PERIYAR UNIVERSITY

NAAC 'A++' Grade - State University - NIRF Rank 56–State Public University Rank 25

SALEM - 636 011, Tamil Nadu, India.

CENTRE FOR DISTANCE AND ONLINE EDUCATION

(CDOE)

M.Sc. MATHEMATICS

SEMESTER - II

ELECTIVE COURSE: MECHANICS

(Candidates admitted from 2024 onwards)

PERIYAR UNIVERSITY

CENTRE FOR DISTANCE AND ONLINE EDUCATION (CDOE)

M.Sc. Mathematics - 2024 admission onwards

ELECTIVE – IV

MECHANICS

Prepared by:

Centre for Distance and Online Education (CDOE)

Periyar University

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Contents

MECHANICS

OBJECTIVE: The objective of this course is to understand the Lagrangian and Hamiltonian equations for dynamical systems.

UNIT I: Mechanical Systems

The Mechanical system – Generalized coordinates – Constraints – Virtual work – Energy and Momentum.

UNIT II : Lagrange's Equations

Derivation of Lagrange's Equations – Examples – Integrals of the motion.

UNIT III: Hamilton's Equations

Hamilton's Principle – Hamilton's Equations – other variational principles.

UNIT IV: Hamilton – Jacobi Theory

Hamilton Principle Function – Hamilton-Jacobi Equation – Separability.

UNIT V: Canonical Transformation

Differential forms and Generating Functions – Special Transformations – Lagrange and Poisson Brackets.

TEXT BOOK:

D.T. Greenwood, "Classical Dynamics", Prentice Hall of India, New Delhi, 1985.

REFERENCES:

1. H. Goldstein, "Classical Mechanics", 2nd Edition, Narosa Publishing House, New Delhi.

2. R.D. Gregory, "Classical Mechanics", Cambridge University Press, 2006.

3. J.L.Synge and B.A.Griffth, "Principles of Mechanics", 3rd Edition, McGraw Hill Book Co., New York, 1970.

Unit 1

INTRODUCTORY CONCEPTS

Objectives

After the successful completion of this unit; the students are expected

- To recall the basic concepts of velocity, accelartion, linear momentum and force.
- To classify contact forces and body forces.
- To understand the fundamental concepts of constrains and constrained motion.
- To gain the knowledge about on Principle of virtual work and D'Alembert's principle.
- To analyse and work with problems related to Principle of virtual work.

1. Introduction

Dear students,

in the under graduate course we have studied statics and dynamics in the vector form.

In this post graduate course, let us introduce the classical dynamics.

For what reason do we learn classical mechanics?. First of all, we are living in a time of engineering, technological and scientific Era. So therefore, the knowledge about engineering is the most essential aspects of learning the mechanical systems.

Secondly, astronomical aspects of understanding the cellestial bodies such as planets, stars, galaxies and man made spacecraft like projectiles are all described by classical mechanics. Thirdly, a significant amount of mathematics was created to solve mechanical problems.

Mechanics: Mechanics is the science that deals with the action of forces on bodies.

Dynamics: Dynamics is the study of the motions of interacting bodies. It describes these motions interms of postulated laws. Motion means change of position of the moving particle. The motion of a particle is therefore the motion of a point in space.

Statics: Statics is the study of a particle acted by force and kept at rest in equilibrium.

Matter: Matter is any thing which occupies space and can be perceived by senses.

Body: A boby is a portion of matter limited in all directions, having a finite shape of size and occupying some definite space.

Particle: A particle is an idealized material body having its mass concentrated at a point. We shall assume that mass of each particle remains constant.

Rigid body: A rigid body is a system of particles, the distance between which remain unchanged. It may also be regarded as a continous distribution of matter.

Frame of reference: A frame of reference is a rigid body in which axises of coordinates are taken.

Newton's laws of motion: 1. Every particle continues to move in a state of uniform motion in a straight line or remains at rest, unless acted upon by an external force.

2. The time rate of change of linear momentum of a particle is proportional to the force acting on it and is in the direction of this force.

3. The forces of action and reaction between two interacting bodies are equal in magnitude and opposite in direction and are collinear.

1.1 The Mechanical System

A mechanical system consisting of N particle, where a particles is an idealized material body having its mass concentrated at a point. The motion of a particle is therefore the motion of a point in space. A point has no geometrical elements i.e, we cannot specify the orientation of the particle nor can be associate any particular rotational motion with it.

1.1.1 Equations of motion

The differential equation of a motion of a system of N particles can be obtained by applying Newton's laws of motion to the particle individually. For a single particle of mass m subjected to a force F we obtain from Newton's second law the vector equation

$$\vec{F} = m\vec{a}, \tag{1.1}$$

where 'm' is a mass and 'a' is acceleration due to the gravity,

$$\begin{aligned} \vec{F} &= m\left(\frac{d\vec{v}}{dt}\right) = \frac{d}{dt}(ma) = \frac{d}{dt}(\vec{P}) \\ \vec{F} &= \dot{\vec{P}}, \end{aligned} \tag{1.2}$$

where the linear momentum P is given by,

$$P = m\vec{v}, \tag{1.3}$$

and $\vec{a} = \frac{d\vec{v}}{dt} = \dot{\vec{v}}$ is the acceleration.

$$\begin{aligned} \vec{F} &= m\vec{a} = m\left(\frac{d\vec{v}}{dt}\right) \\ &= m\frac{d}{dt}\left(\frac{dr}{dt}\right) = m\frac{d^2r}{dt^2} \\ &= m\ddot{\vec{r}}. \end{aligned}$$

Thus the equation of motion is a differential equation of second order.
The equation of motion for the system of 'N' particles is given by,

$$\begin{aligned} \vec{F_i} &= m_i\ddot{\vec{r_i}} \\ i.e., \quad m_i\ddot{\vec{r_i}} &= \vec{F_i} + \vec{R_i} \qquad (i = 1, 2, ...N), \end{aligned} \tag{1.4}$$

where m_i is the mass of the i^{th} particle, $\vec{F_i}$ is the applied forces(sum of all other forces), $\vec{R_i}$ is the constraint force(that ensures the geometrical conditions). Thus we have broken the total force acting on the particle into two vector components $\vec{F_i}$ and $\vec{R_i}$.

