathbf{u})\} = \mathbf{a} \cdot \{\mathbf{f}'(\mathbf{z})(\mathbf{y} - \mathbf{x})\}$$

where $\mathbf{z} = \mathbf{x} + \theta\mathbf{u} \in L(\mathbf{x}, \mathbf{y})$. Thus, we obtain (1.15).

Note. *If S is convex, then $L(\mathbf{x}, \mathbf{y}) \subseteq S$ for all $\mathbf{x}, \mathbf{y} \in S$, the above MVT holds for all $\mathbf{x}, \mathbf{y} \in S$.*

Examples

1. If f is real-valued $(m = 1)$, then we can take $a = 1$ in (1.15) to obtain

$$f(\mathbf{y}) - f(\mathbf{x}) = f'(\mathbf{z})\,(\mathbf{y} - \mathbf{x}) = \nabla f(\mathbf{z}) \cdot (\mathbf{y} - \mathbf{x}).$$

2. If f is vector-valued $(m > 1)$, and if $\mathbf{a} \in \mathbb{R}^m$ is a unit vector, (1.15) and the Cauchy-Schwarz inequality give us

$$\|\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x})\| \leq \|\mathbf{f}'(\mathbf{z})(\mathbf{y} - \mathbf{x})\|$$

   Using (1.9), we obtain

$$\|\mathbf{f}'(\mathbf{z})(\mathbf{y} - \mathbf{x})\| \leq M\|\mathbf{y} - \mathbf{x}\|$$

   and therefore

$$\|\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x})\| \leq M\|\mathbf{y} - \mathbf{x}\|$$

   where $M = \sum_{k=1}^{m} \|\nabla f_k(\mathbf{z})\|$. Note that M depends on $\mathbf{z}$ and hence on $\mathbf{x}$ and $\mathbf{y}$.

3. If S is convex and if all the partial derivatives $D_j f_k$ are bounded on S, then there is a constant $A > 0$ such that

$$\|\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x})\| \leq A\|\mathbf{y} - \mathbf{x}\|.$$

   In other words, $\mathbf{f}$ satisfies a Lipschitz condition on S.

Let us sum up

In this section, we have proved the mean-value theorem.

1.9 A Sufficient Condition for Differentiability

We have also seen that neither the existence of all partial derivatives nor the existence of all directional derivatives suffices to establish differentiability (since neither implies continuity). The next theorem shows that continuity of all but one of the partials does imply differentiability.

Theorem 1.9.1. *Assume that one of the partial derivatives $D_1\mathbf{f}, \ldots, D_n\mathbf{f}$ exists at $\mathbf{c}$ and that the remaining $n-1$ partial derivatives exist in some n-ball $B(c)$ and are continuous at $\mathbf{c}$. Then $\mathbf{f}$ is differentiable at $\mathbf{c}$.*

Proof. If $\mathbf{f} = (f_1, \ldots, f_m)$, then $\mathbf{f}$ is differentiable at $\mathbf{c}$ if and only if, each component f_k (real-valued) is differentiable at $\mathbf{c}$. Therefore, it suffices to prove the theorem when $\mathbf{f}$ is real-valued.

Assume that $D_1 f(c)$ exists and that the partials $D_2 f, \ldots, D_n f$ exist and are continuous at $\mathbf{c}$.

The only candidate for $f'(\mathbf{c})$ is the gradient vector $\nabla f(\mathbf{c})$.

To prove that $\mathbf{f}$ is differentiable at $\mathbf{c}$, it is enough to show that

$$f(\mathbf{c}+\mathbf{v}) - f(\mathbf{c}) = \nabla f(\mathbf{c}) \cdot \mathbf{v} + o(\|\mathbf{v}\|) \quad \text{as } \mathbf{v} \to \mathbf{0}$$

The idea is to express the difference $f(\mathbf{c}+\mathbf{v}) - f(\mathbf{c})$ as a sum of n terms, where the k th term is an approximation to $D_k f(\mathbf{c}) v_k$.

Let $\mathbf{v} = \lambda \mathbf{y}$, where $\|\mathbf{y}\| = 1$ and $\lambda = \|\mathbf{v}\|$. We shall take λ small enough so that $\mathbf{c}+\mathbf{v}$ lies in the ball $B(c)$ in which the partial derivatives $D_2 f, \ldots, D_n f$ exist. Since $\mathbf{y} \in \mathbb{R}^n$, we can express $\mathbf{y}$ as

$$\mathbf{y} = y_1 \mathbf{u}_1 + \cdots + y_n \mathbf{u}_n$$

where $\mathbf{u}_k$ is the k-th unit coordinate vector.

Now we write the difference $f(\mathbf{c}+\mathbf{v})-$ $f(c)$ as a telescoping sum as follows:

$$f(\mathbf{c}+\mathbf{v}) - f(\mathbf{c}) = f(\mathbf{c}+\lambda\mathbf{y}) - f(\mathbf{c}) = \sum_{k=1}^{n} \left\{ f\left(\mathbf{c}+\lambda\mathbf{v}_k\right) - f\left(\mathbf{c}+\lambda\mathbf{v}_{k-1}\right) \right\}, \qquad (1.16)$$

where

$$\mathbf{v}_0 = \mathbf{0}, \quad \mathbf{v}_1 = y_1\mathbf{u}_1, \quad \mathbf{v}_2 = y_1\mathbf{u}_1 + y_2\mathbf{u}_2, \ldots, \mathbf{v}_n = y_1\mathbf{u}_1 + \cdots + y_n\mathbf{u}_n$$

The first term in the sum is $f\left(\mathbf{c}+\lambda\mathbf{v}_1\right) - f(\mathbf{c}+\lambda\mathbf{v}_0) = f\left(\mathbf{c}+\lambda y_1\mathbf{u}_1\right) - f(\mathbf{c})$. Since $D_1 f(\mathbf{c})$ exists, we can write

$$f\left(\mathbf{c}+\lambda y_1\mathbf{u}_1\right) - f(\mathbf{c}) = \lambda y_1 D_1 f(\mathbf{c}) + \lambda y_1 E_1(\lambda)$$

where $E_1(\lambda) \to 0$ as $\lambda \to 0$.

For $k \geq 2$, the k-th term in the sum is

$$f\left(\mathbf{c} + \lambda \mathbf{v}_{k-1} + \lambda y_k \mathbf{u}_k\right) - f\left(\mathbf{c} + \lambda \mathbf{v}_{k-1}\right) = f\left(\mathbf{b}_k + \lambda y_k \mathbf{u}_k\right) - f\left(\mathbf{b}_k\right)$$

where $\mathbf{b}_k = \mathbf{c} + \lambda \mathbf{v}_{k-1}$. The two points $\mathbf{b}_k$ and $\mathbf{b}_k + \lambda y_k \mathbf{u}_k$ differ only in their k-th component, and we can apply the one-dimensional mean-value theorem for derivatives to write

$$f\left(\mathbf{b}_k + \lambda y_k \mathbf{u}_k\right) - f\left(\mathbf{b}_k\right) = \lambda y_k D_k f\left(\mathbf{a}_k\right) \tag{1.17}$$

where $\mathbf{a}_k$ lies on the line segment joining $\mathbf{b}_k$ to $\mathbf{b}_k + \lambda y_k \mathbf{u}_k$. Note that $\mathbf{b}_k \to \mathbf{c}$ and hence $\mathbf{a}_k \to \mathbf{c}$ as $\lambda \to 0$. Since each $D_k f$ is continuous at $\mathbf{c}$ for $k \geq 2$ we can write

$$D_k f\left(\mathbf{a}_k\right) = D_k f(\mathbf{c}) + E_k(\lambda), \qquad \text{where } E_k(\lambda) \to 0 \text{ as } \lambda \to 0$$

Using this in (1.17), we find that (1.16) becomes

$$f(\mathbf{c} + \mathbf{v}) - f(\mathbf{c}) = \lambda \sum_{k=1}^{n} D_k f(\mathbf{c}) y_k + \lambda \sum_{k=1}^{n} y_k E_k(\lambda) = \nabla f(\mathbf{c}) \cdot \mathbf{v} + \|\mathbf{v}\| E(\lambda)$$

where

$$E(\lambda) = \sum_{k=1}^{n} y_k E_k(\lambda) \to 0 \text{ as } \|\mathbf{v}\| \to 0$$

This completes the proof.

1.10 A Sufficient Condition for Equality of Mixed Partial Derivatives

The partial derivatives $D_1 \mathbf{f}, \ldots, D_n \mathbf{f}$ of a function from $\mathbb{R}^n$ to $\mathbb{R}^m$ are themselves functions from $\mathbb{R}^n$ to $\mathbb{R}^m$ and they, in turn, can have partial derivatives. These are called second-order partial derivatives. We use the following notation :

$$D_{r,k} \mathbf{f} = D_r\left(D_k \mathbf{f}\right) = \frac{\partial^2 \mathbf{f}}{\partial x_r \partial x_k}$$

Higher-order partial derivatives are similarly defined.

Example 1.10.1. *The example*

$$f(x,y) = \begin{cases} \dfrac{xy\,(x^2-y^2)}{(x^2+y^2)} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

shows that $D_{1,2}f(x,y)$ is not necessarily the same as $D_{2,1}f(x,y)$. In fact, in this example we have

$$D_1 f(x,y) = \frac{y\,(x^4+4x^2y^2-y^4)}{(x^2+y^2)^2}, \qquad \text{if } (x,y) \neq (0,0)$$

and $D_1 f(0,0) = 0$. Hence, $D_1 f(0,y) = -y$ for all y and therefore

$$D_{2,1}f(0,y) = -1, \quad D_{2,1}f(0,0) = -1$$

On the other hand, we have

$$D_2 f(x,y) = \frac{x\,(x^4-4x^2y^2-y^4)}{(x^2+y^2)^2}, \qquad \text{if } (x,y) \neq (0,0)$$

and $D_2 f(0,0) = 0$, so that $D_2 f(x,0) = x$ for all x. Therefore, $D_{1,2}f(x,0) = 1$, $D_{1,2}f(0,0) = 1$, and we see that $D_{2,1}f(0,0) \neq D_{1,2}f(0,0)$.

The following theorem basically gives us a criterion for determining when the two mixed partials $D_{1,2}\mathbf{f}$ and $D_{2,1}\mathbf{f}$ will be equal.

Theorem 1.10.2. *If both partial derivatives $D_r\mathbf{f}$ and $D_k\mathbf{f}$ exist in an n-ball $B(\mathbf{c};\delta)$ and if both are differentiable at $\mathbf{c}$, then*

$$D_{r,k}\mathbf{f}(\mathbf{c}) = D_{k,r}\mathbf{f}(\mathbf{c}) \tag{1.18}$$

Proof. If $\mathbf{f} = (f_1, \ldots, f_m)$, then $D_k\mathbf{f} = (D_k f_1, \ldots, D_k f_m)$. Therefore it suffices to prove the theorem for real-valued f. Also, since only two components are involved in (1.18), it suffices to consider the case $n = 2$. For simplicity, we assume that $\mathbf{c} = (0,0)$. We shall prove that

$$D_{1,2}f(0,0) = D_{2,1}f(0,0)$$

Choose $h \neq 0$ so that the square with vertices $(0,0), (h,0), (h,h)$, and $(0,h)$ lies in the 2-ball $B(0;\delta)$. Consider the quantity

$$\Delta(h) = f(h,h) - f(h,0) - f(0,h) + f(0,0)$$

We will show that $\Delta(h)/h^2$ tends to both $D_{2,1}f(0,0)$ and $D_{1,2}f(0,0)$ as $h \to 0$.

Let $G(x) = f(x,h) - f(x,0)$ and note that

$$\Delta(h) = G(h) - G(0). \tag{1.19}$$

By the one-dimensional Mean-value theorem we have

$$G(h) - G(0) = hG'(x_1) = h\left\{D_1 f(x_1,h) - D_1 f(x_1,0)\right\}$$

where x_1 lies between 0 and h. Since $D_1 f$ is differentiable at $(0,0)$, we have the first-order Taylor formulas

$$D_1 f(x_1,h) = D_1 f(0,0) + D_{1,1} f(0,0)x_1 + D_{2,1} f(0,0)h + \left(x_1^2 + h^2\right)^{1/2} E_1(h)$$

and

$$D_1 f(x_1,0) = D_1 f(0,0) + D_{1,1} f(0,0)x_1 + |x_1| E_2(h)$$

where $E_1(h)$ and $E_2(h) \to 0$ as $h \to 0$. Using these in (1.19) and (1.10) we find

$$\Delta(h) = D_{2,1} f(0,0)h^2 + E(h)$$

where $E(h) = h\left(x_1^2 + h^2\right)^{1/2} E_1(h) + h|x_1| E_2(h)$. Since $|x_1| \leq |h|$, we have

$$0 \leq |E(h)| \leq \sqrt{2}h^2 |E_1(h)| + h^2 |E_2(h)|$$

so

$$\lim_{h\to 0} \frac{\Delta(h)}{h^2} = D_{2,1} f(0,0)$$

Applying the same procedure to the function $H(y) = f(h,y) - f(0,y)$ in place of $G(x)$, we find that

$$\lim_{h\to 0} \frac{\Delta(h)}{h^2} = D_{1,2} f(0,0)$$

which completes the proof.

Theorems 1.10.1 and 1.10.2 leads to the following result:

Theorem 1.10.3. *If both partial derivatives $D_r\mathrm{f}$ and $D_k\mathrm{f}$ exist in an n-ball $B(\mathrm{c})$ and if both $D_{r,k}\mathbf{f}$ and $D_{k,r}\mathbf{f}$ are continuous at c, then*

$$D_{r,k}\mathbf{f}(\mathbf{c}) = D_{k,r}\mathbf{f}(\mathbf{c}).$$

1.11 Taylor's Formula for Functions from $\mathbb{R}^n$ to $\mathbb{R}^1$

The Taylor formula (or Taylor series) is a powerful tool in calculus for approximating a function near a point using polynomials. It expresses a smooth function as an infinite sum of terms based on the function's derivatives at a single point. When truncated, it provides polynomial approximations to the function, which become more accurate as more terms are included.

Taylor's formula for real-valued functions defined on $[a, b]$ can be stated as follows:

Theorem 1.11.1. *Let f be a function having finite n-th derivative $f^{(n)}$ everywhere in an open interval (a, b) and assume that $f^{(n-1)}$ is continuous on the closed interval $[a, b]$. Assume that $c \in [a, b]$. Then, for every x in $[a, b],\ x \neq c$, there exists a point x_1 interior to the interval joining x and c such that*

$$f(x) = f(c) + \sum_{k=1}^{n-1} \frac{f^{(k)}(c)}{k!}(x-c)^k + \frac{f^{(n)}(x_1)}{n!}(x-c)^n$$

The above Taylor's theorem can be extended to real-valued functions f defined on subsets of $\mathbb{R}^n$. In order to state the general theorem in a form which resembles the one-dimensional case, we introduce special symbols as follows:

If all second-order partial derivatives of f exist at $\mathbf{x} \in \mathbb{R}^n$, and if $\mathbf{t} = (t_1, \ldots, t_n)$ is an arbitrary point in $\mathbb{R}^n$, we write

$$f''(\mathbf{x}; \mathbf{t}) = \sum_{i=1}^{n} \sum_{j=1}^{n} D_{i,j} f(\mathbf{x}) t_j t_i.$$

We also define

$$f'''(\mathbf{x}; \mathbf{t}) = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} D_{i,j,k} f(\mathbf{x}) t_k t_j t_i$$

if all third-order partial derivatives exist at $\mathbf{x}$. The symbol $f^{(m)}(\mathbf{x}; \mathbf{t})$ is similarly defined if all m-th order partials exist.

These sums are analogous to the formula

$$f'(\mathbf{x}; \mathbf{t}) = \sum_{i=1}^{n} D_i f(\mathbf{x}) t_i,$$

for the directional derivative of a function which is differentiable at $\mathbf{x}$.

Theorem 1.11.2. (Taylor's formula). *Assume that f and all its partial derivatives of order $< m$ are differentiable at each point of an open set S in $\mathbb{R}^n$. If $\mathbf{a}, \mathbf{b} \in S$ such that $L(\mathbf{a}, \mathbf{b}) \subseteq S$, then there is a point $\mathrm{z} \in L(\mathbf{a}, \mathbf{b})$ such that*

$$f(\mathbf{b}) - f(\mathbf{a}) = \sum_{k=1}^{m-1} \frac{1}{k!} f^{(k)}(\mathbf{a}; \mathbf{b} - \mathbf{a}) + \frac{1}{m!} f^{(m)}(\mathbf{z}; \mathbf{b} - \mathbf{a})$$

Proof. Since S is open, there is a $\delta > 0$ such that a $+t(\mathbf{b} - \mathbf{a}) \in S$ for all real $t \in (-\delta, 1 + \delta)$. Define g on $(-\delta, 1 + \delta)$ by

$$g(t) = f[\mathbf{a} + t(\mathbf{b} - \mathbf{a})].$$

Then $f(\mathbf{b}) - f(\mathbf{a}) = g(1) - g(0)$. By applying the one-dimensional Taylor formula to g, we have

$$g(1) - g(0) = \sum_{k=1}^{m-1} \frac{1}{k!} g^{(k)}(0) + \frac{1}{m!} g^{(m)}(\theta), \quad \text{where } 0 < \theta < 1 \tag{1.20}$$

Note that g is a composite function given by $g(t) = f[\mathbf{p}(t)]$, where $\mathbf{p}(t) = \mathbf{a} + t(\mathbf{b} - \mathbf{a})$. The k-th component of $\mathbf{p}$ has derivative $p_k'(t) = b_k - a_k$. Applying the chain rule, we see that $g'(t)$ exists in the interval $(-\delta, 1 + \delta)$ and is given by

$$g'(t) = \sum_{j=1}^{n} D_j f[\mathbf{p}(t)]\,(b_j - a_j) = f'(\mathbf{p}(t); \mathbf{b} - \mathbf{a})$$

Applying the chain rule again, we obtain

$$g''(t) = \sum_{i=1}^{n} \sum_{j=1}^{n} D_{i,j} f[\mathbf{p}(t)]\,(b_j - a_j)\,(b_i - a_i) = f''(\mathbf{p}(t); \mathbf{b} - \mathbf{a})$$

Similarly, we find that $g^{(m)}(t) = f^{(m)}(\mathbf{p}(t); \mathbf{b} - \mathbf{a})$. When these are used in (1.20) we obtain the theorem, since the point $\mathbf{z} = \mathbf{a} + \theta(\mathbf{b} - \mathbf{a}) \in L(\mathbf{a}, \mathbf{b})$.

Summary

In this chapter, we have studied the concept of partial, directional and total derivatives. We have seen that the total derivatives can be expressed in term of partial derivatives. Further, we have generalized some of the important theorems of calculus like chain rule, mean-value theorem, and Taylor's formula to functions depending on n-variables.

Check your progress

1. If $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at c, then

   (A) $f'(c) : \mathbb{R}^n \to \mathbb{R}^m$ is linear (B) $f'(c) : \mathbb{R}^n \to \mathbb{R}^n$ is linear

   (C) $f'(c) : \mathbb{R}^m \to \mathbb{R}^n$ is linear (D) $f'(c) : \mathbb{R}^m \to \mathbb{R}^m$ is linear

2. If $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at c, then

   (A) $T_c(u) = f'(c; u)$ (B) $T_c(u) = f'(c)$

   (C) $T_c(u) = f$ (D) $T(u) = D_k f(c)$

3. If $f : \mathbb{R}^n \to \mathbb{R}^m$ is linear, then

   (A) $f'(c)$ exists for every c and $f'(c) = f$

   (B) $f'(c)$ exists for some c and $f'(c) = f$

   (C) $f'(c)$ need not exists at c

   (D) None of the above

4. If $f(x, y) = \log(x^2 + y^2),\ (x, y) \neq (0, 0)$, then

   (A) $D_{1,2}f(x, y) = D_{2,1}f(x, y)$ for all $x, y \in \mathbb{R}$

   (B) $D_{1,2}f(x, y) \neq D_{1,1}f(x, y)$ for all $x, y \in \mathbb{R}$

   (C) $D_{1,2}f(x, y) = D_{2,1}f(x, y)$ for some $x, y \in \mathbb{R}$

   (D) $D_{1,2}f(x, y) \neq D_{1,1}f(x, y)$ for some $x, y \in \mathbb{R}$

5. The limit $\lim\limits_{h \to 0} \dfrac{h}{|h|} =$

   (A) 0 (B) 1 (C) doesn't exist (D) -1

Glossary

- $S \subseteq \mathbb{R}^n$ means that S is a subset of $\mathbb{R}^n$.

- $\mathbf{u}_k$, the k-th unit coordinate vector means $\mathbf{u}_k = (0, \ldots, 1, \ldots, 0)$, where is 1 is in the k-th position.

- $\mathbf{f} : S \to \mathbb{R}^m$ means that $\mathbf{f}$ is a function defined on a set $S \subset \mathbb{R}^n$ with values in $\mathbb{R}^m$.

- $\mathbf{f} = (f_1, f_2, \ldots, f_m)$ means that $\mathbf{f}(\mathbf{c}) = (f_1(\mathbf{c}), \ldots, f_n(\mathbf{c}))$, where $f_i : S \to \mathbb{R},\ i = 1, 2, \ldots, n$.

- If $n = 1$, then we will use $\mathbf{f} = f$.

- $B(\mathbf{c}; r) = \{x \in \mathbb{R}^n : \|x - \mathbf{c}\| < r\}$ will be called an n-ball in $\mathbb{R}^n$

- $D_{\boldsymbol{k}} f(c) = \dfrac{\partial f}{\partial x_k}$, a partial derivative of f with respect to x_k.

- **Open n-ball:** Let $a \in \mathbb{R}^n$ and $r > 0$. The set $B(a; r) = \{x \in \mathbb{R}^n : \|x - a\| < r\}$ is called an open $n-$ball of radius r and center a.

- **Interior point:** Let S be a subset of $\mathbb{R}^n$. A point $a \in S$ is said to be an interior point of S, if there is an open n-ball $B(a; r)$ with center at a such that $B(a; r) \subseteq S$.

- **Open set:** A set $S \subset \mathbb{R}^b$ is said to be open if all its points are interior points.

- A function $\mathbf{f} : \mathbb{R}^{\boldsymbol{n}} \to \mathbb{R}^m$ is said to be **linear** if

$$\mathbf{f}(a\mathbf{x} + b\mathbf{y}) = a\mathbf{f}(\mathbf{x}) + b\mathbf{f}(\mathbf{y})$$

  for every pair of vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^{\boldsymbol{n}}$ and every pair of scalars a and b.

Self-Assessment Questions

1. If f is differentiable at c, then show that f is continuous at c.

2. State and prove the chain rule.

3. State and prove the mean value theorem for differentiable functions.

4. Establish sufficient condition for differentiability.

5. Establish sufficient condition for equality of mixed partial dierivatives.

6. State and prove Talyor's formula.

Exercises

1. Calculate all first order partial derivatives and the directional derivative for $f(x) = x \cdot L(x)$ where $L : \mathbb{R}^n \to \mathbb{R}^n$ is a linear function.

2. If $f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & if \quad (x,y) \neq (0,0) \\ 0 & if \quad (x,y) = (0,0), \end{cases}$ then show that $D_{2,1}f(0,0) \neq D_{1,2}f(0,0)$.

3. If $T : \mathbb{R}^n \to \mathbb{R}^m$ and $S : \mathbb{R}^m \to \mathbb{R}^p$ are linear transformations with the domain of S containing the range of f, then show that the composition $S \circ T$ is linear map from $\mathbb{R}^n$ into $\mathbb{R}^p$. Also calculate the matrix $m(S \circ T)$.

4. Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $f(x,y) = (\sin x \cos y, \sin x \sin y, \cos x \cos y)$. Determine the Jacobian matrix $Df(x,y)$.

Answers for check your progress

1. (A) 2. (A) 3. (A) 4. 5. (C)

Section 1.2:

(a) Let $x = (x_1, x_2, \cdots, x_n)$ and $a = (a_1, a_2, \cdots, a_n)$. Then

$$f(x) = a \cdot x = a_1x_1 + a_2x_2 + \cdots + a_nx_n.$$

The first-order partial derivatives of f are given by,

$$D_kf(x) = a_k \quad (k = 1, 2, 3, \cdots, n)$$

The directional derivative of f at x along u is given by

$$\begin{aligned} f'(x;u) &= \lim_{h\to 0} \frac{f(x+hu) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{a \cdot (x+hu) - a \cdot x}{h} \\ &= \lim_{h\to 0} \frac{a \cdot x + ha \cdot u - a \cdot x}{h} \\ &= a \cdot u \end{aligned}$$

(b) i) The first order partial derivatives of f are given by,

Let

$$\begin{aligned}\|x\|^4 = (\|x\|^2)^2 = \left(\sum_{i=1}^n x_i^2\right)^2 &= \sum_{i=1}^n x_i^4 + \sum_{\substack{i,j=1\\ i\neq j}}^n x_i^2 x_j^2 \|x\|^4 \\ &= x_k^4 + \sum_{\substack{i=1\\ i\neq k}}^n x_i^4 + 2x_k^2 \sum_{\substack{i=1\\ i\neq k}}^n x_i^2 + \sum_{\substack{i,j=1\\ i\neq j, i\neq k, j\neq k}}^n x_i^2 x_j^2\end{aligned}$$

Let $k = 1, 2, \cdots, n$, then

$$D_k f(x) = 4x_k^3 + 4x_k \sum_{i=1}^n x_i^2$$

Thus, $D_k f(x)$ exists and is continuous. Hence, f is differentiable and thus, it has directional derivative in every direction.

ii) The directional derivative of f at x along u:

$$\begin{aligned}f'(x;u) = f'(x)(u) &= f'(x)(u_1 e_1 + u_2 e_2 + \cdots + u_n e_n) \\ &= \sum_{k=1}^n u_k f'(x)(e_k) \\ &= \sum_{k=1}^n u_k f'(x; e_k) \\ &= \sum_{k=1}^n u_k D_k f(x) \\ &= 4\|x\|^2 \sum_{k=1}^n u_k x_k \\ f'(x;u) &= 4\|x\|^2 (x \cdot u).\end{aligned}$$

(c) $f(x) = x \cdot L(x)$ and L is a linear function.

i) The first order partial derivatives of f are given by,

$$\begin{aligned}
D_k f(x) &= \lim_{h\to 0} \frac{f(x+he_k)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+he_k)\cdot L(x+he_k) - x\cdot L(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+he_k)\cdot [L(x)+hL(e_k)] - x\cdot L(x)}{h} \\
&= \lim_{h\to 0} \frac{x\cdot L(x) + hx\cdot L(e_k) + he_k\cdot L(x) + h^2 e_k\cdot L(e_k) - x\cdot L(x)}{h} \\
&= \lim_{h\to 0} \frac{h[x\cdot L(e_k) + e_k\cdot L(x) + he_k\cdot L(e_k)]}{h} \\
D_k f(x) &= x\cdot L(e_k) + e_k\cdot L(x).
\end{aligned}$$

Let $u = (u_1, u_2, \cdots, u_n) = u_1e_1 + u_2e_2 + \cdots + u_ne_n$.

ii) The directional derivative of f at x along u is given by

$$\begin{aligned}
f'(x;u) &= \sum_{k=1}^{n} u_k D_k f(x) = \sum_{k=1}^{n} u_k[x\cdot L(e_k) + e_k\cdot L(x)] \\
&= \sum_{k=1}^{n} xu_k\cdot L(e_k) + \sum_{k=1}^{n} u_ke_k\cdot L(x) \\
&= x\sum_{k=1}^{n} u_k\cdot L(e_k) + \sum_{k=1}^{n} u_ke_k\cdot L(x) \\
&= x\cdot \sum_{k=1}^{n} L(u_ke_k) + \sum_{k=1}^{n} u_ke_k\cdot L(x) \\
f'(x;u) &= x\cdot L(u) + L(x)\cdot u
\end{aligned}$$

(d)

$$\begin{aligned}
f(x) &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j \\
&= a_{kk}x_k^2 + \sum_{i\neq k} a_{ik}x_ix_k + \sum_{i\neq k} a_{ki}x_kx_i + \sum_{i\neq k, j\neq k, i\neq j} a_{ij}x_ix_j
\end{aligned}$$

Then, we have

$$\begin{aligned}
D_k f(x) &= 2a_{kk}x_k + \sum_{i\neq k} a_{ik}x_i + \sum_{i\neq k} a_{ki}x_i \\
&= 2a_{kk}x_k + 2\sum_{i\neq k} a_{ik}x_i \qquad (since,\ a_{ik} = a_{ki}) \\
D_k f(x) &= 2\sum_{i\neq k} a_{ik}x_i
\end{aligned}$$

Thus, $D_k f(x)$ exists and is continuous. Hence, f is differentiable and thus, it has directional derivative in every direction. Then,

$$\begin{aligned} f'(x;u) &= \sum_{k=1}^{n} u_k D_k f(x) \\ &= 2\sum_{k=1}^{n} u_k \sum_{i=1}^{n} a_{ik} x_i \\ &= 2\sum_{i=1}^{n}\sum_{i=1}^{n} a_{ik} x_i u_k \\ &= 2x^T A\, u. \end{aligned}$$

where $A = (a_{ij})_{i=1,j=1}^{n}$.

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Suggested Readings

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Unit 2

Implict Functions and Extremum Problems

Objectives

After reading this unit learners will be able to understand the important theorems of multivariable calculus as follows:

- Inverse function theorem
- Implicit function theorem
- Extremum problems

2.1 Introduction

Given a simple equation

$$f(x,t)=0 \tag{2.1}$$

the basic question to be asked to whether we can determine x as a function of t. If yes, then we have

$$x=g(t),$$

for some function g and we say that g is defined "implicitly" by (2.1).

In a similar manner, when we have a system of several equations involving n several variables we can think of whether can we solve these equations for some of the variables in terms of the remaining $n-1$ variables. This is the same type of problem as above, except that x and t are replaced by vectors, and f and g are replaced by vector-valued functions. Under suitable conditions, a solution always exists. Basically, the implicit function theorem gives these sufficient conditions and some idea about the solution.

For example, a system of n linear equations of the form

$$\sum_{j=1}^{n} a_{ij}x_j = t_i \quad (i = 1, 2, \ldots, n) \tag{2.2}$$

where the a_{ij} and t_i are given numbers and $x_1, \ldots, x_n$ represent unknowns has a unique solution if, and only if, the determinant of the coefficient matrix $A = [a_{ij}]$ is nonzero.

Note. *The determinant of a square matrix $A = [a_{ij}]$ is denoted by $\det A$ or $\det [a_{ij}]$. If det $[a_{ij}] \neq 0$, the solution of (2.2) can be obtained by Cramer's rule which expresses each x_k as a quotient of two determinants, say $x_k = A_k/D$, where $D = \det [a_{ij}]$ and A_k is the determinant of the matrix obtained by replacing the k-th column of $[a_{ij}]$ by $t_1, \ldots, t_n$. In particular, if each $t_i = 0$, then each $x_k = 0$.*

Next we show that the system (2.2) can be written in the form (2.1). Each equation in (2.2) has the form

$$f_i(\mathbf{x}, \mathbf{t}) = 0 \qquad \text{where } \mathbf{x} = (x_1, \ldots, x_n)\,, \quad \mathbf{t} = (t_1, \ldots, t_n)$$

and

$$f_i(\mathbf{x}, \mathbf{t}) = \sum_{j=1}^{n} a_{ij}x_j - t_i.$$

Therefore the system in (2.2) can be expressed as one vector equation

$$\mathbf{f}(\mathbf{x}, \mathbf{t}) = \mathbf{0},$$

where $\mathbf{f} = (f_1, \ldots, f_n)$. If $D_j f_i$ denotes the partial derivative of f_i with respect to the j-th coordinate x_j, then $D_j f_i(\mathbf{x}, \mathbf{t}) = a_{ij}$. Thus the coefficient matrix $A = [a_{ij}]$ in (2.2) is a Jacobian matrix. Linear algebra tells us that (2.2) has a unique solution if the determinant of this Jacobian matrix is nonzero.

In the general implicit function theorem, the nonvanishing of the determinant of a Jacobian matrix also plays a role. This comes about by approximating $\mathbf{f}$ by a linear function. The equation $\mathbf{f}(\mathbf{x}, \mathbf{t}) = \mathbf{0}$ gets replaced by a system of linear equations whose coefficient matrix is the Jacobian matrix of $\mathbf{f}$.

Notation. If $\mathbf{f} = (f_1, \ldots, f_n)$ and $\mathbf{x} = (x_1, \ldots, x_n)$, the Jacobian matrix $\mathrm{Df}(\mathbf{x}) = [D_j f_i(x)]$ is an $n \times n$ matrix. Its determinant is called a Jacobian determinant and is denoted by $J_{\mathrm{f}}(\mathbf{x})$. Thus,

$$J_{\mathbf{f}}(\mathbf{x}) = \det \mathbf{Df}(\mathbf{x}) = \det [D_j f_i(\mathbf{x})]$$

The notation

$$\frac{\partial\,(f_1, \ldots, f_n)}{\partial\,(x_1, \ldots, x_n)}$$

is also used to denote the Jacobian determinant $J_{\mathrm{f}}(\mathbf{x})$.

The next theorem gives a relation between the Jacobian determinant of a complex-valued function and its derivative.

Theorem 2.1.1. *If $f = u + iv$ is a complex-valued function with a derivative at a point z in* $\mathbf{C}$*, then $J_f(z) = |f'(z)|^2$.*

Proof. We have $f'(z) = D_1 u + i D_1 v$, so $|f'(z)|^2 = (D_1 u)^2 + (D_1 v)^2$. Also,

$$J_f(z) = \det \begin{bmatrix} D_1 u & D_2 u \\ D_1 v & D_2 v \end{bmatrix} = D_1 u D_2 v - D_1 v D_2 u = (D_1 u)^2 + (D_1 v)^2$$

by the Cauchy-Riemann equations.

Check your progress

1. A system of algebraic equations $AX = B$ has a unique solution if and only if

   (A) the determinant of the coefficient matrix $A = [a_{ij}]$ is nonzero.

   (B) the determinant of the coefficient matrix $A = [a_{ij}]$ is zero.

   (C) the coefficient matrix $A = [a_{ij}]$ is positive definite.

   (D) the each entries a_{ij} of the coefficient matrix A is nonzero.

2.2 Functions with nonzero Jacobian determinant

In this section, we give some important properties of functions with non vanishing Jacobian determinant at certain points. These results will be used to prove the implicit function theorem.

Theorem 2.2.1. *Let $B = B(\mathbf{a}; r)$ be an n-ball in $\mathbb{R}^n$, let ∂B denote its boundary,*

$$\partial B = \{\mathbf{x} : \|\mathbf{x} - \mathbf{a}\| = r\}$$

and let $\bar{B} = B \cup \partial B$ denote its closure. Let $\mathbf{f} = (f_1, \dots, f_n)$ be continuous on $\bar{B}$, and assume that all the partial derivatives $D_j f_i(\mathbf{x})$ exist if $\mathbf{x} \in B$. Assume further that $\mathbf{f}(\mathbf{x}) \neq \mathbf{f}(\mathbf{a})$ if $\mathbf{x} \in \partial B$ and that the Jacobian determinant $J_{\mathbf{f}}(\mathbf{x}) \neq 0$ for each $\mathbf{x}$ in B. Then $\mathbf{f}(B)$, the image of B under $\mathbf{f}$, contains an n-ball with center at $\mathbf{f}(\mathbf{a})$.

Proof.

1. First let us define a real-valued function g on ∂B by:

   $$g(\mathbf{x}) = \|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a})\| \quad \text{if } \mathbf{x} \in \partial B$$

   Since $f(x) \neq f(a)$ if $\mathbf{x} \in \partial B$, $g(\mathbf{x}) > 0$ for each $\mathbf{x}$ in ∂B. Also, since $\mathbf{f}$ is continuous on $\bar{B}$, the function g is continuous on ∂B . Since ∂B is closed and bounded, it is compact and so g attains an absolute minimum (call it m) for some point on ∂B. Note that $m > 0$ since g is positive on ∂B.

2. Let $T = B\left(\mathbf{f}(\mathbf{a}); \frac{m}{2}\right)$, an $n-$ball with centre at $\mathbf{f}(a)$ and radius $\dfrac{m}{2}$.

   Thus the theorem will be proved if $T \subseteq \mathbf{f}(B)$.

   Let us now prove that $\mathbf{y} \in \mathbf{f}(B)$ whenever $\mathbf{y} \in T$.

   For $\mathbf{y}$ in T, let us define a real-valued function h on $\bar{B}$ by

   $$h(\mathbf{x}) = \|\mathbf{f}(\mathbf{x}) - \mathbf{y}\| \quad \text{if } \mathbf{x} \in \bar{B}$$

   Then h is continuous on the compact set $\bar{B}$ and hence attains its absolute minimum on $\bar{B}$.

3. We shall now show that h attains its minimum inside the open n-ball B.

   Note that at the center point $\mathbf{a}$, we have $h(\mathbf{a}) = \|\mathbf{f}(\mathbf{a}) - \mathbf{y}\| < m/2$ since $\mathbf{y} \in T$. Hence the minimum value of h in $\bar{B}$ must also be less that $m/2$.

   If $\mathbf{x} \in \partial B$, we have

$$\begin{aligned} h(\mathbf{x}) = \|\mathbf{f}(\mathbf{x}) - \mathbf{y}\| &= \|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a}) - (\mathbf{y} - \mathbf{f}(\mathbf{a}))\| \\ &\geq \|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a})\| - \|\mathbf{f}(\mathbf{a}) - \mathbf{y}\| \\ &> g(\mathbf{x}) - \frac{m}{2} \\ &\geq m - \frac{m}{2} \\ &= \frac{m}{2}. \end{aligned}$$

   Hence, h cannot attain its minimum on the boundary ∂B. Hence there is an interior point $\mathbf{c}$ in B at which h attains its minimum.

4. At this point the square of h also has a minimum. Since

$$h^2(\mathbf{x}) = \|\mathbf{f}(\mathbf{x}) - \mathbf{y}\|^2 = \sum_{i=1}^{n} [f_i(\mathbf{x}) - y_i]^2$$

   and since each partial derivative $D_k\left(h^2\right)$ must be zero at $\mathbf{c}$, we must have

$$\sum_{i=1}^{n} [f_i(\mathrm{c}) - y_i]\, D_k f_i(\mathbf{c}) = 0 \text{ for } k = 1, 2, \ldots, n.$$

   But this is a system of linear equations whose determinant $J_{\mathrm{f}}(\mathbf{c}) \neq 0$, since $\mathbf{c} \in B$. Therefore $f_i(\mathbf{c}) = y_i$ for each i, or $\mathbf{f}(\mathbf{c}) = \mathbf{y}$. That is, $\mathbf{y} \in f(B)$. Hence $T \subseteq \mathbf{f}(B)$ and the proof is complete.

Definition 2.2.2. *Let (S, d_S) and (T, d_T) be metric spaces. A function $f : S \to T$ is called an* **open mapping** *if, for every open set A in S, the image $f(A)$ is open in T.*

The next theorem gives a sufficient condition for a mapping that takes open sets onto open sets.

Theorem 2.2.3. (Open Mapping Theorem) *Let A be an open subset of $\mathbb{R}^n$ and assume that $\mathbf{f} : A \to \mathbb{R}^n$ is continuous and has finite partial derivatives $D_j f_i$ on A. If $\mathbf{f}$ is one-to-one on A and if $J_{\mathbf{f}}(\mathbf{x}) \neq 0$ for each $\mathbf{x}$ in A, then $\mathbf{f}(A)$ is open.*

Proof. If $\mathbf{b} \in \mathbf{f}(A)$, then $\mathbf{b} = \mathbf{f}(\mathbf{a})$ for some $\mathbf{a}$ in A. There is an n-ball $B(\mathbf{a}; r) \subseteq A$ on which $\mathbf{f}$ satisfies the hypotheses of the previous theorem. So, $\mathbf{f}(B)$ contains an n-ball with center at b. Therefore, b is an interior point of $f(A)$, so $f(A)$ is open.

The next theorem shows that a function with continuous partial derivatives is locally one-to-one near a point where the Jacobian determinant does not vanish.

Theorem 2.2.4. *Assume that $\mathbf{f} = (f_1, \ldots, f_n)$ has continuous partial derivatives $D_j f_i$ on an open set S in $\mathbb{R}^n$, and that the Jacobian determinant $J_{\mathbf{f}}(\mathbf{a}) \neq 0$ for some point $\mathbf{a}$ in S. Then there is an n-ball $B(\mathbf{a})$ on which $\mathbf{f}$ is one-to-one.*

Proof. Let $\mathbf{Z}_1, \ldots, \mathbf{Z}_n$ be n points in S and let $\mathbf{Z} = (\mathbf{Z}_1; \ldots; \mathbf{Z}_n)$ denote that point in $\mathbb{R}^{n^2}$ whose first n components are the components of $\mathbf{Z}_1$, whose next n components are the components of $\mathbf{Z}_2$, and so on.

Let us define a real-valued function h by

$$h(\mathbf{Z}) = \det\left[D_j f_i\left(\mathbf{Z}_i\right)\right]$$

This function is continuous at those points $\mathbf{Z}$ in $\mathbb{R}^{n^2}$ where $h(\mathbf{Z})$ is defined because each $D_j f_i$ is continuous on S and a determinant is a polynomial in its n^2 entries. Let $\mathbf{Z}$ be the special point in $\mathbb{R}^{n^2}$ obtained by putting

$$\mathbf{Z}_1 = \mathbf{Z}_2 = \cdots = \mathbf{Z}_n = \mathbf{a}.$$

Then $h(\mathbf{Z}) = J_{\mathbf{f}}(\mathbf{a}) \neq 0$ and hence, by continuity, there is some n-ball $B(\mathbf{a})$ such that

$$\det\left[D_j f_i\left(\mathbf{Z}_i\right)\right] \neq 0$$

if each $\mathbf{Z}_i \in B(\mathbf{a})$. We will prove that $\mathbf{f}$ is one-to-one on $B(\mathbf{a})$.

Suppose that $\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{y})$ for some pair of points $\mathbf{x} \neq \mathbf{y}$ in $B(\mathbf{a})$. Then we must arrive at a contradiction.

Since $B(\mathbf{a})$ is convex, the line segment $L(\mathbf{x}, \mathbf{y}) \subseteq B(\mathbf{a})$ and we can apply the Mean-Value Theorem to each component f_i of $\mathbf{f}$ to write

$$0 = f_i(\mathbf{y}) - f_i(\mathbf{x}) = \nabla f_i\left(\mathbf{Z}_i\right) \cdot (\mathbf{y} - \mathbf{x}) \quad \text{for } i = 1, 2, \ldots, n$$

where each $\mathbf{Z}_i \in L(\mathbf{x}, \mathbf{y})$ and hence $\mathbf{Z}_i \in B(\mathbf{a})$. (The Mean-Value Theorem is applicable because $\mathbf{f}$ is differentiable on S.) But this is a system of linear equations of the form

$$\sum_{k=1}^{n} (y_k - x_k)\, a_{ik} = 0 \quad \text{with} \quad a_{ik} = D_k f_i\left(\mathbf{Z}_i\right).$$

The determinant of this system is not zero, since $\mathbf{Z}_i \in B(\mathbf{a})$. Hence $y_k - x_k = 0$ for each k, and this contradicts the assumption that $\mathbf{x} \neq \mathbf{y}$. We have thus shown that $\mathbf{x} \neq \mathbf{y}$ implies $\mathbf{f}(\mathbf{x}) \neq \mathbf{f}(\mathbf{y})$ and hence that $\mathbf{f}$ is one-to-one on $B(\mathbf{a})$.