Forces acting on the body

Forces that act on the body may be classified according to the mode of application as follows,

1. Contact forces (applied force $\vec{F}$) are transmitted to the body by a direct push or pull.

2. Body or field forces (constraint force $\vec{R}$) are associated with action at a distance and are represented by gravitational, electrical (or) other fields.

Note-1 Body forces are applied through the body, but contact forces are applied only at its boundary surface. The forces $\vec{R_i}$ associated with the geometrical constrains are always contact forces. However the applied force $\vec{F_i}$ may be either the body (or) contact type (or) the combination of forces.

Note-2 Instead of writing a single vector $\vec{r_i} = x_i\vec{i} + y_i\vec{j} + z_i\vec{k}$ for each particles, its more convenient to write three scalar equations. Using the cartesian co-ordinates (x_i, y_i, z_i) are represented the position of the i^{th} particle is in the form,

$$\begin{aligned} m_i\ddot{\vec{r}} &= \vec{F_i} + \vec{R_i}, \\ m_i\ddot{x}_i &= \vec{F_x} + \vec{R_x}, \\ m_i\ddot{y}_i &= \vec{F_{iy}} + \vec{R_{iy}}, \\ m_i\ddot{z}_i &= \vec{F_{iz}} + \vec{R_{iz}}, \quad (i = 1, 2..., N), \end{aligned} \tag{1.5}$$

where F_{ix} and r_{ix} are the x components of F_i and R_i, respectively, and where $F_{iy}, R_{iy}, F_{iz}, R_{iz}$ are defined similarly.

Dear students, in this subsection we are going to discuss about generalised co-ordinates and configuration space. First let us define the degrees of freedom.

1.1.2 Degrees of freedom

The number of degrees of freedom is equal to the number of co-ordinates minus the number of independent equation of constraints.

(ie.,) No. of degrees of freedom = No. of co-ordinates - No. of independent equation of constraints.

The degrees of freedom gives the minimum number of independent generalised coordinates required to describe the mechanical system completely.

Example: If a configuration of a system of N particles is described by using $3N$ cartesian co-ordinates and if there are l independent equation of constraints, then there are $(3N - l)$ degrees of freedom.

Problem: Consider the triangular body formed by rigid rods with particles attached at the corner. Find the degrees of freedom.

Solution: Degrees of freedom $= 3N - l = 9 - 3 = 6$. Now the system has 9 cartesian coordinates and 3 independent constraints.

1.1.3 Generalized co-ordinates

The wide variety of possible co-ordinate transformations, any set of parameters which gives an unambiguous representation of the configuration of the system serve as a system of co-ordinates in a more general sense. These parameters are known as generalized co-ordinates.

Co-ordinate transformation

The values of each set of co-ordinates are simply a group of numbers. The process of obtaining one set of numbers from the other is known as coordinate transformation.

Example: Consider the transformation equations relating the cartesian co-ordinates $x_1, x_2, ..., x_{3N}$ to the generalized co-ordinates $q_1, q_2, ..., q_n$ are given by

$$
\begin{aligned}
x_1 &= x_1(q_1, q_2, ..., q_n, t) \\
x_2 &= x_2(q_1, q_2, ..., q_n, t) \\
. \quad & . \quad . \\
. \quad & . \quad . \\
x_{3N} &= x_{3N}(q_1, q_2, ..., q_n, t).
\end{aligned} \tag{1.6}
$$

If the x has l equation of the constraints and the q's have m equation of constrains then equating the number of degrees of freedom we get

$$3N - l = n - m. \tag{1.7}$$

There should be always a one-to-one correspondence between points in the domain of x and the points in the domain of q at any time t. The necessary and sufficient condition that solve for q, a function x and t is called the Jacobian determinant transformation to be not equal to zero which is,

$$\frac{\partial(x_1, x_2, ..., x_{3N})}{\partial(q_1, q_2, ..., q_n)} \neq 0.$$

Problem: Find the transformation equations by considering a particle which is constrained to move in a fixed circular path of radius 'a'.

Solution: The equation of constraints is,

$$a = \sqrt{(x_1 - 0)^2 + (x_2 - 0)^2} = \sqrt{x_1^2 + x_2^2}.$$

Cartesian to generalized co-ordinate: Let the generalized cocordinates be q_1 and q_2, where q_1 denotes the polar angle, q_2 denotes the radius (constant). (ie.,) $q_2 = a$. The transformation equations are,

$$x_1 = q_2 \cos q_1,\ x_2 = q_2 \sin q_1.$$

Generalized to cartesian co-ordinate: The Jacobian for this transformation is,

$$\frac{\partial(x_1, x_2)}{\partial(q_1, q_2)} = \begin{vmatrix} \frac{\partial x_1}{\partial q_1} & \frac{\partial x_1}{\partial q_2} \\ \frac{\partial x_2}{\partial q_1} & \frac{\partial x_2}{\partial q_2} \end{vmatrix} = \begin{vmatrix} -q_2 \sin q_1 & \cos q_1 \\ q_2 \cos q_1 & \sin q_1 \end{vmatrix}$$
$$= q_2 \sin^2 q_1 - q_2 \cos^2 q_1 = -q_2 \neq 0.$$

Hence q's can be expressed in terms of x except when $q_2 = 0$.