Note. *The above theorem is a local theorem and not a global theorem. The nonvanishing of $J_{\mathbf{f}}(\mathbf{a})$ guarantees that $\mathbf{f}$ is one-to-one on a neighborhood of $\mathbf{a}$. It does not follow that $\mathbf{f}$ is one-to-one on S, even when $J_{\mathbf{f}}(\mathbf{x}) \neq 0$ for every $\mathbf{x}$ in S. The following example illustrates this point. Let f be the complex-valued function defined by $f(z) = e^z$ if $z \in \mathbf{C}$. If $z = x + iy$ we have*

$$J_f(z) = |f'(z)|^2 = |e^z|^2 = e^{2x}$$

Thus $J_f(z) \neq 0$ for every z in C. However, f is not one-to-one on $\mathbf{C}$ because $f\left(z_1\right) = f\left(z_2\right)$ for every pair of points z_1 and z_2 which differ by $2\pi i$.

The next theorem gives a global property of functions with nonzero Jacobian determinant.

Theorem 2.2.5. *Let A be an open subset of $\mathbb{R}^n$ and assume that $\mathbf{f} : A \to \mathbb{R}^n$ has continuous partial derivatives $D_j f_i$ on A. If $J_{\mathbf{f}}(\mathbf{x}) \neq 0$ for all $\mathbf{x}$ in A, then $\mathbf{f}$ is an open mapping.*

Proof. Let S be any open subset of A. If $\mathbf{x} \in S$ there is an n-ball $B(\mathbf{x})$ in which $\mathbf{f}$ is one-to-one by previous theorem. Hence by Theorem 2.2.4, the image $\mathbf{f}(B(\mathbf{x}))$ is open in $\mathbb{R}^n$. But we can write $S = \bigcup\limits_{\mathbf{x} \in S} B(\mathbf{x})$. Applying $\mathbf{f}$ we find $\mathbf{f}(S) = \bigcup\limits_{\mathbf{x} \in S} \mathbf{f}(B(\mathbf{x}))$, so $\mathbf{f}(S)$ is open.

Definition 2.2.6. *If a function $\mathbf{f} = (f_1, \ldots, f_n)$ has continuous partial derivatives on a set S, we say that $\mathbf{f}$ is continuously differentiable on S, and we write $\mathbf{f} \in C'$ on S.*

Note. *Continuous differentiability at a point implies differentiability at that point.*

Theorem 2.2.4 shows that a continuously differentiable function with a nonvanishing Jacobian at a point a has a local inverse in a neighborhood of a. The next theorem gives some local differentiability properties of this local inverse function.

Check your progress

1. A function f from a metric space (S, d_S) to another (T, d_T) is called an open mapping if

   (A) for every open set A in S, $f(A)$ is open in T.

   (B) f is a continuous mapping of S into T.

   (C) for every closed set A in S, $f(A)$ is closed in T.

   (D) f is a bijective mapping.

2. A function $f = (f_1, f_2, \cdots, f_n)$ is said to continuously differentiable on $S \subset \mathbb{R}^n$ if

   (A) f has continuous partial derivatives (B) f has all partial derivatives

   (C) f has directional derivatives (D) each f_i continuous and differentiable

2.3 The Inverse Function Theorem

The inverse function theorem is a fundamental result in multivariable calculus that provides conditions under which a function has a local inverse near a point. It finds applications in coordinate transformations, in particular, in change of variables in integrals and in proving the implicit function theorem.

We recall the following result about differentiable functions of a single variable:

Let f be a continuously differentiable function of a single variable x on an open interval, and suppose that at some point a in the interval, $f(a)$ is nonzero. Then f maps a sufficiently small interval around the point a one to one onto an interval around the point $f(a)$. Denote the inverse function of f by g, then g is differentiable at $f(a)$, and

$$g'(f(a)) = \frac{1}{f'(a)}.$$

Theorem 2.3.1. *Assume* $\mathbf{f} = (f_1, \ldots, f_n) \in C'$ *on an open set* S *in* $\mathbb{R}^n$*, and let* $T = \mathrm{f}(S)$*. If the Jacobian determinant* $J_{\mathrm{f}}(\mathbf{a}) \neq 0$ *for some point a in* S*, then there are two open sets* $X \subseteq S$ *and* $Y \subseteq T$ *and a uniquely determined function* $\mathbf{g}$ *such that*

a) $\mathbf{a} \in X$ *and* $\mathbf{f}(\mathbf{a}) \in Y$*,*

b) $Y = \mathbf{f}(X)$*,*

c) $\mathbf{f}$ *is one-to-one on* X*,*

d) $\mathbf{g}$ *is defined on* $Y, \mathbf{g}(Y) = X$*, and* $\mathbf{g}[\mathbf{f}(\mathbf{x})] = \mathbf{x}$ *for every* $\mathbf{x}$ *in* X*,*

e) $\mathbf{g} \in C'$ *on* Y.

Proof.

1. The function J_{f} defined by $J_{\mathrm{f}}(x) = \det[D_j f_i(x)]$ is continuous on S and, since $J_{\mathrm{f}}(\mathbf{a}) \neq 0$, there is an n-ball $B_1(\mathrm{a})$ such that $J_{\mathrm{f}}(\mathbf{x}) \neq 0$ for all $\mathbf{x}$ in $B_1(\mathrm{a})$. By Theorem 2.2.4, there is an n-ball $B(a) \subseteq B_1(a)$ on which $\mathbf{f}$ is one-to-one.

2. Let B be an n-ball with center at a and radius smaller than that of $B(\mathbf{a})$. Then, by Theorem 2.2.1, $\mathbf{f}(B)$ contains an n-ball denoted as Y with center at $\mathbf{f}(\mathrm{a})$. Let $X = \mathbf{f}^{-1}(Y) \cap B$. Then X is open since both $\mathbf{f}^{-1}(Y)$ and B are open.

3. The set $\bar{B}$ (the closure of B) is compact and $\mathbf{f}$ is one-to-one and continuous on $\bar{B}$. Hence, there exists a function g (the inverse function f^{-1}) defined on $f(\bar{B})$ such that
$$g[f(x)] = \mathbf{x} \text{ for all } \mathbf{x} \in \bar{B}.$$
Moreover, g is continuous on $\mathrm{f}(\bar{B})$. Since $X \subseteq \bar{B}$ and $Y \subseteq \mathrm{f}(\bar{B})$, we have

   a) $\mathbf{a} \in X$ and $\mathbf{f}(\mathbf{a}) \in Y$,

   b) $Y = \mathbf{f}(X)$,

   c) $\mathbf{f}$ is one-to-one on X,

   d) $\mathbf{g}$ is defined on $Y, \mathbf{g}(Y) = X$, and $\mathbf{g}[\mathbf{f}(\mathbf{x})] = \mathbf{x}$ for every $\mathbf{x}$ in X.

The uniqueness of g follows from (d).

4. To prove (e), we define a real-valued function h by the equation

$$h(\mathbf{Z})=\det\left[D_j f_i\left(\mathbf{Z}_i\right)\right],$$

where $\mathbf{Z}_1, \ldots, \mathbf{Z}_n$ are n points in S, and $\mathbf{Z}=\left(\mathbf{Z}_1; \ldots; \mathbf{Z}_n\right)$ is the corresponding point in $\mathbb{R}^{\boldsymbol{n}^2}$. Then, arguing as in the proof of Theorem 2.2.4, there is an n-ball $B_2(\mathbf{a})$ such that

$$h(\mathbf{Z})=\det[D_j f_i(\mathbf{Z}_i)] \neq 0$$

if each $\mathbf{Z}_i \in B_2(\mathbf{a})$. We can now assume that, the n-ball $B(\mathbf{a})$ was chosen so that $B(\mathbf{a}) \subseteq B_2(\mathbf{a})$. Then $\bar{B} \subseteq B_2(\mathbf{a})$ and $h(\mathbf{Z}) \neq 0$ if each $\mathbf{Z}_i \in \bar{B}$.

5. To prove (e), let us write $\mathbf{g}=\left(g_1, \ldots, g_n\right)$ and show that each $g_k \in C'$ on Y.

To prove this let us take $\mathbf{y} \in Y$ and show that $D_r g_k(y)$ exists.

Consider the difference quotient $\dfrac{g_k\left(\mathbf{y}+t \mathbf{u}_r\right)-g_k(\mathbf{y})}{t}$, where $\mathbf{u}_r$ is the r-th unit coordinate vector. Since Y is open, $\mathbf{y}+t \mathbf{u}_r \in Y$ if t is sufficiently small.

6. Let $\mathbf{x}=\mathbf{g}(\mathbf{y})$ and let $\mathbf{x}'=\mathbf{g}\left(\mathbf{y}+t \mathbf{u}_r\right)$.

Then both $\mathbf{x}$ and $\mathbf{x}'$ are in X and $\mathbf{f}\left(\mathbf{x}'\right)-\mathbf{f}(\mathbf{x})=t \mathbf{u}_r$. Hence $f_i\left(\mathbf{x}'\right)-f_i(\mathbf{x})$ is 0 if $i \neq r$, and is t if $i=r$.

By the Mean-Value Theorem, we have

$$\frac{f_i\left(\mathbf{x}'\right)-f_i(\mathbf{x})}{t}=\nabla f_i\left(\mathbf{Z}_i\right) \cdot \frac{\mathbf{x}'-\mathbf{x}}{t} \quad \text{for } i=1,2, \ldots, n$$

where each $\mathbf{Z}_i$ is on the line segment joining $\mathbf{x}$ and $\mathbf{x}'$; hence $\mathbf{Z}_i \in B$. Note that the expression on the left is 1 or 0 , according to whether $i=r$ or $i \neq r$. Since

$$\det\left[D_j f_i\left(\mathbf{Z}_i\right)\right]=h(\mathbf{Z}) \neq 0,$$

the system of n linear equations with n unknowns $\dfrac{x_j'-x_j}{t}$ has a unique solution.

7. Using Cramer's rule for solving for the k-th unknown, we obtain an expression for $\dfrac{g_k\left(\mathbf{y}+t\mathbf{u}_r\right)-g_k(\mathbf{y})}{t}$ as a quotient of determinants. As $t \to 0$, the point $\mathbf{x} \to \mathbf{x}'$, since g is continuous, and hence each $\mathbf{Z}_i \to \mathbf{x}$, since $\mathbf{Z}_i$ is on the segment joining $\mathbf{x}$ to $\mathbf{x}'$. The determinant which appears in the denominator has for its limit the number $\det\left[D_j f_i(x)\right] = J_{\mathrm{f}}(\mathbf{x})$, and this is nonzero, since $\mathbf{x} \in X$. Therefore, the following limit exists:

$$\lim_{t \to 0} \frac{g_k\left(\mathbf{y}+t\mathbf{u}_r\right)-g_k(\mathbf{y})}{t} = D_r g_k(\mathbf{y})$$

This establishes the existence of $D_r g_k(\mathbf{y})$ for each $\mathbf{y}$ in Y and each $r = 1, 2, \ldots, n$. Moreover, this limit is a quotient of two determinants involving the derivatives $D_j f_i(\mathbf{x})$. Continuity of the $D_j f_i$ implies continuity of each partial $D_r g_k$. This completes the proof of (e).

Check your progress

1. Check whether the function $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $F(x_1, x_2) = \begin{pmatrix} x^2+y \\ y^2+x \end{pmatrix}$ has an inverse near the point $(1, 1)$.

2.4 The Implicit Function Theorem

The implicit function theorem in multivariable calculus provides conditions under which a system of equations can be solved for some variables in terms of others, near a point where certain regularity conditions are met. It generalizes the idea of solving equations implicitly for one variable in terms of others.

Theorem 2.4.1. (Implicit function theorem) *Let* $\mathbf{f} = (f_1, \ldots, f_n)$ *be a vector-valued function defined on an open set* S *in* $\mathbb{R}^{n+k}$ *with values in* $\mathbb{R}^n$*. Suppose* $\mathbf{f} \in C'$ *on* S*. Let* $(\mathbf{x}_0; \mathbf{t}_0)$ *be a point in* S *for which* $\mathbf{f}\left(\mathbf{x}_0; \mathbf{t}_0\right) = \mathbf{0}$ *and for which the* $n \times n$ *determinant* $\det\left[D_j f_i\left(\mathbf{x}_0; \mathbf{t}_0\right)\right] \neq 0$*. Then there exists a* k*-dimensional open set* T_0 *containing* t_0 *and one, and only one, vector-valued function* g*, defined on* T_0 *and having values in* $\mathbb{R}^n$*, such that* *a)* $\mathbf{g} \in C'$ *on* T_0,

b) $g(t_0) = x_0$,

c) $\mathbf{f}(\mathbf{g}(\mathbf{t}); \mathbf{t}) = \mathbf{0}$ *for every* $\mathbf{t}$ *in* T_0.

Proof. We shall apply the inverse function theorem to a certain vector-valued function $\mathbf{F} = (F_1, \ldots, F_n; F_{n+1}, \ldots, F_{n+k})$ defined on S and having values in $\mathbb{R}^{n+k}$. The function F is defined as follows: For $1 \leq m \leq n$, let $F_m(\mathbf{x}; \mathbf{t}) = f_m(\mathbf{x}; \mathbf{t})$, and for $1 \leq m \leq k$, let $F_{n+m}(\mathbf{x}; \mathbf{t}) = t_m$. We can then write $\mathbf{F} = (\mathbf{f}; \mathbf{I})$, where $\mathbf{f} = (f_1, \ldots, f_n)$ and where I is the identity function defined by $I(t) = \mathbf{t}$ for each $\mathbf{t}$ in $\mathbb{R}^k$. The Jacobian $J_F(\mathbf{x}; \mathbf{t})$ then has the same value as the $n \times n$ determinant $\det[D_j f_i(\mathbf{x}; \mathbf{t})]$ because the terms which appear in the last k rows and also in the last k columns of $J_{\mathbf{F}}(\mathbf{x}; \mathbf{t})$ form a $k \times k$ determinant with ones along the main diagonal and zeros elsewhere; the intersection of the first n rows and n columns consists of the determinant $\det[D_j f_i(\mathbf{x}; \mathbf{t})]$, and

$$D_i F_{n+j}(\mathbf{x}; \mathbf{t}) = 0 \quad \text{for } 1 \leq i \leq n, \quad 1 \leq j \leq k$$

Hence the Jacobian $J_F(\mathbf{x}_0; \mathbf{t}_0) \neq 0$. Also, $\mathbf{F}(\mathbf{x}_0; \mathbf{t}_0) = (\mathbf{0}; \mathbf{t}_0)$. Therefore, by inverse function theorem, there exist open sets X and Y containing $(\mathbf{x}_0; \mathbf{t}_0)$ and $(\mathbf{0}; \mathbf{t}_0)$, respectively, such that F is one-to-one on X, and $X = F^{-1}(Y)$. Also, there exists a local inverse function $\mathbf{G}$, defined on Y and having values in X, such that

$$\mathbf{G}[\mathbf{F}(\mathbf{x}; \mathbf{t})] = (\mathbf{x}; \mathbf{t})$$

and such that $\mathbf{G} \in C'$ on Y.

Now $\mathbf{G}$ can be reduced to components as follows: $\mathbf{G} = (\mathbf{v}; \mathbf{w})$ where $\mathbf{v} = (v_1, \ldots, v_n)$ is a vector-valued function defined on Y with values in $\mathbb{R}^n$ and $\mathbf{w} = (w_1, \ldots, w_k)$ is also defined on Y but has values in $\mathbb{R}^k$. We can now determine $\mathbf{v}$ and $\mathbf{w}$ explicitly. The equation $\mathbf{G}[\mathbf{F}(\mathbf{x}; \mathbf{t})] = (\mathbf{x}; \mathbf{t})$, when written in terms of the components $\mathbf{v}$ and $\mathbf{w}$, gives us the two equations

$$\mathbf{v}[\mathbf{F}(\mathbf{x}; \mathbf{t})] = \mathbf{x} \quad \text{and} \quad \mathbf{w}[\mathbf{F}(\mathbf{x}; \mathbf{t})] = \mathbf{t}.$$

But now, every point $(\mathbf{x};\mathbf{t})$ in Y can be written uniquely in the form $(\mathbf{x};\mathbf{t}) = \mathbf{F}(\mathbf{x}';\mathbf{t}')$ for some ($\mathbf{x}';\mathbf{t}'$) in X, because $\mathbf{F}$ is one-to-one on X and the inverse image $\mathbf{F}^{-1}(Y)$ contains X. Furthermore, by the manner in which F was defined, when we write $(\mathbf{x};\mathbf{t}) = \mathbf{F}(\mathbf{x}';\mathbf{t}')$, we must have $\mathbf{t}' = \mathbf{t}$. Therefore,

$$\mathbf{v}(\mathbf{x};\mathbf{t}) = \mathbf{v}\left[\mathbf{F}(\mathbf{x}';\mathbf{t})\right] = \mathbf{x}' \quad \text{and} \quad \mathbf{w}(\mathbf{x};\mathbf{t}) = \mathbf{w}\left[\mathbf{F}(\mathbf{x}';\mathbf{t})\right] = \mathbf{t}.$$

Hence the function $\mathbf{G}$ can be described as follows: Given a point $(\mathbf{x};\mathbf{t})$ in Y, we have $\mathbf{G}(\mathbf{x};\mathbf{t}) = (\mathbf{x}';\mathbf{t})$, where $\mathbf{x}'$ is that point in $\mathbb{R}^n$ such that $(\mathbf{x};\mathbf{t}) = \mathbf{F}(\mathbf{x}';\mathbf{t})$. This statement implies that

$$\mathbf{F}[\mathbf{v}(\mathbf{x};\mathbf{t});\mathbf{t}] = (\mathbf{x};\mathbf{t}) \quad \text{for every } (\mathbf{x};\mathbf{t}) \text{ in } Y.$$

Now we are ready to define the set T_0 and the function g in the theorem. Let

$$T_0 = \left\{\mathbf{t} : \mathbf{t} \in \mathbb{R}^k, \quad (\mathbf{0};\mathbf{t}) \in Y\right\}$$

and for each $\mathbf{t}$ in T_0 define $g(t) = v(0;t)$. The set T_0 is open in $\mathbb{R}^k$. Moreover, $\mathbf{g} \in C'$ on T_0 because $\mathbf{G} \in C'$ on Y and the components of g are taken from the components of G. Also,

$$\mathbf{g}(\mathbf{t}_0) = \mathbf{v}(\mathbf{0};\mathbf{t}_0) = \mathbf{x}_0$$

because $(0;t_0) = F(x_0;t_0)$. Finally, the equation $F[\mathbf{v}(\mathbf{x};\mathbf{t});\mathbf{t}] = (\mathbf{x};\mathbf{t})$, which holds for every $(\mathbf{x};\mathbf{t})$ in Y, yields (by considering the components in $\mathbb{R}^n$) the equation $\mathbf{f}[\mathbf{v}(\mathbf{x};\mathbf{t});\mathbf{t}] = \mathbf{x}$. Taking $\mathbf{x} = \mathbf{0}$, we see that for every t in T_0, we have $\mathbf{f}[\mathbf{g}(\mathbf{t});\mathbf{t}] = \mathbf{0}$, and this completes the proof of statements (a), (b), and (c). It remains to prove that there is only one such function g. But this follows at once from the one-to-one character of $\mathbf{f}$. If there were another function, say $\mathbf{h}$, which satisfied (c), then we would have $\mathbf{f}[\mathbf{g}(\mathbf{t});\mathbf{t}] = \mathbf{f}[\mathbf{h}(\mathbf{t});\mathbf{t}]$, and this would imply $(g(t);t) = (h(t);t)$, or $g(t) = h(t)$ for every t in T_0.

Check your progress

2. Verify that the system of two equations

$$F_1(x, y, z) = x^2 + y^2 + z^2 - 1 = 0,$$
$$F_2(x, y, z) = x + z - 1 = 0.$$

can be solved for y and z as functions of x.

2.5 Extrema of real-valued functions of one variable

This section will examine real-valued functions f and identify points where f has a local extremum, or a local minimum or maximum.

If f is a function of one variable, a necessary condition for a function f to have a local extremum at an interior point c of an interval is that $f'(c) = 0$, provided that $f'(c)$ exists. However, this is not sufficient condition, as we can see by taking $f(x) = x^3, c = 0$. Now, we derive a sufficient condition.

Theorem 2.5.1. *For some integer $n \geq 1$, let f have a continuous nth derivative in the open interval (a, b). Suppose also that for some interior point c in (a, b) we have*

$$f'(c) = f''(c) = \cdots = f^{(n-1)}(c) = 0, \quad \textit{but} \quad f^{(n)}(c) \neq 0$$

Then for n even, f has a local minimum at c if $f^{(n)}(c) > 0$, and a local maximum at c if $f^{(n)}(c) < 0$. If n is odd, there is neither a local maximum nor a local minimum at c.

Proof. Since $f^{(n)}(c) \neq 0$, there exists an interval $B(c)$ such that for every x in $B(c)$, the derivative $f^{(n)}(x)$ will have the same sign as $f^{(n)}(c)$. Now by Taylor's formula for one dimension (Theorem 1.11.1), for every x in $B(c)$ we have

$$f(x) - f(c) = \frac{f^{(n)}(x_1)}{n!}(x - c)^n, \quad \text{where } x_1 \in B(c)$$

If n is even, this equation implies $f(x) \geq f(c)$ when $f^{(n)}(c) > 0$, and $f(x) \leq f(c)$ when $f^{(n)}(c) \leq 0$. If n is odd and $f^{(n)}(c) > 0$, then $f(x) > f(c)$ when $x > c$, but $f(x) < f(c)$ when $x < c$, and there can be no extremum at c. A similar statement holds if n is odd and $f^{(n)}(c) < 0$. This proves the theorem.

2.6 Extrema of real-valued functions of several variables

Necessary condition for a function to have a local maximum or a local minimum at an interior point a of an open set is that each partial derivative $D_k f(\mathbf{a})$ must be zero at that point. We can also state this in terms of directional derivatives by saying that $f'(\mathbf{a};\mathbf{u})$ must be zero for every direction $\mathbf{u}$.

The converse of the above statement is not true, however. For example, let us consider a function of two real variables:

$$f(x,y) = \left(y - x^2\right)\left(y - 2x^2\right)$$

Here we have $D_1 f(0,0) = D_2 f(0,0) = 0$. Now $f(0,0) = 0$, but the function assumes both positive and negative values in every neighborhood of $(0,0)$, so there is neither a local maximum nor a local minimum at $(0,0)$.

This example illustrates another interesting phenomenon. If we take a fixed straight line through the origin and restrict the point (x,y) to move along this line toward $(0,0)$, then the point will finally enter the region above the parabola $y = 2x^2$ (or below the parabola $y = x^2$ in which $f(x,y)$ becomes and stays positive for every $(x,y) \neq (0,0)$. Therefore, along every such line, f has a minimum at $(0,0)$, but the origin is not a local minimum in any two-dimensional neighborhood of $(0,0)$.

Definition 2.6.1. *If f is differentiable at a and if $\nabla f(\mathbf{a}) = 0$, the point a is called a stationary point of* f. *A stationary point is called a saddle point if every n-ball $B(\mathbf{a})$ contains points $\mathbf{x}$ such that $f(\mathbf{x}) > f(\mathbf{a})$ and other points such that $f(\mathbf{x}) < f(\mathbf{a})$.*

In the foregoing example, the origin is a saddle point of the function.

To determine whether a function of n variables has a local maximum, a local minimum, or a saddle point at a stationary point a, we must determine the algebraic sign of $f(\mathbf{x}) - f(\mathrm{a})$ for all $\mathbf{x}$ in a neighborhood of a. As in the one-dimensional case, this is done with the help of Taylor's formula.

Take $m = 2$ and $\mathbf{y} = \mathbf{a} + \mathbf{t}$ in the Taylor's formula. If the partial derivatives of f are differentiable on an n-ball $B(\mathbf{a})$ then

$$f(\mathbf{a}+\mathbf{t}) - f(\mathbf{a}) = \nabla f(\mathbf{a}) \cdot \mathbf{t} + \frac{1}{2} f''(\mathbf{z}; \mathbf{t}) \tag{2.3}$$

where z lies on the line segment joining a and $a + t$, and

$$f''(\mathbf{z}; \mathbf{t}) = \sum_{i=1}^{n} \sum_{j=1}^{n} D_{i,j} f(\mathbf{z}) t_i t_j$$

At a stationary point we have $\nabla f(\mathrm{a}) = 0$ so (2.3) becomes

$$f(\mathbf{a}+\mathbf{t}) - f(\mathbf{a}) = \frac{1}{2} f''(\mathbf{z}; \mathbf{t})$$

Therefore, as $\mathbf{a}+\mathbf{t}$ ranges over $B(\mathbf{a})$, the algebraic sign of $f(\mathbf{a}+\mathbf{t}) - f(\mathbf{a})$ is determined by that of $f''(z; t)$. We can write (2.3) in the form

$$f(\mathbf{a}+\mathbf{t}) - f(\mathbf{a}) = \frac{1}{2} f''(\mathbf{a}; \mathbf{t}) + \|\mathbf{t}\|^2 E(\mathbf{t}) \tag{2.4}$$

where

$$\|\mathbf{t}\|^2 E(\mathbf{t}) = \frac{1}{2} f''(\mathbf{z}; \mathbf{t}) - \frac{1}{2} f''(\mathbf{a}; \mathbf{t})$$

The inequality

$$\|\mathbf{t}\|^2 |E(\mathbf{t})| \leq \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} |D_{i,j} f(\mathbf{z}) - D_{i,j} f(\mathbf{a})| \, \|\mathbf{t}\|^2$$

shows that $E(\mathbf{t}) \to 0$ as $\mathbf{t} \to \mathbf{0}$ if the second-order partial derivatives of f are continuous at-a. Since $\|\mathbf{t}\|^2 E(\mathbf{t})$ tends to zero faster than $\|\mathbf{t}\|^2$, it seems reasonable to expect that the algebraic sign of $f(\mathbf{a}+\mathbf{t}) - f(\mathbf{a})$ should be determined by that of $f''(\mathbf{a}; \mathbf{t})$. This is what is proved in the next theorem.

Theorem 2.6.2. (Second-derivative test for extrema) *Assume that the second-order partial derivatives $D_{i,j} f$ exist in an n-ball $B(\mathbf{a})$ and are continuous at $\mathbf{a}$, where $\mathbf{a}$ is a stationary point of f. Let*

$$Q(\mathbf{t}) = \frac{1}{2} f''(\mathbf{a}; \mathbf{t}) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} D_{i,j} f(\mathbf{a}) t_i t_j \tag{2.5}$$

a) If $Q(\mathbf{t}) > 0$ for all $\mathbf{t} \neq 0$, f has a relative minimum at $\mathbf{a}$.

b If $Q(\mathbf{t}) < 0$ *for all* $\mathbf{t} \neq 0$, f *has a relative maximum at* $\mathbf{a}$.

c) If $Q(\mathbf{t})$ *takes both positive and negative values, then* f *has a saddle point at* a.

Proof. First let us note that the function Q is continuous at each point t in $\mathbf{R}^n$.

Let $S = \{\mathbf{t} : \|\mathbf{t}\| = 1\}$ denote the boundary of the n-ball $B(0;1)$.

If $Q(\mathbf{t}) > 0$ for all $\mathbf{t} \neq \mathbf{0}$, then $Q(\mathbf{t})$ is positive on S.

Since S is compact, Q has a minimum on S (call it m), and $m > 0$.

Now $Q(c\mathbf{t}) = c^2 Q(\mathbf{t})$ for every real c.

Taking $c = 1/\|\mathbf{t}\|$ where $\mathbf{t} \neq 0$ we see that $ct \in S$ and hence $c^2 Q(\mathbf{t}) \geq m$, so $Q(\mathbf{t}) \geq m\|t\|^2$. Using this in (2.4) we find

$$f(\mathbf{a}+\mathbf{t}) - f(\mathbf{a}) = Q(\mathbf{t}) + \|\mathbf{t}\|^2 E(\mathbf{t}) \geq m\|\mathbf{t}\|^2 + \|\mathbf{t}\|^2 E(\mathbf{t})$$

Since $E(\mathbf{t}) \to 0$ as $\mathbf{t} \to \mathbf{0}$, there is a positive number r such that $|E(\mathbf{t})| < \frac{1}{2}m$ whenever $0 < \|\mathbf{t}\| < r$. For such $\mathbf{t}$ we have $0 \leq \|\mathbf{t}\|^2 |E(\mathbf{t})| < \frac{1}{2}m\|\mathbf{t}\|^2$, so

$$f(\mathbf{a}+\mathbf{t}) - f(\mathbf{a}) > m\|\mathbf{t}\|^2 - \frac{1}{2}m\|\mathbf{t}\|^2 = \frac{1}{2}m\|\mathbf{t}\|^2 > 0$$

Therefore f has a relative minimum at a, which proves (a). To prove (b) we use a similar argument, or simply apply part (a) to $-f$.

Finally, we prove (c). For each $\lambda > 0$ we have, from (4),

$$f(\mathbf{a}+\lambda\mathbf{t}) - f(\mathbf{a}) = Q(\lambda\mathbf{t}) + \lambda^2\|\mathbf{t}\|^2 E(\lambda\mathbf{t}) = \lambda^2\left\{Q(\mathbf{t}) + \|\mathbf{t}\|^2 E(\lambda\mathbf{t})\right\}$$

Suppose $Q(\mathbf{t}) \neq 0$ for some $\mathbf{t}$. Since $E(\mathbf{y}) \to 0$ as $\mathbf{y} \to \mathbf{0}$, there is a positive r such that

$$\|t\|^2 E(\lambda\mathbf{t}) < \frac{1}{2}|Q(\mathbf{t})| \quad \text{if } 0 < \lambda < r$$

Therefore, for each such λ the quantity $\lambda^2\{Q(\mathbf{t}) + \|t\|^2 E(\lambda\mathbf{t})\}$ has the same sign as $Q(\mathbf{t})$. Therefore, if $0 < \lambda < r$, the difference $f(\mathbf{a}+\lambda\mathbf{t}) - f(\mathbf{a})$ has the same sign as $Q(\mathbf{t})$. Hence, if $Q(\mathbf{t})$ takes both positive and negative values, it follows that f has a saddle point at $\mathbf{a}$.

Note. A real-valued function Q defined on $\mathbb{R}^n$ by an equation of the type

$$Q(\mathbf{x}) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_i x_j$$

where $\mathbf{x} = (x_1, \ldots, x_n)$ and the a_{ij} are real is called a quadratic form. The form is called symmetric if $a_{ij} = a_{ji}$ for all i and j, positive definite if $\mathbf{x} \neq 0$ implies $Q(\mathbf{x}) > 0$, and negative definite if $\mathbf{x} \neq 0$ implies $Q(\mathbf{x}) < 0$.

In general, it is not easy to determine whether a quadratic form is positive or negative definite. For the case $n = 2$, the following criterion can be used:

Theorem 2.6.3. *Let f be a real-valued function with continuous second-order partial derivatives at a stationary point $\mathbf{a}$ in $\mathbf{R}^2$. Let*

$$A = D_{1,1}f(\mathbf{a}), \quad B = D_{1,2}f(\mathbf{a}), \quad C = D_{2,2}f(\mathbf{a})$$

and let

$$\Delta = \det \begin{bmatrix} A & B \\ B & C \end{bmatrix} = AC - B^2$$

Then we have:

a) If $\Delta > 0$ and $A > 0$, f has a relative minimum at a.

b) If $\Delta > 0$ and $A < 0$, f has a relative maximum at $\mathbf{a}$.

c) If $\Delta < 0$, f has a saddle point at $\mathbf{a}$.

Proof. In the two-dimensional case we can write the quadratic form in (2.5) as follows:

$$Q(x, y) = \frac{1}{2}\left\{Ax^2 + 2Bxy + Cy^2\right\}$$

If $A \neq 0$, this can also be written as

$$Q(x, y) = \frac{1}{2A}\left\{(Ax + By)^2 + \Delta y^2\right\}$$

If $\Delta > 0$, the expression in brackets is the sum of two squares, so $Q(x, y)$ has the same sign as A. Therefore, statements (a) and (b) follow at once from parts (a) and (b) of Theorem 2.6.2.

If $\Delta < 0$, the quadratic form is the product of two linear factors. Therefore, the set of points (x, y) such that $Q(x, y) = 0$ consists of two lines in the xy-plane intersecting at $(0,0)$. These lines divide the plane into four regions; $Q(x,y)$ is positive in two of these regions and negative in the other two. Therefore f has a saddle point at a.

Note. If $\Delta = 0$, there may be a local maximum, a local minimum, or a saddle point at a.

Check your progress

1. A stationary point $\mathbf{a}$ is a saddle point if for every $n-$ball $B(\mathbf{a})$ contains points x such that

   (A) $\nabla f(\mathbf{a}) = 0$ (B) $f(x) > f(\mathbf{a})$

   (C) $f(x) < f(\mathbf{a})$ (D) All the above

2. At the point $(0,0)$, the function $f(x,y) = (y-x^2)(y-2x^2)$ has a

   (A) local minimum (B) local maximum

   (C) saddle point (D) $\nabla f(0,0) \neq 0$

Excerises

1. If $x(r,\theta) = r\cos\theta, y(r,\theta) = r\sin\theta$, show that $\dfrac{\partial(x,y)}{\partial(r,\theta)} = r$

2. If $x(r,\theta,\phi) = r\cos\theta, y(r,\theta,\phi) = r\sin\theta,\ z = r\cos\phi$, show that $\dfrac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = -r^2\sin\phi$.

3. Let A be an open subset of $\mathbb{R}^n$ and assume that $f : A \to \mathbb{R}^n$ has continuous partial derivatives $D_j f_i$ on A. If $J_f(x) \neq 0$ for all x in A, then show that f is an open mapping.

Answers for check your progress

1. Consider the function $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$F(x,y) = \begin{pmatrix} x^2+1 \\ y^2+x \end{pmatrix}.$$

   The Jacobian matrix at a point (x,y) is

$$J_F(x,y) = \begin{pmatrix} 2x & 1 \\ 1 & 2y \end{pmatrix}.$$

At $(1,1)$, the Jacobian is

$$J_F(x,y) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

The determinant of $J_f(x,y)$ is nonzero. Hence, by the inverse unction theorem, F has a local inverse near $(1,1)$.

2. Consider the system of two equations

$$F_1(x,y,z) = x^2 + y^2 + z^2 - 1 = 0,$$

$$F_2(x,y,z) = x + z - 1 = 0.$$

This defines two equations in three variables (x,y,z). The implicit function theorem can be used to solve for z and y as functions of x, near a point where certain conditions hold.

We compute the Jacobian matrix of F with respect to y and z

$$J_F(x_0,y_0,z_0) = \begin{pmatrix} 2y_0 & 2z_0 \\ 0 & 1 \end{pmatrix}.$$

If we evaluate this at a point where the determinant is nonzero, say $(x_0,y_0,z_0) = (0,1,0)$, then the matrix is invertible, and by the implicit function theorem, we can locally solve for y and z as functions of x.

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Suggested Readings

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2. P. D. Lax and M. S. Terrell Multivariable Calculus with Applications, Springer, 2017.

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Unit 3

Line Integrals

Objectives

After reading this unit, learners will be able to

- understand the the notion of line integrals which are important in both pure and applied mathematics
- prove the basic properties of line integrals
- develop a deep understanding of vector fields and line integrals.
- apply first and second fundamental theorem of calculus for line integrals to problems in physics.

3.1 Introduction

Line integrals are of fundamental importance in both pure and applied mathematics. They occur in connection with work, potential energy, heat flow, change in entropy, circulation of a fluid, and other physical situations in which the behavior of a vector or scalar field is studied along a curve. We use integrals to find the total amount of some quantity on a curve or surface in space. Examples include total mass of a wire, work along a curve, total charge on a surface, and flux across a surface.

We are already familiar with the concept of Riemann-integral $\int_a^b f(x)dx$ for real-valued functions and vector-valued functions defined and bounded on finite intervals.

Here, we are going to replace the interval $[a, b]$ by a curve in n-space described by a vector-valued function α, and the integrand is a vector field $\boldsymbol{f}$ defined and bounded on this curve. The resulting integral will be called a **line integral, a curvilinear integral, or a contour integral**, and is denoted by $\int \boldsymbol{f} \cdot d\boldsymbol{\alpha}$ or by some similar symbol. The dot is used purposely to suggest an inner product of two vectors. The curve is called a **path of integration**.

3.2 Paths and line integrals

First let us recall the definition of a curve.

Let $J = [a, b]$ be a finite closed interval in $\mathbb{R}^1$ and let $\boldsymbol{\alpha}$ be a vector-valued function defined on J, i.e., $\boldsymbol{\alpha} : J \to \mathbb{R}^n$.

1. As t runs over J, the function values $\alpha(t)$ trace out a set of points in n-space called the **graph** of the function.

2. If α is continuous on J, the graph is called a **curve** or called as a **curve described by $\boldsymbol{\alpha}$**.

In the study of curves, it is noted that different functions can trace out the same curve in different ways, for example, in different directions or with different velocities. In the study of line integrals we are concerned not only with the set of points on a curve but also the direction in which the curve is traced out, that is, with the function $\boldsymbol{\alpha}$ itself. Such a function will be called a continuous path.

Definition 3.2.1. *Let $J = [a, b]$ be a finite closed interval in $\mathbb{R}^1$.*

1. *A function $\boldsymbol{\alpha} : J \to \mathbb{R}^n$ which is continuous on J is called a* **continuous path** *in n-space.*

2. *The path is called* **smooth** *if the derivative $\boldsymbol{\alpha}'$ exists and is continuous (or $\boldsymbol{\alpha}$ is continuously differentiable) in the open interval (a, b).*

3. *The path is called* **piecewise smooth** *if the interval* $[a, b]$ *can be partitioned into a finite number of subintervals in each of which the path is smooth.*

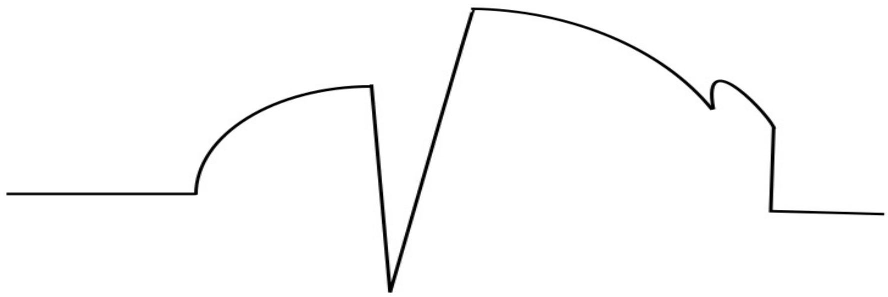

Figure 3.1: A piecewise smooth path in a plane

In the follwing figurem the curve has a tangent line at all but a finite number of its points. These exceptional points subdivide the curve into arcs, along each of which the tangent line turns continuously.

Definition 3.2.2. (Line Integral) *Let* $\boldsymbol{\alpha}$ *be a piecewise smooth path in n-space defined on* $[a, b]$*, and let* $\boldsymbol{f}$ *be a vector field defined and bounded on the graph of* $\boldsymbol{\alpha}$*. The line integral of* $\boldsymbol{f}$ *along* $\boldsymbol{\alpha}$ *is denoted by* $\int \boldsymbol{f} \cdot d\boldsymbol{\alpha}$ *and is defined by*

$$\int \boldsymbol{f} \cdot d\boldsymbol{\alpha} = \int_a^b f[\boldsymbol{\alpha}(t)] \cdot \boldsymbol{\alpha}'(t)dt \tag{10.1}$$

provided the integral on the right exists, either as a proper or improper integral.

Note. *Note that the line integrals are defined in terms of ordinary integrals. In general, the* dot *product* $f[\alpha(t)] \cdot \alpha'(t)$ *is bounded on* $[a, b]$ *and continuous except possibly at a finite number of points, in which case the integral exists as a proper integral.*

3.3 Other notations for line integrals

1. If C denotes the graph of $\boldsymbol{\alpha}$, the line integral $\int \boldsymbol{f} \cdot d\boldsymbol{\alpha}$ is also written as $\int_C \boldsymbol{f} \cdot d\boldsymbol{\alpha}$ and is called the integral of $\boldsymbol{f}$ along C.

2. If $\boldsymbol{A} = \boldsymbol{\alpha}(a)$ and $\boldsymbol{B} = \boldsymbol{\alpha}(b)$ denote the end points of C, the line integral is sometimes written as $\displaystyle\int_a^b \boldsymbol{f}$ or as $\displaystyle\int_a^b \boldsymbol{f} \cdot d\boldsymbol{\alpha}$ and is called the line integral of $\boldsymbol{f}$ from $\boldsymbol{a}$ to $\boldsymbol{b}$ along $\boldsymbol{\alpha}$.

3. When $\boldsymbol{A} = \boldsymbol{B}$ the path is said to be closed. The symbol $\oint$ is often used to indicate integration along a closed path.

4. When $\boldsymbol{f}$ and $\boldsymbol{\alpha}$ are expressed in terms of their components, say
$$\boldsymbol{f} = (f_1, \ldots, f_n) \quad \text{and} \quad \alpha = (\alpha_1, \ldots, \alpha_n)$$
the integral on the right of (10.1) becomes a sum of integrals,
$$\sum_{k=1}^{n} \int_a^b f_k[\boldsymbol{\alpha}(t)]\alpha_k'(t)dt$$

5. In the case of two-dimension, the path α is usually described by a pair of parametric equations,
$$x = \alpha_1(t), \quad y = \alpha_2(t)$$
and the line integral $\int_C \boldsymbol{f} \cdot d\boldsymbol{\alpha}$ is written as $\int_C f_1 dx + f_2 dy$, or as $\int_C f_1(x,y)dx + f_2(x,y)dy$.

6. In the case of three-dimension, we use three parametric equations,
$$x = \alpha_1(t), \quad y = \alpha_2(t), \quad z = \alpha_3(t)$$
and we write the line integral
$$\int_C \boldsymbol{f} \cdot d\boldsymbol{\alpha} = \int_C f_1 dx + f_2 dy + f_3 dz = \int_C f_1(x,y,z)dx + f_2(x,y,z)dy + f_3(x,y,z)dz$$

When the notation $\displaystyle\int_a^b f$ is used it should be kept in mind that the integral depends not only on the end points $\boldsymbol{a}$ and $\boldsymbol{b}$ but also on the path $\boldsymbol{\alpha}$ joining them. In this case the line integral is also written as $\displaystyle\int f_1 d\alpha_1 + \cdots + f_n d\alpha_n$.

The following examples illustrates that the value of the integral is independent of the parametric representation used to describe the curve.

Example 3.3.1. *Let $\boldsymbol{f}$ be a two-dimensional vector field given by $\boldsymbol{f}(x,y) = \sqrt{y}\,\boldsymbol{i} + (x^3 + y)\,\boldsymbol{j}$ for all (x,y) with $y \geq 0$. Calculate the line integral of $\boldsymbol{f}$ from $(0,0)$ to $(1,1)$ along each of the following paths:*

(a) the line with parametric equations $x = t, y = t, 0 \leq t \leq 1$;

(b) the path with parametric equations $x = t^2, y = t^3, 0 \leq t \leq 1$.

Solution. For the path in part (a) we take $\alpha(t) = t\boldsymbol{i} + t\boldsymbol{j}$. Then $\alpha'(t) = \boldsymbol{i} + \boldsymbol{j}$ and $\boldsymbol{f}[\boldsymbol{\alpha}(t)] = \sqrt{t}\boldsymbol{i} + (t^3 + t)\,\boldsymbol{j}$. Therefore the dot product of $\boldsymbol{f}[\boldsymbol{\alpha}(t)]$ and $\boldsymbol{\alpha}'(t)$ is equal to $\sqrt{t} + t^3 + t$ and we find

$$\int_{(0,0)}^{(1,1)} f \cdot d\boldsymbol{\alpha} = \int_0^1 \left(\sqrt{t} + t^3 + t\right) dt = \frac{17}{12}$$

For the path in part (b) we take $\boldsymbol{\alpha}(t) = t^2\boldsymbol{i} + t^3\boldsymbol{j}$. Then $\boldsymbol{\alpha}'(t) = 2t\boldsymbol{i} + 3t^2\boldsymbol{j}$ and $f[\alpha(t)] = t^{3/2}\boldsymbol{i} + (t^6 + t^3)\,\mathrm{j}$. Therefore

$$f[\alpha(t)] \cdot \alpha'(t) = 2t^{5/2} + 3t^8 + 3t^5$$

so,

$$\int_{(0,0)}^{(1,1)} f \cdot d\boldsymbol{\alpha} = \int_0^1 \left(2t^{5/2} + 3t^8 + 3t^5\right) dt = \frac{59}{42}$$

This example shows that the integral from one point to another may depend on the path joining the two points.