The transformation equations are,

$$\tan q_1 = \frac{x_2}{x_1}$$
$$q_1 = \tan^{-1}\left((\frac{x_2}{x_1})\right), q_2 = \sqrt{x_1^2 + x_2^2} = a,$$

where $0 < q_1 < 2\pi, 0 < q_2 < \infty$. These transformation equation apply at all points on the finite x_1, x_2 plane except at the origin.

1.1.4 Configuration Space

The configuration system of N particles is specified by giving the values of $3N$ cartesian co-ordinates. If the system has l-independent equation of constraint, it is possible to find n independent generalized co-ordinates $q_1, q_2, ..., q_n$, where $n = 3N - l$. Here a set of n numbers namely the values of nq's are completely known, then we can specify the configuration of the system. It is convenient to think of the n numbers as the co-ordinates of a single point in an n-dimensional space which is known as the configuration space.

Let us sum up

1. We have derived equations of motion of a mechanical system consisting of N particles.
2. Introducing the concept of inertial frame.
3. Also we have studied applied force, contact force, constraint force, body force.
4. We have introduced the concept of degrees of freedom.
5. We have defined generalized co-ordinates.
6. Also discussed the configuration space.

Check your progress

1. What is a Particle?
2. State Newton's Second Law?
3. What is meant by constrain force?
4. Define Inertial frame?
5. What is applied force?
6. What is degrees of freedom?
7. Define Generalized Coordinates?
8. What is configuration space?

In the next section, we will discuss about the various constrains and constrain forces.

1.2 Constraint Force and Constrained Motion

Dear students, in this section we will introduce the constrained motion of the particle subject to Holonomic and Non-holonomic constrains. Also we will discuss Bilateral and Unilateral constrains, Sceleronomic and Rheonomic with illustrative examples. When a system of N particles have less than $3N$ degrees of freedom then there must be some constraints. Constraints are those equations which place geometrical restrictions upon the possible motion of the particle and rest in corresponding forces of constraint.

1.2.1 Holonomic constraints

Constraints of the form

$$\phi_j(q_1, q_2, ..., q_n, t) = 0, \ (j = 1, 2, ..., k) \tag{1.8}$$

called holonomic constraints. Where $q_1, q_2, ..., q_n$ are generalized co-ordinates that there are k independent equation of constrains, t denotes the time.

Holonomic System: A system whose constraints equation are all of the Holonomic constraints then the system is called Holonomic system.

Example: A particle constraints to move along any curve on a given surface is an example of Holonomic constraints. Consider the motion of two particles x, y plane are connected by a rigid rod of length l. The corresponding equation of constraint is,

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 - l^2 = 0.$$

Sceleronomic constraints: Constraint equations which has no time t explicitly then the constraints are known as sceleronomic constraints.

Sceleronomic system: A mechanical system is sceleronomic, if 1. None of the constraint equation contains time explicity. 2. The transformation equation must give the

x's as function of q's only,

$$x_1 = x_1(q_1, q_2, ..., q_n)$$
$$x_2 = x_2(q_1, q_2, ..., q_n)$$
$$. \quad . \quad .$$
$$. \quad . \quad .$$
$$x_{3N} = x_{3N}(q_1, q_2, ..., q_n).$$

Rhenomic constraint: Constraint equations which has time t explicitly. Then the constraints are known as Rhenomic constraints.

Example: Consider two particles connected by a rigid rod of length l. The length of the rod has been given as a explicit function of time, then the constraints equation are Rhenomic system.

1.2.2 Non-holonomic constraints

Dear students, in this subsection let us introduce motion of the dynamical system subject to non-holonomic constraints. Basically, non-holonomis constranits are expressed in the differential forms or inequalities as in the case of unilateral constraints (for eg. Motion of air molecules in a cubic container).

A system of m constraints which are written as non-integrable, differentiable expression of the form,

$$\sum_{i=1}^{n} a_{ji} dq_i + a_{jt} dt = 0, (j = 1, 2, ..., m), \tag{1.9}$$

where a is a function of q's and t's constraints of this type is called non holonomic constraints.

Example: Consider that the particles can slide on the horizontal xy plane without friction. The system is changed, however, by the addition of a nonholonomic constraint in the form of knife-edge supports at two particles. These supports move with the problem and are oriented perpendicular to the rod at either particle. Hence, the velocity of the center of the rod must be perpendicular to theory, resulting in the constraint equation

$$\dot{x} = -\dot{y}\tan\theta \quad or \quad \cos\theta dx + \sin\theta dy = 0. \tag{1.10}$$

1.2.3 Unilateral constraints

The constraints which can be written as a inequality of the form,

$$f(q_1, q_2, ..., q_n, t) \leq 0, \tag{1.11}$$

are called unilateral constraint.

Example: Suppose that a free particle is contained within a fixed hollow sphere of radius r which is centered at the origin. Then, using (x, y, z) as the generalized coordinates of the particle, the unilaterial constraint is given by

$$x^2 + y^2 + z^2 - r^2 \leq 0. \tag{1.12}$$

Let us sum up

1. We have studied the constrained motion of the particle under subject to various constraints

2. We have discussed following type of constraints namely Non-holonomic, Bilateral, Unilateral, Sceleronomic and Rheonomic with illustrative examples.