Now let us carry out the calculation for part (b) once more, using the same curve but with a different parametric representation. The same curve can be described by the function

$$\beta(t) = ti + t^{3/2}j, \quad \text{where} \quad 0 \le t \le 1$$

This leads to the relation

$$f[\beta(t)] \cdot \beta'(t) = \left(t^{3/4}\boldsymbol{i} + \left(t^3 + t^{3/2}\right)\mathbf{j}\right) \cdot \left(\boldsymbol{i} + \tfrac{3}{2}t^{1/2}\boldsymbol{j}\right) = t^{3/4} + \tfrac{3}{2}t^{7/2} + \frac{3}{2}t^2$$

and

$$\int_0^1 f[\beta(t)] \cdot \beta'(t) dt = \frac{59}{42}.$$

Check your progress

1. The line integral of a vector field f along a piecewise smooth path α is defined by

   (A) $\displaystyle\int_a^b f'(\alpha(t))\alpha_k'(t)dt$ (B) $\displaystyle\sum_{k=1}^{n}\int_a^b f_k'(\alpha(t))\alpha_k(t)dt$

   (C) $\displaystyle\int_a^b f'(\alpha(t))\alpha'(t)dt$ (D) $\displaystyle\sum_{k=1}^{n}\int_a^b f_k(\alpha(t))\alpha_k'(t)dt$

2. The value of the line integral $\int_C (3x^2 - 2y)ds$ along the line from $(3, 6)$ to $(-1, 1)$ is

   (A) $8\sqrt{53}$ (B) $6\sqrt{53}$ (C) $-8\sqrt{53}$ (D) $-8\sqrt{53}$

3. The value of the line integral of the vector field $f(x, y) = (x^2 + y^2)i + (x^2 - y^2)j$ from $(0, 0)$ to $(2, 0)$ along the curve $y = 1 - |1 - x|$ is

   (A) $\frac{4}{3}$ (B) $-\frac{2}{3}$ (C) $\frac{4}{3}$ (D) $\frac{2}{5}$

4. The value of the line integral of $f(x, y) = \sqrt{y}i + (x^3 + y)j$ from $(0, 0)$ to $(1, 1)$ along the path $x = t, y = t^{3/2}$ is

   (A) $\frac{1}{42}$ (B) $\frac{39}{42}$ (C) $\frac{59}{42}$ (D) doesn't exist

5. Which of the following statement is true for line integrals?

   (A) The line integral of a continuous gradient is independent of the path in any open connected set.

   (B) The line integral of a continuous gradient is is zero around every piecewise smooth closed path.

   (C) The value of the integral independent on the path joining a to b.

   (D) For some vector fields, the integral depends only on the end points a and b and not on the path which joins them.

Problems

1. Compute the value of the line integral $\int_C (x^2 - 2xy)dx + (y^2 - 2xy)dy$, where C is a path from $(-2, 4)$ to $(1, 1)$ along the parabola $y = x^2$.

   Solution. Let us take

$$\alpha(t) = (t, t^2).$$

   Given that

$$\int_c f \cdot d\alpha = \int_c f_1(x, y)dx + f_2(x, y)dy$$

   where, $f_1(x, y) = x^2 - 2xy, \ f_2(x, y) = y^2 - 2xy$

Take, $x = t = \alpha_1(t),\ y = t^2 = \alpha_2(t)$. We have

$$\alpha(t) = (\alpha_1(t), \alpha_2(t)) = (t, t^2) = ti + t^2 j$$

$$\alpha'(t) = (1, 2t) = 1i + 2tj$$

3.4 Basic properties of line integrals

Since line integrals are defined in terms of usual integrals, they inherit many of the properties of usual integrals which will be stated as follows:

Theorem 3.4.1. *Suppose $\boldsymbol{f}$ and $\boldsymbol{g}$ are vector fields defined and bounded on the graph of $\boldsymbol{\alpha}$. The following properties hold:*

1. *Linearity Property of line integral:*

$$\int (a\boldsymbol{f} + b\boldsymbol{g}) \cdot d\boldsymbol{\alpha} = a \int \boldsymbol{f} \cdot d\boldsymbol{\alpha} + b \int \boldsymbol{g} \cdot d\boldsymbol{\alpha}$$

2. *Additive property with respect to the path of integration:*

$$\int_C \boldsymbol{f} \cdot d\boldsymbol{\alpha} = \int_{C_1} \boldsymbol{f} \cdot d\boldsymbol{\alpha} + \int_{C_2} \boldsymbol{f} \cdot d\boldsymbol{\alpha}$$

where the two curves C_1 and C_2 make up the curve C. That is, C is described by a function α defined on an interval $[a, b]$, and the curves C_1 and C_2 are those traced out by $\alpha(t)$ as t varies over subintervals $[a, c]$ and $[c, b]$, respectively, for some c satisfying $a < c < b$.

Next we examine the behavior of line integrals under a change of parameter. Let α be a continuous path defined on an interval $[a, b]$, let u be a real-valued function that is differentiable, with u' never zero on an interval $[c, d]$, and such that the range of u is $[a, b]$. Then the function β defined on $[c, d]$ by the equation

$$\beta(t) = \alpha[u(t)]$$

is a continuous path having the same graph as $\boldsymbol{\alpha}$. Two paths α and β so related are called equivalent. They are said to provide different parametric representations of the same curve. The function u is said to define a change of parameter.

Let C denote the common graph of two equivalent paths $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. If the derivative of u is always positive on $[c, d]$ the function u is increasing and we say that the two paths $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ trace out C in the same direction. If the derivative of u is always negative we say that α and β trace out C in opposite directions. In the first case the function u is said to be orientation-preserving; in the second case u is said to be orientation-reversing. An example is shown in Figure 10.2.

The next theorem shows that the value of a line integral remains unchanged under a change of parameter that preserves orientation and it reverses its sign if the change of parameter reverses orientation. We assume both intergals $\int \boldsymbol{f} \cdot d\boldsymbol{\alpha}$ and $\int \boldsymbol{f} \cdot d\boldsymbol{\beta}$ exist. In the above figures, the first one says that the function h preserves orientation and in

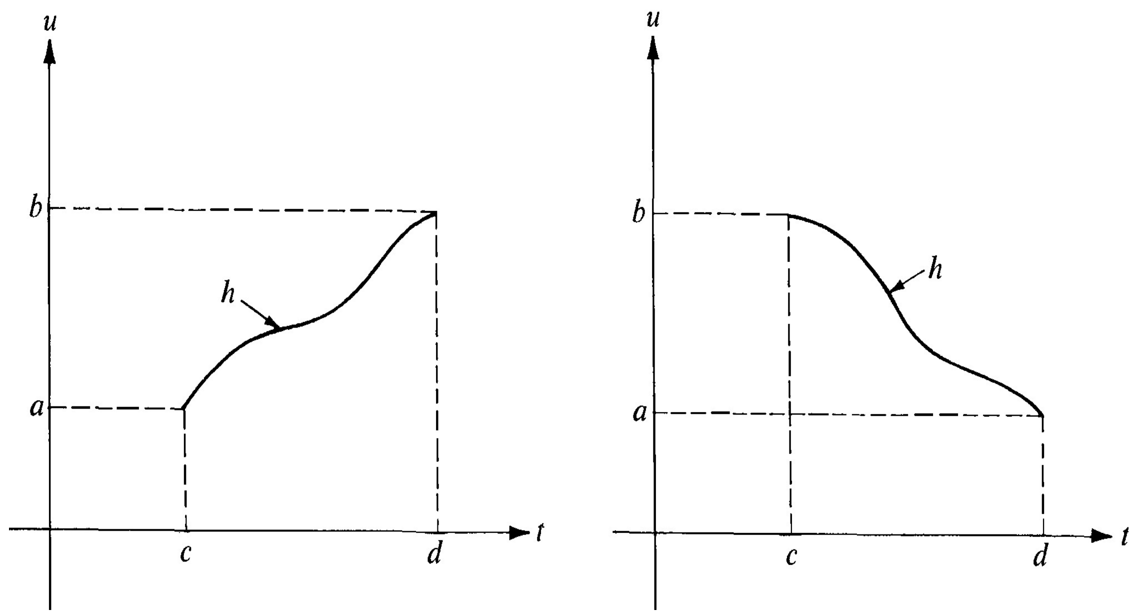

Figure 3.2: A change of parameter defind by $u = h(t)$

the second figure, the function h reverses the orientation.

Theorem 3.4.2. Behavior of a line integral under a change of parameter. *Let α and β be equivalent piecewise smooth paths. Then we have*

$$\int_C \boldsymbol{f} \cdot d\boldsymbol{\alpha} = \int_C \boldsymbol{f} \cdot d\boldsymbol{\beta}$$

if α and β trace out C in the same direction; and

$$\int_C f \cdot d\boldsymbol{\alpha} = -\int_C f \cdot d\boldsymbol{\beta}$$

if $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ trace out C in opposite directions.

Proof. It is sufficient to prove the theorem for smooth paths alone. For piecewise smooth paths, the results follows from the additive property with respect to the path of integration.

We apply the chain rule to prove the required result. The paths $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are related by an equation of the form $\boldsymbol{\beta}(t) = \boldsymbol{\alpha}[u(t)]$, where u is defined on an interval $[c, d]$ and $\boldsymbol{\alpha}$ is defined on an interval $[a, b]$. From the chain rule we have

$$\boldsymbol{\beta}'(t) = \boldsymbol{\alpha}'[u(t)]u'(t)$$

Therefore we find

$$\int_C \boldsymbol{f} \cdot d\boldsymbol{\beta} = \int_c^d \boldsymbol{f}[\boldsymbol{\beta}(t)] \cdot \boldsymbol{\beta}'(t)dt = \int_c^d \boldsymbol{f}(\boldsymbol{\alpha}[u(t)]) \cdot \boldsymbol{\alpha}'[u(t)]u'(t)dt$$

Taking $v = u(t), dv = u'(t)dt,$ we obtain

$$\int_C \boldsymbol{f} \cdot d\boldsymbol{\beta} = \int_{u(c)}^{u(d)} \boldsymbol{f}(\boldsymbol{\alpha}(v)) \cdot \boldsymbol{\alpha}'(v)dv = \pm \int_a^b \boldsymbol{f}(\boldsymbol{\alpha}(v)) \cdot \boldsymbol{\alpha}'(v)dv = \pm \int_C \boldsymbol{f} \cdot d\boldsymbol{\alpha}$$

where the +ve sign is used if $a = u(c)$ and $b = u(d)$, i.e., when α and β trace out C in the same direction and the -ve sign is used if $a = u(d)$ and $b = u(c)$ i.e., when α and β trace out C in opposite directions.

Check your progress

1. Prove the linearity property of line integrals.

3.5 Line integrals with respect to arc length

Let $\boldsymbol{\alpha}$ be a path with $\boldsymbol{\alpha}'$ continuous on an interval $[a, b]$. The graph of $\boldsymbol{\alpha}$ is a rectifiable curve and the corresponding arc-length function s is given by the integral

$$s(t) = \int_a^t \|\boldsymbol{\alpha}'(u)\| \, du$$

The derivative of arc length is given by

$$s'(t) = \|\boldsymbol{\alpha}'(t)\|$$

Let φ be a scalar field defined and bounded on C, the graph of $\boldsymbol{\alpha}$. The line integral of φ with respect to arc length along C is denoted by the symbol $\int_C \varphi ds$ and is defined by the equation

$$\int_C \varphi ds = \int_a^b \varphi[\boldsymbol{\alpha}(t)]s'(t)dt$$

whenever the integral on the right exists.

Now consider a scalar field φ given by $\varphi[\boldsymbol{\alpha}(t)] = \boldsymbol{f}[\boldsymbol{\alpha}(t)] \cdot \boldsymbol{T}(t)$, the dot product of a vector field $\boldsymbol{f}$ defined on C and the unit tangent vector $\boldsymbol{T}(t) = d\boldsymbol{\alpha}/ds$. In this case the line integral $\int_C \varphi ds$ is the same as the line integral $\int_C \boldsymbol{f} \cdot d\boldsymbol{\alpha}$ because

$$\boldsymbol{f}[\boldsymbol{\alpha}(t)] \cdot \boldsymbol{\alpha}'(t) = \boldsymbol{f}[\boldsymbol{\alpha}(t)] \cdot \frac{d\boldsymbol{\alpha}}{ds}\frac{ds}{dt} = \boldsymbol{f}[\boldsymbol{\alpha}(t)] \cdot \boldsymbol{T}(t)s'(t) = \varphi[\boldsymbol{\alpha}(t)]s'(t)$$

When $\boldsymbol{f}$ denotes a velocity field, the dot product $\boldsymbol{f} \cdot \boldsymbol{T}$ is the tangential component of velocity, and the line integral $\int_C \boldsymbol{f} \cdot \boldsymbol{T} ds$ is called the **flow integral of $\boldsymbol{f}$ along** C. When C is a closed curve the flow integral is called the **circulation of $\boldsymbol{f}$ along** C.

3.6 Open connected sets. Independence of the path

Definition 3.6.1. *Let S be an open set in $\mathbb{R}^n$. The set S is called* **connected** *if every pair of points in S can be joined by a piecewise smooth path whose graph lies in S. That is, for every pair of points $\boldsymbol{A}$ and $\boldsymbol{B}$ in S there is a piecewise smooth path $\boldsymbol{\alpha}$ defined on an interval $[a, b]$ such that $\boldsymbol{\alpha}(t) \in S$ for each t in $[a, b]$ satisfying $\boldsymbol{\alpha}(a) = \boldsymbol{A}$ and $\boldsymbol{\alpha}(b) = \boldsymbol{B}$.*

Example 3.6.2. *Three examples of open connected sets in the plane are shown in Figure 10.3. Examples in 3-space analogous to these would be (a) a solid ellipsoid, (b) a solid polyhedron, and (c) a solid torus; in each case only the interior points are considered.*

Definition 3.6.3. *An open set S is said to be* **disconnected** *if S is the union of two or more disjoint non-empty open sets.*

If $\mathbf{f}$ is a vector field that is continuous on an open connected set S. Choose two points $\mathbf{a}$ and $\mathbf{b}$ in S. In general, the value of the integral of $\mathbf{f}$ depends on the path joining $\mathbf{a}$ and $\mathbf{b}$.

Definition 3.6.4. *For vector fields, we say that the integral is* **independent of the path** *from a to b if the line integral of* $\mathbf{f}$ *depends only on the end points a and b and not on the path which joins them.*

Definition 3.6.5. *We say that the line integral of f is* **independent of the path in** S *if it is independent of the path from a to b for every pair of points a and b in S.*

3.7 Second fundamental theorems of calculus to line integrals.

The fundamental theorem of calculus is an important result in calculus that connects differentiation and integration, showing that they are essentially inverse processes.

The fundamental theorem of line integrals extends the fundamental theorem of calculus to line integrals over vector fields. It provides a way to evaluate the line integral of a gradient field along a curve, simplifying the computation of such integrals.

The second fundamental theorem for real functions, states that

$$\int_a^b \varphi'(t)dt = \varphi(b) - \varphi(a)$$

provided that φ' is continuous on some open interval containing both a and b. To extend this result to line integrals we need a slightly stronger version of the theorem in which continuity of φ' is assumed only in the open interval (a, b).

This theorem is particularly useful in physics and engineering for simplifying the work done by conservative forces and in other applications involving potential fields.

Lemma 3.7.1. (Zero-Derivative theorem) *If $f'(x) = 0$ for each x in an open interval I, then f is constant on I.*

Theorem 3.7.2. *Let φ be a real function that is continuous on a closed interval $[a, b]$ and assume that the integral $\displaystyle\int_a^b \varphi'(t)dt$ exists. If φ' is continuous on the open interval (a, b), we have*

$$\int_a^b \varphi'(t)dt = \varphi(b) - \varphi(a)$$

Proof. For each x in $[a,b]$ define $f(x) = \int_a^x \varphi'(t)dt$. We wish to prove that

$$f(b) = \varphi(b) - \varphi(a) \tag{3.1}$$

Then, f is continuous on the closed interval $[a,b]$ and also f is differentiable on the open interval (a,b), with $f'(x) = \varphi'(x)$ for each x in (a,b). Therefore, by the zero-derivative theorem, the difference $f - \varphi$ is constant on the open interval (a,b). By continuity, $f - \varphi$ is also constant on the closed interval $[a,b]$. In particular, $f(b) - \varphi(b) = f(a) - \varphi(a)$. But since $f(a) = 0$, this proves (3.1).

Now, we are ready to prove the second fundamental theorem of calculus for line integrals for vector fields. We can also note that there are vector fields having line integrals independent of the path.

Theorem 3.7.3. *Let φ be a differentiable scalar field with a continuous gradient $\nabla\varphi$ on an open connected set S in $\mathbb{R}^n$. Then for any two points $\boldsymbol{a}$ and $\boldsymbol{b}$ joined by a piecewise smooth path $\boldsymbol{\alpha}$ in S we have*

$$\int_a^b \nabla\varphi \cdot d\boldsymbol{\alpha} = \varphi(\boldsymbol{b}) - \varphi(\boldsymbol{a})$$

Proof. Choose any two points $\boldsymbol{a}$ and $\boldsymbol{b}$ in S and join them by a piecewise smooth path $\boldsymbol{\alpha}$ in S defined on an interval $[a,b]$. Assume first that $\boldsymbol{\alpha}$ is smooth on $[a,b]$. Then the line integral of $\nabla\varphi$ from $\boldsymbol{a}$ to $\boldsymbol{b}$ along $\boldsymbol{\alpha}$ is given by

$$\int_a^b \nabla\varphi \cdot d\boldsymbol{\alpha} = \int_a^b \nabla\varphi[\boldsymbol{\alpha}(t)] \cdot \boldsymbol{\alpha}'(t)dt$$

By the chain rule we have

$$\nabla\varphi[\boldsymbol{\alpha}(t)] \cdot \boldsymbol{\alpha}'(t) = g'(t)$$

where g is the composite function defined on $[a,b]$ by the formula

$$g(t) = \varphi[\boldsymbol{\alpha}(t)]$$

The derivative g' is continuous on the open interval (a,b) because $\nabla\varphi$ is continuous on S and α is smooth. Therefore, by applying Theorem 3.8.2 to g, we obtain

$$\int_a^b \nabla\varphi \cdot d\boldsymbol{\alpha} = \int_a^b g'(t)dt = g(b) - g(a) = \varphi[\boldsymbol{\alpha}(b)] - \varphi[\boldsymbol{\alpha}(a)] = \varphi(\boldsymbol{b}) - \varphi(\boldsymbol{a})$$

This proves the theorem if $\boldsymbol{\alpha}$ is smooth.

When $\boldsymbol{\alpha}$ is piecewise smooth we partition the interval $[a, b]$ into a finite number (say r) of subintervals $[t_{k-1}, t_k]$, in each of which $\boldsymbol{\alpha}$ is smooth, and we apply the result just proved to each subinterval. This gives us

$$\int_a^b \nabla\varphi = \sum_{k=1}^{r} \int_{\boldsymbol{\alpha}(t_{k-1})}^{\boldsymbol{\alpha}(t_k)} \nabla\varphi = \sum_{k=1}^{r} \{\varphi\left[\boldsymbol{\alpha}\left(t_k\right)\right] - \varphi\left[\boldsymbol{\alpha}\left(t_{k-1}\right)\right]\} = \varphi(\boldsymbol{b}) - \varphi(\boldsymbol{a})$$

as required.

Note. *We note that the linear integral of a gradient is independent of the path in a any open connected set S in which the gradient is continuous.*

Check your progress

1. Which of the following statement is true for line integrals?

   (A) The line integral of a continuous gradient is independent of the path in any open connected set.

   (B) The line integral of a continuous gradient is is zero around every piecewise smooth closed path.

   (C) The value of the integral independent on the path joining a to b.

   (D) For some vector fields, the integral depends only on the end points a and b and not on the path which joins them.

3.8 The first fundamental theorem of calculus for line integrals

Let us first recall the first fundamental theorem for real-valued functions, which states that the derivative of indefinite integral of a continuous function f is equal to f, i.e.,. if

$$\varphi(x) = \int_a^x f(t)dt,$$

then at the points of continuity of f, we have $\varphi'(x) = f(x)$.

Now, we extend the above theorem to line integrals.

Theorem 3.8.1. *Let $\boldsymbol{f}$ be a vector field that is continuous on an open connected set S in $\mathbb{R}^n$, and assume that the line integral of $\boldsymbol{f}$ is independent of the path in S. Let a be a fixed point of S and define a scalar field φ on S by the equation*

$$\varphi(\boldsymbol{x}) = \int_a^{\boldsymbol{x}} \boldsymbol{f} \cdot d\boldsymbol{\alpha}$$

where $\boldsymbol{\alpha}$ is any piecewise smooth path in S joining $\boldsymbol{a}$ to $\boldsymbol{x}$. Then the gradient of φ exists and is equal to f; that is,

$$\nabla\varphi(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x}) \quad \textit{for every } \boldsymbol{x} \textit{ in } S$$

Proof. We shall prove that the partial derivative $D_k\varphi(x)$ exists and

$$D_k\varphi(x) = f_k(x),$$

the k-th component of $\boldsymbol{f}(\boldsymbol{x})$, for each $k = 1, 2, \ldots, n$ and each $\boldsymbol{x}$ in S.