Check your progress

9. Define Holonomic constraints with an example.

10. What is Non-holonomic constraints.

11. Define Unilateral constraints.

1.3 D'Alembert's Principle of Virtual Work

Dear students, in this section we will discuss about virtual work and Principle of virtual work and D'Alembert's principle of virtual work. We also discuss the Langrage's modified D'Alembert's principle which will be used to derive Langrages's equation of motion.

The concept of virtual work is fundamental in the study of analytical mechanics.

1.3.1 Virtual displacement

Let us suppose that the configuration of a system of N particles can be given by $3N$ cartesian co-ordinates $x_1, x_2, ..., x_{3N}$, which are measured relative to an inertial frame and subject to constraints. Further let $\delta x_1, \delta x_2, ..., \delta x_{3N}$ denote the infinitesimal displacement which are virtual or imaginary. That is they are assumed to occur with out passage of time. This small change δx in the configuration of the system is called the virtual displacement.

Problem 1: Show that "a virtual displacement is not in general, a possible real displacement".

Solution: Cartesian co-ordinates: 1. Consider a system subjected to k holonomic constraints of the form,

$$\phi_j(x_1, x_2, ..., x_{3N}, t) = 0, \ (j = 1, 2, ..., k). \tag{1.13}$$

The total differentiation of ϕ_j is given by

$$\sum_{i=1}^{3N} \frac{\partial \phi_j}{\partial x_i} dx_i + \frac{\partial \phi_j}{\partial t} dt = 0. \tag{1.14}$$

A virtual displacement takes the form,

$$\sum_{i=1}^{3N} \frac{\partial \phi_j}{\partial x_i} \delta x_i = 0, \quad (j = 1, 2, ..., k), \tag{1.15}$$

(* the time is held fixed, dt is omitted).

2. Consider a system subjected to 'm' non-holonomic constraints of the form,

$$\sum_{i=1}^{3N} a_{ji} dx_i + a_{jt} dt = 0 \quad (j = 1, 2, ..., m). \tag{1.16}$$

Now the virtual displacement takes the form,

$$\sum_{i=1}^{3N} a_{ji} \delta x_i = 0, \tag{1.17}$$

(1.14) and (1.15) imply that the holonomic constraint must also be sceleronomic.

$$(ie.,)\frac{\partial \phi_j}{\partial t} = 0, \quad (j = 1, 2, ..., k). \tag{1.18}$$

Similarly (1.16) and (1.17) imply that

$$a_{jt} = 0. \tag{1.19}$$

Since (1.18) and (1.19) cannot happen for a real displacement it follows that virtual displacement is not in general, a possible real displacement.

1.3.2 Virtual Velocity

In general, a virtual displacement is not a possible real displacement. It is sometimes convenient to assume that a set of $\delta x'$s conforming to the instantaneous constraints occurs during an interval δt . The corresponding ratio of the form $\frac{\delta x}{\delta t}$ have the dimensions of velocity is called the virtual velocity. In general, virtual velocities are not possible velocities for the actual system. Infact, a virtual velocity of a moving particle is consistent with the constraints is also a possible velocity.

1.3.3 Virtual Work

Let us consider a system of N particles. Let $x_1, x_2, ..., x_{3N}$ be the cartesian co-ordinates. Let $F_1, F_2, ..., F_N$ be the forces applied to the corresponding co-ordinates. Let $\delta_1, \delta_2, ..., \delta_{3N}$ denotes the virtual displacements.

$$\delta W = \sum_{j=1}^{3N} F_j \delta x_j \quad (or) \quad \delta W = \sum_{i=1}^{3N} F_i \cdot \delta r_i,$$

here $\vec{r_i}$ is the position vector of the particle and $\vec{F_i}$ is the applied on the i^{th} particle.

Virtual work of constraint force: Let the total force acting on the i^{th} particle be separated into an applied force $(\vec{F_i})$ and a constraint force $(\vec{R_i})$. The virtual work due to the constraint force is given by,

$$\delta W_c = \sum_{i=1}^{N} \vec{R_i} \cdot \delta \vec{r_i}.$$

Workless constraint: A workless constraint is any bilateral constraint such that, the virtual work of the corresponding constraint force is zero for any virtual displacement which is consistent with the constraints. That is,

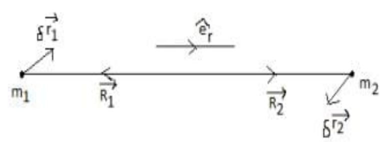
$$\delta W_c = 0 \quad (or) \quad \sum_{i=1}^{N} \vec{R_i} \cdot \delta\vec{r_i} = 0.$$

Examples of workless constraints: 1. Rigid inter connections between particles.

2. Sliding motion on a frictionless surface.

3. Rolling contact without slipping.

Rigid inter conections between particles: Let us consider two particles of mass m_1 and m_2 connected by a rigid mass less rod, $\vec{R_1} = -\vec{R_2}$. By Newtons 3rd law



$$\vec{R_1} = -\vec{R_2}\hat{e_r}, \tag{1.20}$$

where $\hat{e_r}$ is the unit vector directed along the rod. Since the rod is rigid, the displacement component of the particles in the direction of the rod must be equal

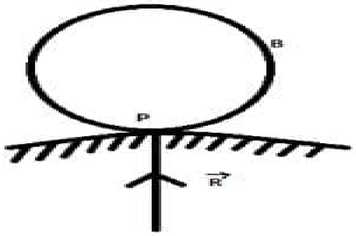
$$\hat{e_r}\delta\vec{r_1} = \hat{e_r}\delta\vec{r_2}. \tag{1.21}$$

Now the virtual work is given by,

$$\delta W_c = \sum_{i=1}^{2} \vec{R_i}\delta\vec{r_i} = \vec{R_1}\delta\vec{r_1} + \vec{R_2}\delta\vec{r_2} = -\vec{R_2}\hat{e_r}\delta\vec{r_1} + \vec{R_2}\hat{e_r}\delta\vec{r_2} = -\vec{R_2}\hat{e_r}\delta\vec{r_1} + \vec{R_2}\hat{e_r}\delta\vec{r_1} = 0.$$

Hence the rigid inter connection between particle is a work less constraints.

Sliding motion on a frictionless surface: Consider a body B, which slides without friction on the surface. The constraint force $(\vec{R})$ acts normal to the constant point P. Any virtual displacement of P involves sliding in the tangent plane at that point.



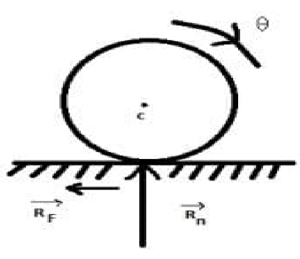
Hence no work is done by the constraint force.

Rolling contact with out slipping: Consider a vertical circular disc which roles with out slipping along a horizontal path. The total force acting on the disc can be separated in to a normal component R_n and the functional component R, which acts tangential to the surface. These force components pass through the instantaneous center c (point of contact). The center does not move as a result of a virtual displacement zero. The virtual velocity of c is 0. Hence the virtual work of the constraint force is zero.

1.3.4 Principle of virtual work

Dear students, in this subsection first let us state and prove the Principle of virtual work.

Theorem: The necessary and sufficient condition for the static equilibrium of an initially motionless scleronomic system which is subject to workless constraints is, that zero virtual work be done by the applied forces in moving through an arbitrary virtual displacement satisfying the constraints.



Proof: Consider a scleronomic system of N particles. If the system is in static equilibrium, then $\vec{F_i} + \vec{R_i} = 0$. Where $\vec{F_i}$ is the applied force, $\vec{R_i}$ is the constraint force acting on the i^{th} particle. The virtual work done by the forces in moving through an arbitrary virtual displacement is zero. $\delta W = 0$

$$\sum_{i=1}^{N} \vec{F_i}\delta\vec{r_i} + \sum_{i=1}^{N} \vec{R_i}\delta\vec{r_i} = 0. \tag{1.22}$$

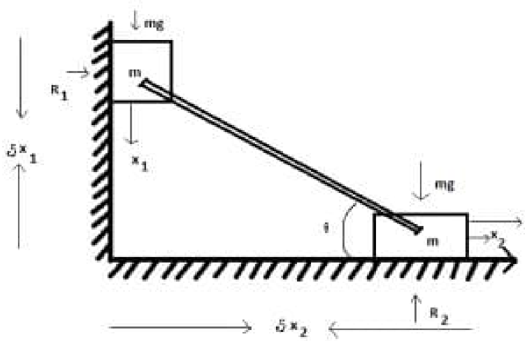
Since the co-system is subjected to work less constraints, the virtual work done by these constraints is zero. $\delta W = 0$

$$\sum_{i=1}^{N} \vec{R_i}\delta\vec{r_i} = 0. \tag{1.23}$$

Substitute (1.23) in (1.22), we get

$$\sum_{i=1}^{N} \vec{F_i}\delta\vec{r_i} = 0.$$

The virtual work done by the applied force is zero. Conversely, suppose that the same system of particles is initially motionless but not in equilibrium. Then one or more of the particles must have a net force applied to it and in accordance with the Newton's law of motion, the particles will move in the direction of force. Let us consider a virtual displacement $\delta\vec{r_i}$ in the direction of the actual motion of the particle.



Hence the virtual work is positive and is given by, $\delta W > 0$

$$\sum_{i=1}^{N} \vec{F_i}\delta\vec{r_i} + \sum_{i=1}^{N} \vec{R_i}\delta\vec{r_i} > 0. \tag{1.24}$$

Since the constraints are workless,

$$\sum_{i=1}^{N} \vec{R_i}\delta\vec{r_i} = 0. \tag{1.25}$$

Substitute (1.25) in (1.24),we get

$$\sum_{i=1}^{N} \vec{F_i}\delta\vec{r_i} > 0.$$

The reversal of δr's yields a negative virtual work of the system. Since the system is not in equilibrium it is always possible to find a set of virtual displacement with the constraints yielding non-zero virtual work.

Problem 1: Two frictionless blocks of equal mass 'm' are connected by a mass less rigid rod using x_1 and x_2 as co-ordinates, find F_2, if the system is in equilibrium.

Consider a sceleronomic system.

The constraints acting on the system are as follows,

1) External constraints forces due to the wall and floor called R_1 and R_2 respectively.

2) Internal constraint forces are the equal and opposite forces on the rod.

3) The applied forces are the gravitational forces acting on the blocks and external force F_2.

By principle of virtual work, the required condition for static equilibrium is work done by the applied force $= 0$.

$$mg\delta x_1 + F_2\delta x_2 = 0 \tag{1.26}$$

The displacement components along the rod must be equal at the two ends,

$$\delta x_1 \sin\theta - \delta x_2 \cos\theta = 0. \tag{1.27}$$

$(1.29) \times \sin\theta \Rightarrow F_2 \sin\theta \delta x_2 + mg\delta x_1 = 0$

$(1.30) \times mg \Rightarrow -mg\delta x_2 \cos\theta + mg\delta x_1 \sin\theta = 0.$