Let $B(\boldsymbol{x}; r)$ be an n-ball with center at $\boldsymbol{x}$ and radius r lying in S. If $\boldsymbol{y}$ is a unit vector, the point $\boldsymbol{x} + h\boldsymbol{y}$ also lies in S for every real h satisfying $0 < |h| < r$, and we can form the difference quotient

$$\frac{\varphi(\boldsymbol{x} + h\boldsymbol{y}) - \varphi(\boldsymbol{x})}{h}$$

By the additive property of line integrals, we have

$$\varphi(\boldsymbol{x} + h\boldsymbol{y}) - \varphi(\boldsymbol{x}) = \int_{\boldsymbol{x}}^{x+hy} \boldsymbol{f} \cdot d\boldsymbol{\alpha}$$

and the path joining $\boldsymbol{x}$ to $\boldsymbol{x} + h\boldsymbol{y}$ can be any piecewise smooth path lying in S.

In particular, we can choose a line segment described by

$$\boldsymbol{\alpha}(t) = \boldsymbol{x} + th\boldsymbol{y}, \quad \text{where} \quad 0 \le t \le 1$$

Since $\boldsymbol{\alpha}'(t) = h\boldsymbol{y}$, the difference quotient becomes

$$\frac{\varphi(\boldsymbol{x} + h\boldsymbol{y}) - \varphi(\boldsymbol{x})}{h} = \int_0^1 f(\boldsymbol{x} + th\boldsymbol{y}) \cdot \boldsymbol{y} dt \tag{3.2}$$

Now we take $\boldsymbol{y} = \boldsymbol{e}_k$, the k-th unit coordinate vector, and note that the integrand becomes $\boldsymbol{f}(\boldsymbol{x} + th\boldsymbol{y}) \cdot \boldsymbol{y} = f_k\left(\boldsymbol{x} + th\boldsymbol{e}_k\right)$. Then we make the change of variable $u = ht, du = hdt$, and we write (3.2) in the form

$$\frac{\varphi\left(\boldsymbol{x} + h\boldsymbol{e}_k\right) - \varphi(\boldsymbol{x})}{h} = \frac{1}{h}\int_0^h f_k\left(\boldsymbol{x} + u\boldsymbol{e}_{\boldsymbol{k}}\right) du = \frac{g(h) - g(0)}{h} \tag{3.3}$$

where g is the function defined on the open interval $(-r, r)$ by the equation

$$g(t) = \int_0^t f_k\left(\boldsymbol{x} + u\boldsymbol{e_k}\right) du$$

Since each component f_k is continuous on S, the first fundamental theorem for ordinary integrals asserts that $g'(t)$ exists for each t in $(-r, r)$ and that

$$g'(t) = f_k\left(x + te_k\right)$$

In particular, $g'(0) = f_k(x)$. Therefore, if we let $h \to 0$ in (3.3) we find that

$$\lim_{h \to 0} \frac{\varphi\left(\boldsymbol{x} + he_k\right) - \varphi(x)}{h} = \lim_{h \to 0} \frac{g(h) - g(0)}{h} = g'(0) = f_k(x)$$

Thus, the partial derivative $D_k\varphi(\boldsymbol{x})$ exists and $D_k\varphi(\boldsymbol{x}) = f_k(\boldsymbol{x})$.

Let us sum up

In this unit, we introduced the notion of line integrals for vector-valued function. Further, we proved some of the important properties. Moreover, we proved the fundamental theorem of calculus for line integrals.

Check your progress

1. The value of the line integral $\int_C \nabla\varphi \cdot d\alpha$ where $\varphi(x, y) = x^3(3 - y^2) + 4y$ and C is given by $\alpha(t) = (3 - t^2, 5 - t)$ with $-2 \leq t \leq 3$ is

   (A) 100 (B) 150 (C) 120 (D) 125

2. Which of the following statement is true for line integrals?

   (A) The line integral of a continuous gradient is independent of the path in any open connected set.

   (B) The line integral of a continuous gradient is is zero around every piecewise smooth closed path.

   (C) The value of the integral independent on the path joining a to b.

   (D) For some vector fields, the integral depends only on the end points a and b and not on the path which joins them.

Glossary

Let $f : \mathbb{R}^n \to \mathbb{R}^m$.

1. When $n = m = 1$, f is called a real-valued function of a real variable.
2. When $n = 1$ and $m > 1$, f is called a vector-valued function of a real variable.
3. When $n > 1$ and $m = 1$, f is called a real-valued function of a vector variable or a **scalar field**.
4. When $n > 1$ and $m > 1$, f is called a vector-valued function of a vector variable or a **vector field**.

Exercises

In each of Exercises 1 through 8 calculate the line integral of the vector field $\boldsymbol{f}$ along the path described.

1. $\boldsymbol{f}(x, y) = (x^2 - 2xy)\, i + (y^2 - 2xy)\, j$, from $(-1, 1)$ to $(1, 1)$ along the parabola $y = x^2$.
2. $\boldsymbol{f}(x, y) = (2a - y)\boldsymbol{i} + x\boldsymbol{j}$, along the path described by $\boldsymbol{\alpha}(t) = a(t - \sin t)\boldsymbol{i} + a(1 - \cos t)\boldsymbol{j}$, $0 \leq t \leq 2\pi$.
3. $\boldsymbol{f}(x, y, z) = (y^2 - z^2)\, i + 2yzj - x^2k$, along the path described by $\alpha(t) = ti + t^2 j + t^3 k$, $0 \leq t \leq 1$.
4. $\boldsymbol{f}(x, y) = (x^2 + y^2)\, \boldsymbol{i} + (x^2 - y^2)\, \boldsymbol{j}$, from $(0, 0)$ to $(2, 0)$ along the curve $y = 1 - |1 - x|$.
5. $\boldsymbol{f}(x, y) = (x + y)\boldsymbol{i} + (x - y)\boldsymbol{j}$, once around the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ in a counterclockwise direction.
6. $\boldsymbol{f}(x, y, z) = 2xyi + (x^2 + z)\, \boldsymbol{j} + y\boldsymbol{k}$, from $(1, 0, 2)$ to $(3, 4, 1)$ along a line segment.
7. $\boldsymbol{f}(x, y, z) = xi + yj + (xz - y)k$, from $(0, 0, 0)$ to $(1, 2, 4)$ along a line segment.

8. $\boldsymbol{f}(x, y, z) = xi + yj + (xz - y)k$, along the path described by $\alpha(t) = t^2 i + 2tj + 4t^3 k$, $0 \le t \le 1$.

In each of Exercises 9 through 12, compute the value of the given line integral.

9. $\int_C (x^2 - 2xy)\, dx + (y^2 - 2xy)\, dy$, where C is a path from $(-2, 4)$ to $(1, 1)$ along the parabola $y = x^2$.

10. $\int_C \frac{(x+y)dx - (x-y)dy}{x^2+y^2}$, where C is the circle $x^2 + y^2 = a^2$, traversed once in a counterclockwise direction.

11. $\int_C \frac{dx+dy}{|x|+|y|}$, where C is the square with vertices $(1, 0), (0, 1), (-1, 0)$, and $(0, -1)$, traversed once in a counterclockwise direction.

12. $\int_C y dx + z dy + x dz$, where

(a) C is the curve of intersection of the two surfaces $x + y = 2$ and $x^2 + y^2 + z^2 = 2(x + y)$. The curve is to be traversed once in a direction that appears clockwise when viewed from the origin.

(b) C is the intersection of the two surfaces $z = xy$ and $x^2 + y^2 = 1$, traversed once in a direction that appears counterclockwise when viewed from high above the xy-plane.

Calculate the line integral with respect to arc length in each of Exercises 13 through 16.

13. $\int_C (x+y) ds$, where C is the triangle with vertices $(0, 0), (1, 0)$, and $(0, 1)$, traversed in a counterclockwise direction.

14. $\int_C y^2 ds$, where C has the vector equation

$$\alpha(t) = a(t - \sin t)i + a(1 - \cos t)j, \quad 0 \le t \le 2\pi$$

15. $\int_C (x^2 + y^2)\, ds$, where C has the vector equation

$$\alpha(t) = a(\cos t + t \sin t)i + a(\sin t - t \cos t)\boldsymbol{j}, \quad 0 \le t \le 2\pi$$

16. $\int_C zds$, where C has the vector equation

$$\alpha(t) = t\cos ti + t\sin tj + tk, \quad 0 \le t \le t_0$$

Application oriented problems

1. A force field $\boldsymbol{f}$ in 3 -space is given by $\boldsymbol{f}(x, y, z) = x\boldsymbol{i} + y\boldsymbol{j} + (xz - y)\boldsymbol{k}$. Compute the work done by this force in moving a particle from $(0,0,0)$ to $(1,2,4)$ along the line segment joining these two points.

2. Find the amount of work done by the force $f(x, y) = (x^2 - y^2)\, i + 2xyj$ in moving a particle (in a counterclockwise direction) once around the square bounded by the coordinate axes and the lines $x = a$ and $y = a, a > 0$.

3. A two-dimensional force field f is given by the equation $f(x, y) = cxyi + x^6y^2j$, where c is a positive constant. This force acts on a particle which must move from $(0,0)$ to the line $x = 1$ along a curve of the form

$$y = ax^b, \quad \text{where} \quad a > 0 \quad \text{and} \quad b > 0$$

Find a value of a (in terms of c) such that the work done by this force is independent of b.

References

1. Tom M. Apostol, Mathematical Analysis, Second Edition, Addison-Wesley Publishing Company Inc., New York, 1974.

2. T.M. Apostol, Calculus Vol.2, Multi-Variable Calculus and Linear Algebra with Applications to Differential Equations and Probability, Second Edition - Reprint, John Wiley & Sons, 2016.

Suggested Readings

1. R. Ghorpade and B.V. Limaye, A Course in Multivariable Calculus and Analysis, Undergraduate Texts in Mathematics, Springer-Verlag, New York, 2010.

2. P. D. Lax and M. S. Terrell Multivariable Calculus with Applications, Springer, 2017.

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Unit 4

Multiple Integrals

Objectives

After reading this read, learners will be able to

1. define the concept of double integral,
2. find a connection between double integrals and line integrals,
3. evaluate the double integral for a step functions and then for more general functions.

4.1 Introduction

Multiple integrals extend the concept of integration to functions of multiple variables, allowing us to calculate volumes, areas, mass, and other physical properties in higher dimensions. There are different types of multiple integrals depending on the number of variables involved, such as double and triple integrals.

Double integrals are used to integrate functions of two variables over a two-dimensional region (usually in the xy-plane). They are often used to calculate the area, volume under a surface, or mass of a region when density varies over the area.

Triple integrals are used for integrating functions of three variables over a three-dimensional region. They are used to compute volumes, mass, and other quantities in three dimensions.

First we consider rectangular regions; later we consider more general regions with curvilinear boundaries. The integrand is a scalar field f defined and bounded on Q. The resulting integral is called a double integral and is denoted by the symbol

$$\iint_Q f, \quad \text{or by} \quad \iint_Q f(x,y)dxdy$$

As in the one-dimensional case, the symbols dx and dy play no role in the definition of the double integral; however, they are useful in computations and transformations of integrals.

4.2 Partitions of rectangles. Step functions

Let Q be a rectangle which is the Cartesian product of two closed intervals $[a,b]$ and $[c,d]$, i.e.,

$$Q = [a,b] \times [c,d] = \{(x,y) \mid x \in [a,b] \text{ and } y \in [c,d]\}$$

Let P_1 and P_2 be two partition of $[a,b]$ and $[c,d]$ respectively, say

$$P_1 = \{x_0, x_1, \ldots, x_{n-1}, x_n\} \quad \text{and} \quad P_2 = \{y_0, y_1, \ldots, y_{m-1}, y_m\}$$

where $x_0 = a, x_n = b, y_0 = c$, and $y_m = d$. The Cartesian product $P_1 \times P_2$ is said to be a partition of Q. Since P_1 decomposes $[a,b]$ into n subintervals and P_2 decomposes $[c,d]$ into m subintervals, the partition $P = P_1 \times P_2$ decomposes Q into mn subrectangles.

Definition 4.2.1. *A partition P' of Q is said to be* **finer** *than P if $P \subseteq P'$, that is, if every point in P is also in P'.*

Definition 4.2.2. *The Cartesian product of two open subintervals of P_1 and P_2 is a subrectangle with its edges missing. This is called an* **open subrectangle** *of P or of Q.*

Definition 4.2.3. (Step Function) *A function f defined on a rectangle Q is said to be a* **step function** *if a partition P of Q exists such that f is constant on each of the open subrectangles of P.*

A typical graph of a step function is shown below. Most of the graph consists of horizontal rectangular patches. A step function also has well-defined values at each of the boundary

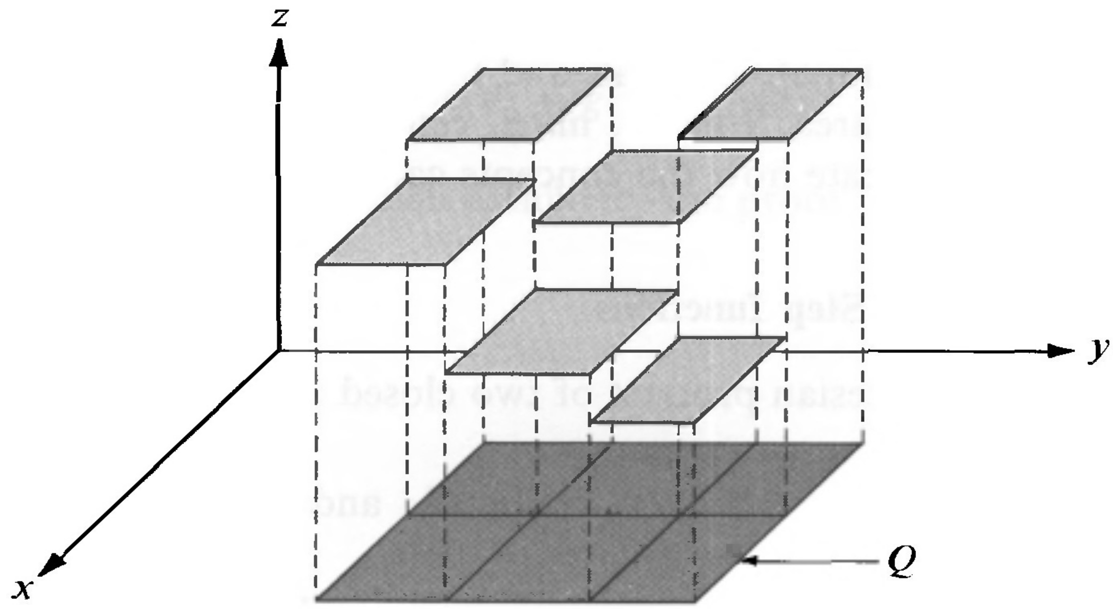

Figure 4.1: Step function defined on a rectangle Q

Check your progress

1. If f and g are two step functions defined on a given rectangle Q, show that the linear combination $c_1 f + c_2 g$ is also a step function.

2. If P and P' are partitions of Q such that f is constant on the open subrectangles of P and g is constant on the open subrectangles of P', then show that $c_1 f + c_2 g$ is constant on the open subrectangles of the union $P \cup P'$ (which we may call a common refinement of P and P').

3. Show that the set of step functions defined on Q forms a **linear space.**

4.3 The double integral of a step function

Let $P = P_1 \times P_2$ be a partition of a rectangle Q into mn subrectangles and let f be a step function that is constant on the open subrectangles of Q. Let the subrectangle determined by $[x_{i-1}, x_i]$ and $[y_{j-1}, y_j]$ be denoted by Q_{ij} and let c_{ij} denote the constant value that f takes at the interior points of Q_{ij}. If f is positive, the volume of the rectangular box with base Q_{ij} and altitude c_{ij} is the product

$$c_{ij} \cdot (x_i - x_{i-1})\,(y_j - y_{j-1})$$

For any step function f, positive or not, the sum of all these products is defined to be the double integral of f over Q. Thus, we have the following definition.

Definition 4.3.1. (Double integral of a step function) *Let f be a step function which takes the constant value c_{ij} on the open subrectangle $(x_{i-1}, x_i) \times (y_{j-1}, y_j)$ of a rectangle Q. The double integral of f over Q is defined by the formula*

$$\iint_Q f = \sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} (x_i - x_{i-1}) (y_j - y_{j-1}) = \sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} \Delta x_i \Delta y_j \tag{11.1}$$

Note. *As in the one-dimensional case, the value of the integral will not differ if the partition P is replaced by any finer partition P'. Thus, the value of the integral is independent of the choice of P so long as f is constant on the open subrectangles of Q.*

Notation. We will write the symbol for the integral as $\iint_Q f(x, y) dx dy$ (or) $\iint_Q f$.

Note. *If $f(x, y) = k$, (a constant) when $a < x < b$ and $c < y < d$, then we have*

$$\iint_Q f = k(b - a)(d - c). \tag{11.2}$$

Since we have

$$b - a = \int_a^b dx \quad \text{and} \quad d - c = \int_c^d dy$$

formula (11.2) can also be written as

$$\iint_Q f = \int_c^d \left[\int_a^b f(x, y) dx \right] dy = \int_a^b \left[\int_c^d f(x, y) dy \right] dx \tag{11.3}$$

The integrals which appear on the right are one-dimensional integrals, and the formula is said to provide an evaluation of the double integral by repeated or iterated integration. In particular, when f is a step function of the type described above, we can write

$$\iint_{Q_{ij}} f = \int_{y_{j-1}}^{y_j} \left[\int_{x_{i-1}}^{x_i} f(x, y) dx \right] dy = \int_{x_{i-1}}^{x_i} \left[\int_{y_{j-1}}^{y_j} f(x, y) dy \right] dx$$

Summing on i and j and using (11.1), we find that (11.3) holds for step functions.

The following further properties of the double integral of a step function are generalizations of the corresponding one-dimensional theorems. They may be proved as direct consequences of the definition in (11.1) or by use of formula (11.3) and the companion theorems for one-dimensional integrals. In the following theorems the symbols s and t denote step functions defined on a rectangle Q. To avoid trivial special cases we assume that Q is a nondegenerate rectangle; in other words, that Q is not merely a single point or a line segment.

Theorem 4.3.2. (Linearity) *For every real c_1 and c_2 we have*

$$\iint_Q [c_1 s(x,y) + c_2 t(x,y)]\, dxdy = c_1 \iint_Q s(x,y)dxdy + c_2 \iint_Q t(x,y)dxdy$$

Theorem 4.3.3. (Additivity) *If Q is subdivided into two rectangles Q_1 and Q_2, then*

$$\iint_Q s(x,y)dxdy = \iint_{Q_1} s(x,y)dxdy + \iint_{Q_2} s(x,y)dxdy$$

Theorem 4.3.4. (Comparison theorem) *If $s(x,y) \leq t(x,y)$ for every (x,y) in Q, we have*

$$\iint_Q s(x,y)dxdy \leq \iint_Q t(x,y)dxdy$$

In particular, if $t(x,y) \geq 0$ for every (x,y) in Q, then

$$\iint_Q t(x,y)dxdy \geq 0.$$

4.4 The definition of the double integral of a function defined and bounded on a rectangle

Let f be a function that is defined and bounded on a rectangle Q, i.e.,

$$|f(x,y)| \leq M \quad \text{if} \quad (x,y) \in Q$$

Then f may be surrounded from above and from below by two constant step functions s and t, where $s(x,y) = -M$ and $t(x,y) = M$ for all (x,y) in Q. Now consider any two step functions s and t, defined on Q, such that

$$s(x,y) \leq f(x,y) \leq t(x,y) \quad \text{for every point } (x,y) \text{ in } Q \tag{4.1}$$

Definition 4.4.1. (Integral of a bounded functon over a rectangle) *If there is one and only one number I such that*

$$\iint_Q s \leq I \leq \iint_Q t \tag{4.2}$$

for every pair of step functions satisfying the inequalities in (**??**)*, this number I is called the double integral of f over Q and is denoted by the symbol*

$$\iint_Q f \quad \text{(or)} \quad \iint_Q f(x,y)dxdy$$

When such an I exists the function ***f* is said to be integrable on** Q**.**

4.5 Upper and lower double integrals

Theorem 4.5.1. *Every function f which is bounded on a rectangle Q has a lower integral $\underline{I}(f)$ and an upper integral $\bar{I}(f)$ satisfying the inequalities*

$$\iint_Q s \leq \underline{I}(f) \leq \bar{I}(f) \leq \iint_Q t$$

for all step functions s and t with $s \leq f \leq t$. The function f is integrable on Q if and only if its upper and lower integrals are equal, in which case we have

$$\iint_Q f = I(f) = \bar{I}(f).$$

Proof. Assume f is bounded on a rectangle Q and let s and t be step functions satisfying

$$s(x,y) \leq f(x,y) \leq t(x,y) \quad \text{for every point } (x,y) \text{ in } Q.$$

We say that s is below f, and t is above f, and we write $s \leq f \leq t$.

Let us define the two sets S and T as follows:

$$S = \left\{ \iint_Q s : s \leq f \right\}$$

$$T = \left\{ \iint_Q t : t \geq f \right\}.$$

Since f is bounded, the sets S and T are nonempty. Also,

$$\iint_Q s \leq \iint_Q t$$

if $s \leq f \leq t$, so every number in S is less than every number in T. Therefore S has a supremum, and T has an infimum, and they satisfy the inequalities

$$\iint_Q s \leq \sup S \leq \inf T \leq \iint_Q t$$

for all s and t satisfying $s \leq f \leq t$. This shows that both numbers $\sup S$ and inf T satisfy (11.5). Therefore, f is integrable on Q if and only if $\sup S = \inf T$, in which case we have

$$\iint_Q f = \sup S = \inf T.$$

The number sup S is called the lower integral of f and is denoted by $\underline{I}(f)$. The number inf T is called the upper integral of f and is denoted by $\overline{I}(f)$. Thus, we have

$$\underline{I}(f) = \sup \left\{ \iint_Q s \mid s \leq f \right\}, \quad \overline{I}(f) = \inf \left\{ \iint_Q t \mid f \leq t \right\}.$$

4.6 Evaluation of a double integral by repeated one-dimensional integration

We evaluate certain double integrals by means of two successive one-dimensional integrations. The result is an extension of formula (11.3), which we have already proved for step functions.