$(+) \qquad\qquad (-)$

$\overline{F_2 \sin\theta \delta x_2 + mg\delta x_2 \cos\theta = 0}$

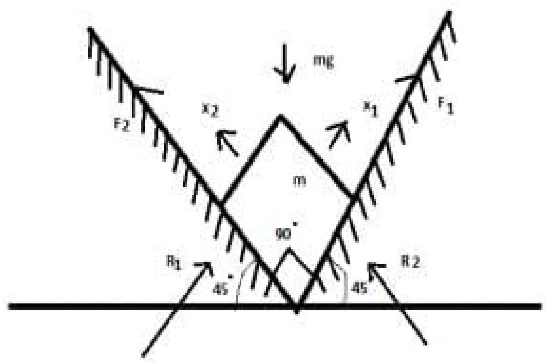
$$\begin{aligned}
F_2 \sin\theta\delta x_2 &= -mg\cos\theta\delta x_2 \\
F_2 &= -mg\frac{\cos\theta}{\sin\theta} \\
F_2 &= -mg\cot\theta.
\end{aligned}$$

This is the force to keep the system initially motionless in static equilibrium.

Note: The forces $\vec{R}_i$ associated with the geometrical constraints are always contact forces. How ever the applied force $\vec{F}_i$ may be of either the body or contact type or the combination of the forces.

Problem 2 : Using a suitable examples, show that the concept of virtual work can be applied to system of virtual unilateral constraints.

Solution: Let us consider a system, consisting of a cube of mass 'm' which is resting in static equilibrium at a corner formed by two mutually perpendicular frictionless



planes.

The unilateral constraint equation are, $x_1 \geq 0, x_2 \geq 0$. At equilibrium position, $x_1 = x_2 = 0$. The applied force on the system are due to gravity.

The components of these forces in the directions of x_1 and x_2 are given by,

$$F_1 = -mg\cos 45^\circ = \frac{-mg}{\sqrt{2}}.$$
$$F_2 = -mg\cos 45^\circ = \frac{-mg}{\sqrt{2}}.$$

The virtual work due to applied force is,

$$\begin{aligned}\delta W &= F_1\delta x_1 + F_2\delta x_2\\ &= \frac{-mg}{\sqrt{2}}\delta x_1 + \frac{-mg}{\sqrt{2}}\delta x_2\\ &= \frac{-mg}{\sqrt{2}}(\delta x_1 + \delta x_2)\\ &\leq 0.\end{aligned}$$

Thus the virtual work $\delta W \leq 0$, for any virtual displacement consistant with unilateral constraints. Virtual work of the constraint force is

$$\delta W_c = R_1\delta x_1 + R_2\delta x_2 \geq 0,$$

where R_1 and R_2 are assumed to be constant due to the virtual displacement.

Calculation of constraint forces R_1 and R_2 using the principle of virtual work

By the principle of virtual work, the total work done by all the forces is equal to zero $\delta W - 0$.

$$R_1\delta x_1 + F_1\delta x_1 + R_2\delta x_2 + F_2\delta x_2 = 0$$
$$(R_1 + F_1)\delta x_1 + (R_2 + F_2)\delta x_2 = 0$$
$$(R_1 - \frac{mg}{\sqrt{2}})\delta x_1 + (R_2 - \frac{mg}{\sqrt{2}})\delta x_2 = 0.$$

Here δx_1 and δx_2 are not constraint and therefore they are completely independent.

$$R_1 - \frac{mg}{\sqrt{2}} = 0, \qquad R_2 - \frac{mg}{\sqrt{2}} = 0$$
$$R_1 = \frac{mg}{\sqrt{2}}, \qquad R_2 = \frac{mg}{\sqrt{2}}.$$
$$\delta W_c = R_1\delta x_1 + R_2\delta x_2 = \frac{mg}{\sqrt{2}}\delta x_1 + \frac{mg}{\sqrt{2}}\delta x_2 = \frac{mg}{\sqrt{2}}(\delta x_1 + \delta x_1) \geq 0.$$

Hence there can be non zero virtual work by the constraint forces in an allowable virtual displacement.

1.3.5 D' Alembert's Principle

If the system is in motion, then $\sum_{i=1}^{N}(\vec{F_i}) - m_i\ddot{\vec{r_i}} = 0$. (or) The sum of all forces, real and inertial acting on each particle of a system is zero.

Proof: Let us consider a system of N particles. The equation of motion for each particle is given by,

$$\vec{F_i} + \vec{R_i} = m_i\ddot{\vec{r_i}}$$
$$\vec{F_i} + \vec{R_i} - m_i\ddot{\vec{r_i}} = 0, \quad (i = 1, 2..., N),$$

where $\vec{F_i}$ is the applied force, $\vec{R_i}$ is the constraint force, $-m_i\ddot{\vec{r_i}}$ is the inertial force, m_i is the mass and $\ddot{\vec{r_i}}$ is an acceleration relative to an inertial force.

If the system is in motion, then $\sum_{i=1}^{N}(\vec{F_i} + \vec{R_i} - m_i\ddot{\vec{r_i}}) = 0$. (or) The sum of all forces, real and inertial acting on each particle of a system is zero.

Lagrangian form of D'Alembert's Principle

Since the priciple of the virtual work applies to system in static equilibrium, let us use principle on the force system including the inertial forces. The total work done by all the forces in an arbtary virtual displacement is

$$\delta W = \sum_{i=1}^{N} (\vec{F_i} + \vec{R_i} - m_i \ddot{\vec{r_i}}) \cdot \delta r_i = 0, \quad (i = 1, 2..., N).$$
$$= \sum_{i=1}^{N} \vec{F_i} \cdot \delta r_i - \sum_{i=1}^{N} m_i \ddot{\vec{r_i}} \cdot \delta r_i + \sum_{i=1}^{N} \vec{R_i} \cdot \delta r_i = 0.$$

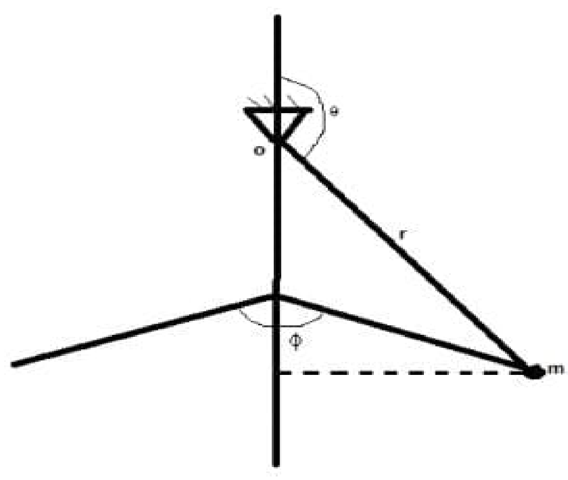
If we now assume that the $\vec{R_i}$ are workless constraints then the virtual work done $\delta W_c = 0$, that is $\sum_{i=1}^{N} \vec{R_i} \cdot \delta r_i = 0$ and if we choose the δr_i to be reversible virtual displacement consistent with the constraints we have

$$\sum_{i=1}^{N} (\vec{F_i} - m_i \ddot{\vec{r_i}}) \cdot \delta r_i = 0$$

This above equation is known as Langrange's form of D'Alembert's principle.

Example: Obtain the equation of motion of a spherical pendulum (or) a particle of mass 'm' is suspended by a mass less rod of length $r = a + b\cos\omega t$ $(a > b > 0)$ to form a spherical pendulum. Find the equation of motion.

Solution: Let us consider the spherical co-ordinates θ and ϕ, where θ is measure from



the upward vertical.

The angle ϕ is measure between a vertical reference plane passing through the support point o and the vertical plane containing the pendulum. The acceleration of a particle whose spherical co-ordinates (r, θ, ϕ) is as follows,

$$\ddot{\vec{r_i}} = (\ddot{\vec{r_i}} - r\dot{\theta}^2 - r\dot{\phi}^2 \sin^2\theta)\vec{e_r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2 \sin\theta\cos\theta)\vec{e_\theta}$$
$$+ (r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\theta}\sin\theta + 2r\dot{theta}\dot{\phi}\cos\theta)\vec{e_\phi}, \quad (1.28)$$

where $\vec{e_r}, \vec{e_\theta}$ and $\vec{e_\phi}$ are unit vectors forming an orthogonal triad. The virtual displacement is given by,

$$\delta\vec{r} = r\delta\theta\hat{e_\theta} + r\sin\theta\delta\phi\vec{e_\phi}. \quad (1.29)$$

The applied gravitational force is given by,

$$\vec{F} = -mg\cos\theta\hat{e_r} + mg\sin\theta\hat{e_\theta}. \quad (1.30)$$

Consider the Lagrange's form of D'Alembert's principle,

$$(\vec{F} - m_i\ddot{\vec{r_i}})\delta\vec{r_i} = 0$$

$$(-mg\cos\theta\hat{e_r} + mg\sin\theta\hat{e_\theta}) - m[(\ddot{\vec{r_i}} - r\dot{\theta}^2\sin^2\theta)\hat{e_r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\theta}^2\sin\theta\cos\theta)\hat{e_\theta}$$
$$+ (r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\theta} + 2r\dot{\theta}\dot{\phi}\cos\theta)\hat{e_\theta}]\}.(r\delta\theta\hat{e_\theta} + r\sin\theta\delta\phi\hat{e_\phi}) = 0$$

$$(mg\sin\theta - mr\ddot{\theta} - 2m\dot{r}\dot{\theta} + mr\dot{\phi}^2\sin\theta\cos\theta).r\delta\theta + (-mr\ddot{\phi}\sin\theta - 2m\dot{r}\dot{\phi}\cos\theta)r\sin\theta\delta\phi = 0$$

$$m(g\sin\theta - r\ddot{\theta} - 2\dot{r}\dot{\theta} + r\dot{\phi}^2\sin\theta\cos\theta).r\delta\theta - m(r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\theta}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta)r\sin\theta\delta\phi = 0$$

$$mr[(g\sin\theta - r\ddot{\theta} - 2\dot{r}\dot{\theta} + r\dot{\phi}^2\sin\theta\cos\theta).\delta\theta - (r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta)\sin\theta\delta\phi] = 0$$

$$[(g\sin\theta - r\ddot{\theta} - 2\dot{r}\dot{\theta} + r\dot{\phi}^2\sin\theta\cos\theta).\delta\theta - (r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta} + 2r\dot{\theta}\dot{\phi}\cos\theta)\sin\theta\delta\phi] = 0$$

$$g\sin\theta - r\ddot{\theta} - 2\dot{r}\dot{\theta} + r\dot{\phi}^2\sin\theta\cos\theta = 0 \quad (1.31)$$

$$r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta} + 2r\dot{\theta}\dot{\phi}\cos\theta = 0. \quad (1.32)$$

Given,

$$r = a + b\cos\omega t$$

$$r = -b\sin\omega t(\omega)$$

$$r = -b\sin\omega\omega t. \quad (1.33)$$

Substitute equation (1.33) in (1.31),

$$g\sin\theta - (a+b\cos\omega t)\ddot{\theta} - 2(-b\omega\sin\omega t)\dot{\theta} + (a+b\cos\omega t)\dot{\phi}^2\sin\theta\cos\theta = 0.$$

$$-(a+b\cos\omega t)\ddot{\theta} + 2b\omega\dot{\theta}\sin\omega t + (a+b\cos\omega t)\sin\theta\cos\theta\dot{\phi}^2 + g\sin\theta = 0.$$

$$(a+b\cos\omega t)\ddot{\theta} - 2b\omega\dot{\theta}\sin\omega t - (a+b\cos\omega t)\sin\theta\cos\theta\dot{\phi}^2 - g\sin\theta = 0.$$