Theorem 4.6.1. *Let f be defined and bounded on a rectangle $Q = [a, b] \times [c, d]$, and assume that f is integrable on Q. For each fixed y in $[c, d]$ assume that the one-dimensional integral $\int_a^b f(x, y)dx$ exists, and denote its value by $A(y)$. If the integral $\int_c^d A(y)dy$ exists it is equal to the double integral $\iint_Q f$. In other words, we have*

$$\iint_Q f(x, y)dxdy = \int_c^a \left[\int_a^b f(x, y)dx \right] dy \tag{4.3}$$

Proof. Choose any two step functions s and t satisfying $s \leq f \leq t$ on Q. Integrating with respect to x over the interval $[a, b]$ we have

$$\int_a^b s(x,y)dx \leq \int_a^b f(x,y)dx \leq \int_a^b t(x,y)dx,$$

and therefore,

$$\int_a^b s(x,y)dx \leq A(y) \leq \int_a^b t(x,y)dx.$$

Since the integral $\displaystyle\int_c^d A(y)dy$ exist, we can integrate with respect to y to obtain

$$\int_c^d \int_a^b s(x,y)dxdy \leq \int_c^d A(y)dy \leq \int_c^d \int_a^b t(x,y)dxdy.$$

Since s and t are arbitrary step functions and f is integrable on Q, we have

$$\iint_Q f = \int_c^d A(y)dy = \int_c^d \left[\int_a^b f(x,y)dx\right] dy.$$

4.7 Integrability of continuous functions

Theorem 4.7.1. (Small-span theorem) *If f is continuous on a rectangle $Q = [a,b] \times [c,d]$, then for every $\epsilon > 0$, there is a partition P of Q into a finite number (say n)of subrectangles $Q_1, Q_2, \ldots, Q_n$ such that the span of f in every rectangle Q_k is less than ϵ, i.e.,*

$$M_k(f) - m_k(f) < \epsilon$$

where $M_k(f) = \sup\limits_{Q_k} |f|$ and $m_k(f) = \inf\limits_{Q_k} |f|$

We will use the above small-span theorem to prove integrability of a function which is continuous on a rectangle.

Theorem 4.7.2. (Integrability of Continuous Functions) *If a function f is continuous on a rectangle $Q = [a,b] \times [c,d]$, then f is integrable on Q. Moreover, the value of the integral can be obtained by iterated integration,*

$$\iint_Q f = \int_c^a \left[\int_a^b f(x,y)dx\right] dy = \int_a^b \left[\int_c^a f(x,y)dy\right] dx \tag{4.4}$$

Proof. Since f is continuous on the rectangle Q (which is compact), f is bounded on Q. If f is bounded on Q, then f has an upper integral and a lower integral.

We shall prove that $I(f) = \bar{I}(f)$.

Choose $\epsilon > 0$.

By the small-span theorem, for this choice of ϵ, there is a partition P of Q into a finite number (say n) of subrectangles $Q_1, \ldots, Q_n$ such that the span of f in every subrectangle is less than ϵ. Denote by $M_k(f)$ and $m_k(f)$, respectively, the absolute maximum and minimum values of f in Q_k. Then we have

$$M_k(f) - m_k(f) < \epsilon$$

where $M_k(f) = \sup_{Q_k} |f|$ and $m_k(f) = \inf_{Q_k} |f|$, for each $k = 1, 2, \ldots, n$. Now let s and t be two step functions defined on the interior of each Q_k as follows:

$$s(\boldsymbol{x}) = m_k(f), \quad t(\boldsymbol{x}) = M_k(f) \quad \text{if} \quad \boldsymbol{x} \in \text{ int } Q_k$$

At the boundary points we define

$$s(\boldsymbol{x}) = m \quad \text{and} \quad t(\boldsymbol{x}) = M,$$

where $m = \inf_Q |f|$ and $M = \sup_Q |f|$.

Then we have $s \leq f \leq t$ for all $\boldsymbol{x}$ in Q. Also, we have

$$\iint_Q s = \sum_{k=1}^{n} m_k(f) a\left(Q_k\right) \quad \text{and} \quad \iint_Q t = \sum_{k=1}^{n} M_k(f) a\left(Q_k\right)$$

where $a\left(Q_k\right)$ is the area of rectangle Q_k. The difference of these two integrals is

$$\iint_Q t - \iint_Q s = \sum_{k=1}^{n} \left\{M_k(f) - m_k(f)\right\} a\left(Q_k\right) < \epsilon \sum_{k=1}^{n} a\left(Q_k\right) = \epsilon a(Q)$$

where $a(Q)$ is the area of Q. Since $\iint_Q s \leq I(f) \leq \bar{I}(f) \leq \iint_Q t$, we obtain the inequality

$$0 \leq \bar{I}(f) - I(f) \leq \epsilon a(Q).$$

Letting $\epsilon \to 0$ we see that $I(f) = \bar{I}(f)$, so f is integrable on Q.

Next we prove that the double integral is equal to the first iterated integral in (4.4). For each fixed y in $[c, d]$ the one-dimensional integral $\int_a^b f(x, y) dx$ exists since the integrand is continuous on Q.

Let $A(y) = \displaystyle\int_a^b f(x, y) dx$.

We shall prove that A is continuous on $[c, d]$. If y and y_1 are any two points in $[c, d]$ we have

$$A(y) - A(y_1) = \int_a^b \{f(x, y) - f(x, y_1)\} dx$$

from which we find

$$|A(y) - A(y_1)| \leq (b - a) \max_{a \leq x \leq b} |f(x, y) - f(x, y_1)| = (b - a) |f(x_1, y) - f(x_1, y_1)|$$

where x_1 is a point in $[a, b]$ where $|f(x, y) - f(x, y_1)|$ attains its maximum. This inequality shows that $A(y) \to A(y_1)$ as $y \to y_1$, so A is continuous at y_1. Therefore the integral $\displaystyle\int_c^d A(y) dy$ exists and, by Theorem 4.6.1, it is equal to $\displaystyle\iint_Q f$. A similar argument works when the iteration is taken in the reverse order.

4.8 Integrability of bounded functions with discontinuities

Let f be defined and bounded on a rectangle Q. In this section, we prove that the double integral exists if f has discontinuities in Q, provided the set of discontinuities is not too large. To measure the size of the set of discontinuities we introduce the concept of bounded set of content zero.

Definition 4.8.1. (Bounded set of Content Zero) *Let A be a bounded subset of the plane. The set A is said to have content zero if for every $\epsilon > 0$ there is a finite set of rectangles whose union contains A and the sum of whose areas does not exceed ϵ.*

In other words, a bounded plane set of content zero can be enclosed in a union of rectangles whose total area is arbitrarily small.

The following statements about bounded sets of content zero are easy consequences of this definition.

(a) Any finite set of points in the plane has content zero.

(b) The union of a finite number of bounded sets of content zero is also of content zero.

(c) Every subset of a set of content zero has content zero.

(d) Every line segment has content zero.

Theorem 4.8.2. *Let f be defined and bounded on a rectangle $Q = [a, b] \times [c, d]$. If the set of discontinuities of f in Q is a set of content zero then the double integral $\iint_Q f$ exists.*

Proof. Let $M > 0$ be such that $|f| \leq M$ on Q.

Let D denote the set of discontinuities of f in Q.

Let $\delta > 0$. Since D has content zero, there is a partition P of Q such that the sum of the areas of all the subrectangles of P which contain points of D is less than δ.

On these subrectangles define step functions s and t as follows:

$$s(x) = -M, \quad t(x) = M$$

On the remaining subrectangles of P, we define s and t as follows:

On the interior of each Q_k, s and t are defined as follows:

$$s(\boldsymbol{x}) = m_k(f), \quad t(\boldsymbol{x}) = M_k(f) \quad \text{if} \quad \boldsymbol{x} \in \text{ int } Q_k$$

At the boundary points we define

$$s(\boldsymbol{x}) = m \quad \text{and} \quad t(\boldsymbol{x}) = M,$$

where $m = \inf_Q |f|$ and $M = \sup_Q |f|$.

. Then we have $s \leq f \leq t$ for all $Q \backslash D$. Also we have

$$\begin{aligned} \iint_{Q\backslash D} s &= \sum_{k=1}^{n} m_k(f) a(Q_k), \\ \iint_{Q\backslash D} t &= \sum_{k=1}^{n} M_k(f) a(Q_k) \end{aligned}$$

where $a(Q_k)$ is the area of rectangle Q_k. Moreover,

$$\begin{aligned}\iint_{Q\setminus D} t - \iint_{Q\setminus D} s &= \sum_{k=1}^{n}[M_k(f) - m_k(f)]a(Q_k), \\ &< \epsilon \sum_{k=1}^{n} a(Q_k) \\ &= \epsilon a(Q\setminus D)\end{aligned}$$

Also,

$$\iint_D t - \iint_D s = \iint_D t - s = 2M \iint_D 1 < 2M\delta$$

Therefore,

$$\begin{aligned}\iint_Q t - \iint_Q s &= \iint_{Q\setminus D} t - s + \iint_D t - s \\ &\leq \epsilon a(Q) + 2M\delta.\end{aligned}$$

The first term, $\epsilon a(Q)$, comes from estimating the integral of $t-s$ over the subrectangles containing only points of continuity of f; the second term, $2M\delta$, comes from estimating the integral of $t - s$ over the subrectangles which contain points of D. Hence,

$$\overline{I}(f) - \underline{I}(f) \leq \iint_Q t - \iint_Q s \leq \epsilon a(Q) + 2M\delta$$

Letting $\epsilon \to 0$ we have

$$0 \leq \bar{I}(f) - I(f) \leq 2M\delta.$$

Since δ is arbitrary, we have $\bar{I}(f) = I(f)$.

So f is integrable on Q.

Let us sum up

In this unit, we introduced the notion of double integration and studied some its important properties.

Check your progress

1. If f is integrable on Q, which of the following is true ?

   (A) $\iint_Q f = \sup S = \inf T$ (B) $\iint_Q s \leq \inf T \leq \sup S \iint_Q t$

   (C) $\iint_Q t \leq \sup S \leq \inf T \leq \iint_Q s$ (D) $\iint_Q t \leq \sup T \leq \inf S \leq \iint_Q s$

2. The value of the double integral $\iint_Q \sin(x+y)dxdy$ where $Q = [0, \pi/2] \times [0, \pi/2]$ is

   (A) 1 (B) 2 (C) $\pi/2$ (D) π

3. The value of the line integral $\oint_C y^2dx + xdy$ where C is the square with vertices $(0,0),\ (2,0),\ (2,2),\ (0,2)$ is

   (A) -4 (B) 4 (C) 8 (D) 2

4. Stokes theorem relates

   (A) a surface integral to a line integral

   (B) a surface integral to a double integral

   (C) a double integral to a line integral

   (D) a volume integral to a surface integral

5. Gauss theorem provides a relationship between

   (A) a triple integral over a solid and a surface integral over the boundary of the solid

   (B) an integral over a surface and a line integral over the boundary of the surface

   (C) a double integral over a plane region R and a line integral over a closed curve forming the boundary of R.

   (D) None of the above.

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2. T.M. Apostol, Calculus Vol.2, Multi-Variable Calculus and Linear Algebra with Applications to Differential Equations and Probability, Second Edition - Reprint, John Wiley & Sons, 2016.

Suggested Readings

1. R. Ghorpade and B.V. Limaye, A Course in Multivariable Calculus and Analysis, Undergraduate Texts in Mathematics, Springer-Verlag, New York, 2010.

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Unit 5

Green's, Stoke's and Gauss's Theorem

Objective

After this unit, learners will be able to

- understand and apply the fundamental theorems that relate different types of integrals in vector calculus.
- apply Green's theorem to convert a line integral around a closed curve into a double integral over the region enclosed by the curve
- apply Stokes' Theorem to compute line integrals or surface integrals in 3D.
- use the Divergence Theorem to simplify complex surface integrals into volume integrals and vice versa.

5.1 Green's theorem in the plane

The second fundamental theorem of calculus for line integrals states that the line integral of a gradient ∇f along a path joining two points A and B may be expressed in terms of the function values $f(A)$ and $f(B)$. A two-dimensional version of the second fundamental theorem is usually referred to as Green's theorem. This expresses a double integral over a plane region R as a line integral taken along a closed curve forming the boundary of R and it can be stated as follows:

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = \oint_C Pdx + Qdy$$

The curve C which appears on the right is the boundary of the region R, and the integration symbol $\oint$ indicates that the curve is to be traversed in the counterclockwise direction.

Two types of assumptions are required for the validity of this identity.

1. First, conditions are imposed on the functions P and Q to ensure the existence of the integrals. The usual assumptions are that P and Q are continuously differentiable on an open set S containing the region R. This implies continuity of P and Q on C as well as continuity of $\partial P/\partial y$ and $\partial Q/\partial x$ on R, although the theorem is also valid under less stringent hypotheses.

2. Second, there are conditions of a geometric nature that are imposed on the region R and its boundary curve C. The curve C may be any rectifiable simple closed curve. The term "rectifiable" means that C has a finite arc length.

Definition 5.1.1. *Suppose C is described by a continuous vector-valued function α defined on an interval $[a, b]$. If $\boldsymbol{\alpha}(a) = \alpha(b)$, the curve is closed.*

Definition 5.1.2. *A closed curve such that $\boldsymbol{\alpha}(t_1) \neq \alpha(t_2)$ for every pair of values $t_1 \neq t_2$ in the half-open interval $(a, b]$ is called a simple closed curve.*

This means that, except for the end points of the interval $[a, b]$, distinct values of t lead to distinct points on the curve. A circle is the prototype of a simple closed curve.

Definition 5.1.3. *Simple closed curves that lie in a plane are usually called Jordan curves.*

Note. *Every Jordan curve C decomposes the plane into two disjoint open connected sets having the curve C as their common boundary. One of these regions is bounded and is called the interior (or inner region) of C. The other is unbounded and is called the exterior (or outer region) of C.*

Green's theorem is valid whenever C is a rectifiable Jordan curve, and the region R is the union of C and its interior. Since we have not defined line integrals along arbitrary rectifiable curves, we restrict our discussion here to piecewise smooth curves.

Theorem 5.1.4. (Green's theorem for plane regions bounded by piecewise smooth Jordan curves) *Let P and Q be scalar fields that are continuously differentiable on an open set S in the xy-plane. Let C be a piecewise smooth Jordan curve, and let R denote the union of C and its interior. Assume R is a subset of S. Then we have the identity*

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \oint_C Pdx + Qdy \tag{5.1}$$

where the line integral is taken around C in the counterclockwise direction.

Proof. The Green's identity is equivalent to the two formulas, namely,

$$\iint_R \frac{\partial Q}{\partial x} dxdy = \oint_C Qdy \tag{5.2}$$

and

$$-\iint_R \frac{\partial P}{\partial y} dxdy = \oint_C Pdx \tag{5.3}$$

In fact, if both of these are true, Green's identity follows by adding the above two equations. Conversely, if Green's identity is true we may obtain (5.2) and (5.3) as special cases by taking $P = 0$ and $Q = 0$, respectively.

Proof for special regions. We shall prove the theorem for a region R of Type I.

Such a region has the form

$$R = \{(x, y) \mid a \le x \le b \text{ and } f(x) \le y \le g(x)\}$$

where f and g are continuous on $[a, b]$ with $f \le g$. The boundary C of R consists of four parts, a lower arc C_1 (the graph of f), an upper arc C_2 (the graph of g), and two vertical line segments, traversed in the directions.

First let us evaluate the double integral $-\iint_R \frac{\partial P}{\partial y} dxdy$ by iterated integration. Integrating first with respect to y, we have

$$\begin{aligned} -\iint_R \frac{\partial P}{\partial y} dxdy &= -\int_a^b \left[\int_{f(x)}^{g(x)} \frac{\partial P}{\partial y} dy\right] dx \\ &= \int_a^b \left[\int_{g(x)}^{f(x)} \frac{\partial P}{\partial y} dy\right] dx \\ &= \int_a^b P[x, f(x)]dx - \int_a^b P[x, g(x)]dx \end{aligned} \tag{5.4}$$

On the other hand, the line integral $\int_C P dx$ can be written as follows:

$$\int_C P dx = \int_{C_1} P dx + \int_{C_2} P dx + \int_{L_1} P dx + \int_{L_2} P dx$$

Since the line integral along each vertical segment is zero, we have

$$\int_C P dx = \int_{C_1} P dx + \int_{C_2} P dx.$$

To evaluate the integral along C_1, we use the vector representation $\alpha(t) = ti + f(t)j$ and obtain

$$\int_{C_1} P dx = \int_a^b P[t, f(t)] dt$$

Next, we use the representation $\alpha(t) = ti + g(t)j$ to evaluate the integral along C_2 and we obtain

$$\int_{C_2} P dx = -\int_a^b P[t, g(t)] dt$$

where negative sign is used to take into account the reversal in direction. Therefore we have

$$\int_C P dx = \int_a^b P[t, f(t)] dt - \int_a^b P[t, g(t)] dt.$$

Comparing this equation with the formula in (5.4) we obtain (5.3).

A similar argument can be used to prove Greens' identity for regions of Type II. In this way a proof of Green's theorem is obtained for regions that are of both Type I and Type II. Once this is done, the theorem can be proved for those regions R that can be decomposed into a finite number of regions that are of both types. "Crosscuts" are introduced and the theorem is applied to each subregion, and the results are added together. The line integrals along the crosscuts cancel in pairs, and the sum of the line integrals along the boundaries of the subregions is equal to the line integral along the boundary of R.

Check your progress

1. The value of the line integral $\oint_C y^2 dx + x dy$ where C is the square with vertices $(0,0),\ (2,0),\ (2,2),\ (0,2)$ is

   (A) -4 (B) 4 (C) 8 (D) 2

5.2 Change of variables in a double integral

In one-dimensional integration theory the method of substitution often enables us to evaluate complicated integrals by transforming them into simpler ones or into types that can be more easily recognized. The method is based on the formula

$$\int_a^b f(x)dx = \int_c^d f[g(t)]g'(t)dt \tag{5.5}$$

where $a = g(c)$ and $b = g(d)$. The above formula is valid under the assumptions that g has a continuous derivative on an interval $[c, d]$ and that f is continuous on the set of values taken by $g(t)$ as t runs through the interval $[c, d]$.

In a similar way, there is a two-dimensional analogue of (5.5) called the formula for making a change of variables in a double integral. It transforms an integral of the form $\iint_S f(x, y)dxdy$, extended over a region S in the xy-plane, into another double integral $\int_T^S F(u, v)dudv$, extended over a new region T in the uv-plane. The exact relationship between the regions S and T and the integrands $f(x, y)$ and $F(u, v)$ will be discussed presently. The method of substitution for double integrals is more elaborate than in the one-dimensional case because there are two formal substitutions to be made, one for x and another for y. This means that instead of the one function g which appears in Equation (5.5), we now have two functions, say X and Y, which connect x, y with u, v as follows:

$$x = X(u, v), \quad y = Y(u, v) \tag{5.6}$$

The two equations in (5.6) define a mapping which carries a point (u, v) in the uv-plane into an image point (x, y) in the xy-plane. A set T of points in the uv-plane is mapped onto another set S in the xy-plane. The mapping can also be described by means of a vector-valued function. From the origin in the xy-plane we draw the radius vector $\boldsymbol{r}$ to a general point (x, y) of S. The vector $\boldsymbol{r}$ depends on both u and v and can be considered a vector-valued function of two variables defined by the equation

$$\boldsymbol{r}(u, v) = X(u, v)\boldsymbol{i} + Y(u, v)\mathbf{j} \quad \text{if} \quad (u, v) \in T \tag{5.7}$$

This equation is called a vector equation of the mapping. As (u, v) runs through the points of T, the endpoint of $\boldsymbol{r}(u, v)$ traces out the points of S.

Sometimes the two equations in (5.6) can be solved for u and v in terms of x and y. When this is possible we may express the result in the form

$$u = U(x, y), \quad v = V(x, y)$$

These equations define a mapping from the xy-plane to the uv-plane, called the inverse mapping of the one defined by (5.6), since it carries points of S back to T. The so-called one-to-one mappings are of special importance. These carry distinct points of T onto distinct points of S; in other words, no two distinct points of T are mapped onto the same point of S by a one-to-one mapping. Each such mapping establishes a one-to-one correspondence between the points in T and those in S and enables us (at least in theory) to go back from S to T by the inverse mapping (which, of course, is also one-to-one).

We shall consider mappings for which the functions X and Y are continuous and have continuous partial derivatives $\partial X/\partial u, \partial X/\partial v, \partial Y/\partial u$, and $\partial Y/\partial v$ on S. Similar assumptions are made for the functions U and V.

The formula for transforming double integrals may be written as

$$\iint_S f(x, y) dx dy = \iint_T f[X(u, v), Y(u, v)] |J(u, v)| du dv \tag{5.8}$$

The factor $J(u, v)$ which appears in the integrand on the right plays the role of the factor $g'(t)$ which appears in the one-dimensional Formula (5.5). This factor is called the Jacobian determinant of the mapping defined by (5.6); it is equal to

$$J(u, v) = \begin{vmatrix} \frac{\partial X}{\partial u} & \frac{\partial Y}{\partial u} \\ \frac{\partial X}{\partial v} & \frac{\partial Y}{\partial v} \end{vmatrix}$$

Sometimes the symbol $\dfrac{\partial(X, Y)}{\partial(u, v)}$ is used instead of $J(u, v)$ to represent the Jacobian determinant.

In special case (5.8) becomes

$$\iint_S dx dy = \iint_T |J(u, v) du dv \tag{5.9}$$

Geometric motivation. Take a region T in the uv-plane, and let S denote the set of points in the xy-plane onto which T is mapped by the vector function r given by (5.7). Now introduce two new vector-valued functions $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$ which are obtained by taking the partial derivatives of the components of $\boldsymbol{r}$ with respect to u and v, respectively. That is, define

$$\boldsymbol{V}_1 = \frac{\partial \boldsymbol{r}}{\partial u} = \frac{\partial X}{\partial u}\boldsymbol{i} + \frac{\partial Y}{\partial u}\boldsymbol{j} \quad \text{and} \quad \boldsymbol{V}_2 = \frac{\partial \boldsymbol{r}}{\partial v} = \frac{\partial X}{\partial v}\boldsymbol{i} + \frac{\partial Y}{\partial v}\boldsymbol{j}$$

These vectors may be interpreted geometrically as follows: Consider a horizontal line segment in the uv-plane (v is constant on such a segment). The vector function $\boldsymbol{r}$ maps this segment onto a curve (called a u-curve) in the xy-plane. If we think of u as a parameter representing time, the vector V_1 represents the velocity of the position $\boldsymbol{r}$ and is therefore tangent to the curve traced out by the tip of $\boldsymbol{r}$. In the same way, each vector V_2 represents the velocity vector of a v-curve obtained by setting $u =$ constant. A u-curve and a v-curve pass through each point of the region S.

Consider now a small rectangle with dimensions Δu and Δv. If Δu is the length of a small time interval, then in time Δu a point of a u-curve moves along the curve a distance approximately equal to the product $\|\boldsymbol{V}_1\|\,\Delta u$ (since $\|\boldsymbol{V}_1\|$ represents the speed and Δu the time). Similarly, in time Δv a point on a v-curve moves a distance nearly equal to $\|V_2\|\,\Delta v$. Hence the rectangular region with dimensions Δu and Δv in the uv-plane is traced onto a portion of the xy-plane that is nearly a parallelogram, whose sides are the vectors $V_1\Delta u$ and $V_2\Delta v$. The area of this parallelogram is the magnitude of the cross product of the two vectors $V_1\Delta u$ and $V_2\Delta v$; this is equal to

$$\|(\boldsymbol{V}_1\Delta u) \times (\boldsymbol{V}_2\Delta v)\| = \|\boldsymbol{V}_1 \times \boldsymbol{V}_2\|\,\Delta u\Delta v$$

If we compute the cross product $\boldsymbol{V}_1 \times \boldsymbol{V}_2$ in terms of the components of $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$ we find

$$\boldsymbol{V}_1 \times \boldsymbol{V}_2 = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial X}{\partial u} & \frac{\partial Y}{\partial u} & 0 \\ \frac{\partial X}{\partial v} & \frac{\partial Y}{\partial v} & 0 \end{vmatrix} = \begin{vmatrix} \frac{\partial X}{\partial u} & \frac{\partial Y}{\partial u} \\ \frac{\partial X}{\partial v} & \frac{\partial Y}{\partial v} \end{vmatrix} \boldsymbol{k} = J(u,v)\boldsymbol{k}$$

Therefore the magnitude of $V_1 \times V_2$ is exactly $|J(u,v)|$ and the area of the curvilinear parallelogram is nearly equal to $|J(u,v)|\Delta u\Delta v$.

If $J(u,v) = 1$ for all points in T, then the "parallelogram" has the same area as the rectangle and the mapping preserves areas. Otherwise, to obtain the area of the parallelogram we must multiply the area of the rectangle by $|J(u,v)|$. This suggests that the Jacobian may be thought of as a "magnification factor" for areas.

Now let P be a partition of a large rectangle R enclosing the entire region T and consider a typical subrectangle of P of, say, dimensions Δu and Δv. If Δu and Δv are small, the Jacobian function J is nearly constant on the subrectangle and hence J acts somewhat like a step function on R. (We define J to be zero outside T.) If we think of J as an actual step function, then the double integral of $|J|$ over R (and hence over T) is a sum of products of the form $|J(u,v)|\Delta u \Delta v$ and the above remarks suggest that this sum is nearly equal to the area of S, which we know to be the double integral $\iint_S dxdy$.

This geometric discussion, which merely suggests why we might expect an equation like (5.9) to hold, can be made the basis of a rigorous proof, but the details are lengthy and rather intricate. As mentioned above, a proof of (5.9), based on an entirely different approach, will be given in a later section.

If $J(u,v) = 0$ at a particular point (u,v), the two vectors V_1 and V_2 are parallel (since their cross product is the zero vector) and the parallelogram degenerates into a line segment. Such points are called singular points of the mapping. As we have already mentioned, transformation formula (5.8) is also valid whenever there are only a finite number of such singular points or, more generally, when the singular points form a set of content zero. This is the case for all the mappings we shall use.

5.3 Surface Integrals

Surface integrals extend the concept of integration to surfaces in three-dimensional space, allowing you to integrate scalar or vector fields over a surface. They play a crucial role in vector calculus, physics, and engineering, particularly in calculating quantities like mass, area, flux, and more. Surface integrals are analogous to line integrals as the integration takes place along a surface rather than along a curve. We defined line integrals in terms of a parametric representation for the curve. Similarly, we shall define surface integrals in terms of a parametric representation for the surface. Then we shall prove that under certain general conditions the value of the integral is independent of the representation.

Definition 5.3.1. *Let $S = \boldsymbol{r}(T)$ be a parametric surface described by a differentiable function $\boldsymbol{r}$ defined on a region T in the uv-plane, and let f be a scalar field defined and bounded on S. The surface integral of f over S is denoted by the symbol $\iint f dS$ (or) by $\iint_S f(x, y, z) dS$, and is defined by the equation*

$$\iint_{\boldsymbol{r}(T)} f dS = \iint_T f[\boldsymbol{r}(u, v)] \left\| \frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v} \right\| dudv$$

whenever the double integral on the right exists.

Let $S = \boldsymbol{r}(T)$ be a simple parametric surface. At each regular point of S let $\boldsymbol{n}$ denote the unit normal having the same direction as the fundamental vector product. That is, let

$$\boldsymbol{n} = \frac{\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v}}{\left\| \frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v} \right\|} \tag{5.10}$$

The dot product $\boldsymbol{F} \cdot \boldsymbol{n}$ represents the component of the flux density vector in the direction of $\boldsymbol{n}$. The mass of fluid flowing through S in unit time in the direction of $\boldsymbol{n}$ is defined to be the surface integral

$$\iint_{\boldsymbol{r}(T_T)} \boldsymbol{F} \cdot \boldsymbol{n} dS = \iint_T \boldsymbol{F} \cdot \boldsymbol{n} \left\| \frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v} \right\| dudv$$

5.4 Change of parametric representation

We turn now to a discussion of the independence of surface integrals under a change of parametric representation. Suppose a function $\boldsymbol{r}$ maps a region A in the uv-plane onto a parametric surface $\boldsymbol{r}(A)$. Suppose also that A is the image of a region B in the st-plane under a one-to-one continuously differentiable mapping $\boldsymbol{G}$ given by

$$\boldsymbol{G}(s,t) = U(s,t)\boldsymbol{i} + V(s,t)\boldsymbol{j} \quad \text{if} \quad (s,t) \in B \tag{5.11}$$

Consider the function $\boldsymbol{R}$ defined on B by the equation

$$\boldsymbol{R}(s,t) = \boldsymbol{r}[\boldsymbol{G}(s,t)] \tag{5.12}$$

Two functions $\boldsymbol{r}$ and $\boldsymbol{R}$ so related will be called smoothly equivalent. Smoothly equivalent functions describe the same surface. That is, $\boldsymbol{r}(A)$ and $\boldsymbol{R}(B)$ are identical as point sets. (This follows at once from the one-to-one nature of $\boldsymbol{G}$.) The next theorem describes the relationship between their fundamental vector products.

Theorem 5.4.1. *Let $\boldsymbol{r}$ and $\boldsymbol{R}$ be smoothly equivalent functions related by Equation (5.12), where $\boldsymbol{G} = U\boldsymbol{i} + V\boldsymbol{j}$ is a one-to-one continuously differentiable mapping of a region B in the st-plane onto a region A in the uv-plane given by Equation (5.11). Then we have*

$$\frac{\partial \boldsymbol{R}}{\partial s} \times \frac{\partial \boldsymbol{R}}{\partial t} = \left(\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v}\right) \frac{\partial(U,V)}{\partial(s,t)} \tag{5.13}$$

where the partial derivatives $\partial\mathbf{r}/\partial u$ and $\partial\mathbf{r}/\partial v$ are to be evaluated at the point $(U(s,t),V(s,t))$. In other words, the fundamental vector product of $\boldsymbol{R}$ is equal to that of $\boldsymbol{r}$, times the Jacobian determinant of the mapping $\boldsymbol{G}$.

Proof. The derivatives $\partial\boldsymbol{R}/\partial s$ and $\partial\boldsymbol{R}/\partial t$ can be computed by differentiation of Equation (5.12). If we apply the chain rule to each component of $\boldsymbol{R}$ and rearrange terms, we find that

$$\frac{\partial \boldsymbol{R}}{\partial s} = \frac{\partial \boldsymbol{r}}{\partial u}\frac{\partial U}{\partial s} + \frac{\partial \boldsymbol{r}}{\partial v}\frac{\partial V}{\partial s} \quad \text{and} \quad \frac{\partial \boldsymbol{R}}{\partial t} = \frac{\partial \boldsymbol{r}}{\partial u}\frac{\partial U}{\partial t} + \frac{\partial \boldsymbol{r}}{\partial v}\frac{\partial V}{\partial t}$$

where the derivatives $\partial\boldsymbol{r}/\partial u$ and $\partial\boldsymbol{r}/\partial v$ are evaluated at $(U(s,t),V(s,t))$. Now we cross multiply these two equations and, noting the order of the factors, we obtain

$$\frac{\partial \boldsymbol{R}}{\partial s} \times \frac{\partial \boldsymbol{R}}{\partial t} = \left(\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v}\right)\left(\frac{\partial U}{\partial s}\frac{\partial V}{\partial t} - \frac{\partial U}{\partial t}\frac{\partial V}{\partial s}\right) = \left(\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v}\right)\frac{\partial(U,V)}{\partial(s,t)}$$

This completes the proof.

5.5 Stoke's Theorem

Stoke's theorem is a generalization of the second fundamental theorem of calculus involving surface integrals. Moreover, Stoke's theorem is a direct extension of Green's theorem which states that

$$\iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \int_C Pdx + Qdy$$

where S is a plane region bounded by a simple closed curve C traversed in the positive (counterclockwise) direction. Stokes' theorem relates a surface integral to a line integral which can be stated as follows:

Theorem 5.5.1. (Stokes' Theorem) *Assume that S is a smooth simple parametric surface, say $S = \boldsymbol{r}(T)$, where T is a region in the uv-plane bounded by a piecewise smooth Jordan curve Γ. Assume also that $\boldsymbol{r}$ is a one-to-one mapping whose components have continuous second-order partial derivatives on some open set containing $T \cup \Gamma$. Let C denote the image of Γ under $\boldsymbol{r}$, and let P, Q, and R be continuously differentiable scalar fields on S. Then we have*

$$\begin{aligned}\iint_S \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy \wedge dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dz \wedge dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy \\ = \int_C Pdx + Qdy + Rdz\end{aligned} \tag{5.14}$$

The curve Γ is traversed in the positive (counterclockwise) direction and the curve C is traversed in the direction inherited from Γ through the mapping function $\boldsymbol{r}$.

Proof. It is sufficint to establish the following:

$$\begin{aligned}\int_C Pdx &= \iint_S \left(-\frac{\partial P}{\partial y} dx \wedge dy + \frac{\partial P}{\partial z} dz \wedge dx\right) \qquad (5.15)\\ \int_C Qdy &= \iint_S \left(-\frac{\partial Q}{\partial z} dy \wedge dz + \frac{\partial Q}{\partial x} dx \wedge dy\right)\\ \int_C Rdz &= \iint_S \left(-\frac{\partial R}{\partial x} dz \wedge dx + \frac{\partial R}{\partial y} dy \wedge dz\right)\end{aligned}$$

Adding the above three equations we get the formula (5.14) in Stokes' theorem. Since the above three equations are similar, we prove only Equation (5.15).

The idea is to express the surface integral on the right as a double integral over T and then we use Green's theorem to express the double integral over T as a line integral over Γ. Finally, we show that this line integral is equal to $\int_C P dx$.

We write

$$\boldsymbol{r}(u,v) = X(u,v)\boldsymbol{i} + Y(u,v)\boldsymbol{j} + Z(u,v)\boldsymbol{k}$$

and express the surface integral over S in the form

$$\iint_S \left(-\frac{\partial P}{\partial y} dx \wedge dy + \frac{\partial P}{\partial z} dz \wedge dx\right) = \iint_T \left\{-\frac{\partial P}{\partial y}\frac{\partial(X,Y)}{\partial(u,v)} + \frac{\partial P}{\partial z}\frac{\partial(Z,X)}{\partial(u,v)}\right\} dudv.$$

Now let p denote the composite function given by

$$p(u,v) = P[X(u,v), Y(u,v), Z(u,v)].$$

The last integrand can be written as

$$-\frac{\partial P}{\partial y}\frac{\partial(X,Y)}{\partial(u,v)} + \frac{\partial P}{\partial z}\frac{\partial(Z,X)}{\partial(u,v)} = \frac{\partial}{\partial u}\left(p\frac{\partial X}{\partial v}\right) - \frac{\partial}{\partial v}\left(p\frac{\partial X}{\partial u}\right).$$

Applying Green's theorem to the double integral over T, we obtain

$$\iint_T \left\{\frac{\partial}{\partial u}\left(p\frac{\partial X}{\partial v}\right) - \frac{\partial}{\partial v}\left(p\frac{\partial X}{\partial u}\right)\right\} dudv = \int_\Gamma p\frac{\partial X}{\partial u}du + p\frac{\partial X}{\partial v}dv$$

where Γ is traversed in the positive direction. We parametrize Γ by a function γ defined on an interval $[a,b]$ and let

$$\alpha(t) = r[\gamma(t)],$$

be a corresponding parametrization of C. Then by expressing each line integral in terms of its parametric representation we find that

$$\int_\Gamma p\frac{\partial X}{\partial u}du + p\frac{\partial X}{\partial v}dv = \int_C P dx$$

which completes the proof of (5.14).

Check your progress

1. Stokes theorem relates

   (A) a surface integral to a line integral

(B) a surface integral to a double integral

(C) a double integral to a line integral

(D) a volume integral to a surface integral

5.6 The Divergence Theorem (Gauss' theorem)

The Gauss Divergence Theorem (often simply called the Divergence Theorem) is a fundamental result in vector calculus that relates the flux of a vector field through a closed surface to the divergence of the vector field inside the surface. It serves as a bridge between the behavior of a vector field on a boundary and its behavior inside a region. It simplifies many problems in physics, engineering, and geometry involving flux such as fluid flow, electromagnetism, conservation laws, etc.

Stokes' theorem relates an integral extended over a surface and a line integral taken over the one or more curves forming the boundary of this surface. The divergence theorem relates a triple integral extended over a solid and a surface integral taken over the boundary of this solid.

Theorem 5.6.1. *Let V be a solid in 3-space bounded by an orientable closed surface S, and let $\boldsymbol{n}$ be the unit outer normal to S. If $\boldsymbol{F}$ is a continuously differentiable vector field defined on V, we have*

$$\iiint_V (\operatorname{div} \boldsymbol{F}) dxdydz = \iint_S \boldsymbol{F} \cdot \boldsymbol{n} dS \tag{5.16}$$

Interpretation: The left-hand side of the theorem, $\displaystyle\iiint_V \nabla \cdot F dV$ represents the total divergence (net source or sink) of the vector field F inside the volume V. The right-hand side, $\displaystyle\iint_S F \cdot n dS$, represents the total flux of the vector field through the boundary surface S. The flux measures how much of the field F flows out of (or into) the volume V through S

. **Proof.** If we express $\boldsymbol{F}$ and $\boldsymbol{n}$ in terms of their components, say

$$\boldsymbol{F}(x, y, z) = P(x, y, z)\boldsymbol{i} + Q(x, y, z)\boldsymbol{j} + R(x, y, z)\boldsymbol{k}$$

and

$$\boldsymbol{n} = \cos\alpha\boldsymbol{i} + \cos\beta\boldsymbol{j} + \cos\gamma\boldsymbol{k}$$

then Equation (5.16) can be written as

$$\iiint_V \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dxdydz = \iint_S (P\cos\alpha + Q\cos\beta + R\cos\gamma)dS \tag{5.17}$$

It suffices to establish the three equations

$$\iiint_V \frac{\partial P}{\partial x} dxdydz = \iint_S P\cos\alpha dS$$
$$\iiint_V \frac{\partial Q}{\partial y} dxdydz = \iint_S Q\cos\beta dS$$
$$\iiint_V \frac{\partial R}{\partial z} dxdydz = \iint_S R\cos\gamma dS$$

and add the results to obtain (5.17). We will first prove the third formula for solids of a very special type.

Assume that

$$V = \{(x, y, z) : g(x, y) \leq z \leq f(x, y) \quad \text{for} \quad (x, y) \text{ in } T\}$$

where T is a connected region in the xy-plane, and f and g are continuous functions on T, with $g(x, y) \leq f(x, y)$ for each (x, y) in T.

Geometrically, this means that T is the projection of V on the xy-plane. Every line through T parallel to the z-axis intersects the solid V along a line segment connecting the surface $z = g(x, y)$ to the surface $z = f(x, y)$. The boundary surface S consists of

1. an upper cap S_1, given by the explicit formula $z = f(x, y)$;
2. a lower part S_2, given by $z = g(x, y)$; and
3. a portion S_3 of the cylinder generated by a line moving parallel to the z-axis along the boundary of T.

The outer normal to S has a nonnegative z-component on S_1, has a nonpositive component on S_2, and is parallel to the xy-plane on S_3. Solids of this type will be called "xy-projectable." They include all convex solids (for example, solid spheres, ellipsoids, cubes) and many solids that are not convex (for example, solid tori with axes parallel to the z-axis).

To prove the result, first we express the triple integral as a double integral extended over the projection T. Then we show that this double integral has the same value as the surface integral in question. We begin with the formula

$$\iiint_V \frac{\partial R}{\partial z} dxdydz = \iint_T \left[\int_{g(x,y)}^{f(x,y)} \frac{\partial R}{\partial z} dz \right] dxdy.$$

We use the second fundamental theorem of calculus to evaluate the one-dimensional integral with respect to z which gives

$$\iiint_V \frac{\partial R}{\partial z} dxdydz = \iint_T \{R[x,y,f(x,y)] - R[x,y,g(x,y)]\} dxdy. \tag{5.18}$$

For the surface integral we can write

$$\iint_S R\cos\gamma dS = \iint_{S_1} R\cos\gamma dS + \iint_{S_2} R\cos\gamma dS + \iint_{S_3} R\cos\gamma dS. \tag{5.19}$$

On S_3 the normal $\boldsymbol{n}$ is parallel to the xy-plane, so $\cos\gamma = 0$ and the integral over S_3 is zero. On the surface S_1 we use the representation

$$\boldsymbol{r}(x,y) = x\boldsymbol{i} + y\boldsymbol{j} + f(x,y)\boldsymbol{k}$$

and on S_2 we use the representation

$$\boldsymbol{r}(x,y) = x\boldsymbol{i} + y\boldsymbol{j} + g(x,y)\boldsymbol{k}.$$

On S_1, the normal $\boldsymbol{n}$ has the same direction as the vector product $\dfrac{\partial \boldsymbol{r}}{\partial x} \times \dfrac{\partial \boldsymbol{r}}{\partial y}$, so we can write

$$\iint_{S_1} R\cos\gamma dS = \iint_{S_1} R dx \wedge dy = \iint_T R[x,y,f(x,y)] dxdy$$

On S_2 the normal $\boldsymbol{n}$ has the direction opposite to that of $\partial \boldsymbol{r}/\partial x \times \partial \boldsymbol{r}/\partial y$ so, by Equation (12.26), we have

$$\iint_{S_2} R\cos\gamma dS = -\iint_{S_2} R dx \wedge dy = -\iint_T R[x,y,g(x,y)] dxdy$$

Therefore Equation (12.56) becomes

$$\iint_S R\cos\gamma dS = \iint_T \{R[x,y,f(x,y)] - R[x,y,g(x,y)]\} dxdy$$

Comparing this with Equation (12.55) we see that

$$\iiint_V \frac{\partial R}{\partial z} dxdydz = \iint_S R\cos\gamma dS$$

In the foregoing proof the assumption that V is xy-projectable enabled us to express the triple integral over V as a double integral over its projection T in the xy-plane. It is clear that if V is yz-projectable we can use the same type of argument to prove the identity

$$\iiint_V \frac{\partial P}{\partial x} dxdydz = \iint_S P\cos\alpha dS$$

and if V is xz-projectable we obtain

$$\iiint_V \frac{\partial Q}{\partial y} dxdydz = \iint_S Q\cos\beta dS$$

Thus we see that the divergence theorem is valid for all solids projectable on all three coordinate planes. In particular the theorem holds for every convex solid.

Check your progress

1. Gauss theorem provides a relationship between

   (A) an integral over a surface and a line integral over the boundary of the surface

   (B) a triple integral and a surface integral over the boundary of the solid

   (C) a double integral over a plane region R and a line integral over a closed curve forming the boundary of R.

   (D) None of the above.

Summary

In this chapter, we have studied three important and famous theorems in integration theory that relates line, surface and volume integrals, namely, Green's, Stoke's and Gauss divergence theorem.

Glossary

1. A parametric representation is a way of describing surfaces where we have three equations involving three variables $x,\ y$ and z in terms of two parameters u and

v:

$$x = X(u, v),\ y = Y(u, v),\ z = Z(u, v)$$

where the point (u, v) vary over some two-dimensional connected set T in the uv-plane and the corresponding points (x, y, z) trace out a surface in xyz-space.

2. If $\mathbf{r}$ is the radius vector from the origin to a point (x, y, z) of the surface, we can write the above parametric equations into one vector equation of the form:

$$r(u, v) = X(u, v)i + Y(u, v)j + Z(u, v)k, \qquad \text{where} \quad (u, v) \in T.$$

This is called a bf vector equation for the surface.

3. If the above parametric equations or the vector equation are assumed to be continuous on T, then the image of T under the mapping $\mathbf{r}$ is called a **parametric surface** and denoted as $\mathbf{r}(T)$.

4. If the function $\mathbf{r}$ is one-to-one on T, the image $\mathbf{r}(T)$ is called a **simple parametric surface**.

Self-Assessment Questions

1. State and prove Greens' theorem.

2. State and prove Stoke's theorem.

3. State and prove Gauss's theorem.

References

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Suggested Readings

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