Substitute equation (1.33) in (1.32),

$$(a+b\cos\omega t)\ddot{\phi}\sin\theta + 2(-b\omega\sin\omega t)\dot{\phi}\sin\theta + 2(a+b\cos\omega t)\dot{\theta}\dot{\phi}\cos\theta = 0.$$

1.3.6 Generalized force

Consider a system of N particles whose positions are specified by the cartesian co-ordinates $(x_1, x_2, ...x_{3N})$. Let $(F_1, F_2, ..., F_{3N})$ be the forces applied at the corresponding co-ordinates. The virtual work done by the forces in an arbitrary virtual displacement is given by

$$\delta W = \sum_{j=1}^{3N} F_j \delta x_j. \tag{1.34}$$

Suppose that the cartesian co-ordinates $x_1, x_2, ..., x_{3N}$ are related to the generalized co-ordinates by equation of the form,

$$x_j = x_j(q_1, q_2, ..., q_n, t).$$

By setting $\delta t = 0$ we get,

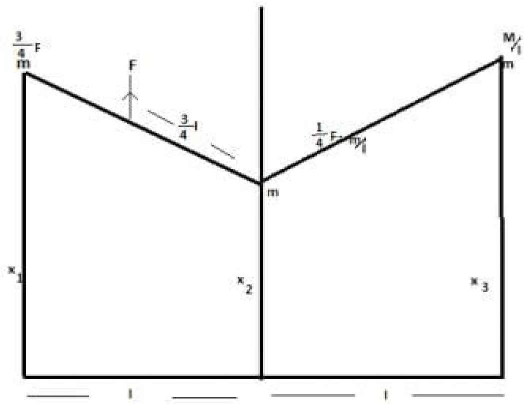
$$\delta x_j = \sum_{i=1}^{n} \frac{\partial x_j}{\partial q_i}\delta q_i, \qquad (j = 1, 2, ..., 3N). \tag{1.35}$$

Here $\frac{\partial x_j}{\partial q_i}$ are functions of q's and t's. Substitute (1.35) in (1.34),

$$\begin{aligned} \delta W &= \sum_{j=1}^{3N} F_j(\sum_{j=1}^{n} \frac{\partial x_j}{\partial q_i}\delta q_i) \\ &= \sum_{j=1}^{n}\sum_{j=1}^{3N} F_j \frac{\partial x_j}{\partial q_i}\delta q_i \\ \delta W &= \sum_{j=1}^{n} Q_i \delta q_i, \end{aligned}$$

where $Q_i = \sum_{j=1}^{3N} F_j \frac{\partial x_j}{\partial q_i}$ is the generalized force associated with the generalized co-ordinate q_i.

Example: The particle are connected by two rigid rods having joint between them to form the given system. The vertical force of a moment M are applied as shown.



The configuration of the system is given by the ordinary co-ordinates (x_1, x_2, x_3) or by the generalized co-ordinates (q_1, q_2, q_3). Where

$$x_1 = q_1 + q_2 + \frac{q_3}{2}.$$

$$x_2 = q_1 - q_2.$$

$$x_3 = q_1 - q_2 + \frac{q_3}{2}.$$

Find the generalized force Q_1, Q_2, Q_3 assuming small motions.

Solution:

$$x_1 = q_1 + q_2 + \frac{q_3}{2} \tag{1.36}$$

$$x_2 = q_1 - q_2 \tag{1.37}$$

$$x_3 = q_1 - q_2 + \frac{q_3}{2} \tag{1.38}$$

$$\frac{\partial(x_1, x_2, x_3)}{\partial(q_1, q_2, q_3)} = \begin{vmatrix} 1 & 1 & \frac{1}{2} \\ 1 & 0 & -1 \\ 1 & -1 & \frac{1}{2} \end{vmatrix} = -3 \neq 0.$$

$(1.36) + (1.37) + (1.38) \Rightarrow$

$$\begin{aligned} x_1 + x_2 + x_3 &= q_1 + q_2 + \frac{q_3}{2} + q_1 - q_2 + q_1 - q_2 + \frac{q_3}{2} \\ &= 3q_1 \\ q_1 &= \frac{1}{3}(x_1 + x_2 + x_3). \end{aligned} \tag{1.39}$$

$(1.36) - (1.38) \Rightarrow$

$$\begin{aligned} x_1 - x_3 &= q_1 + q_2 + \frac{q_3}{2} - q_1 + q_2 - \frac{q_3}{2} \\ &= 2q_2 \\ q_2 &= \frac{1}{2}(x_1 - x_3). \end{aligned} \tag{1.40}$$

Substitute (1.40) in (1.38), we get

$$\begin{aligned} x_2 &= q_1 - q_3 \\ &= \frac{1}{3}(x_1 + x_2 + x_3) - q_3 \\ q_3 &= \frac{1}{3}(x_1 + x_2 + x_3) - x_2 \\ &= \frac{x_1 + x_2 + x_3 - 3x_2}{3} \\ q_3 &= \frac{1}{3}(x_1 - x_2 + x_3). \end{aligned} \tag{1.41}$$

Thus for any set of values of x, we get the corresponding unique set of q's. The force $\vec{F}$ can be replaced by, $\frac{3\vec{F}}{4}$ at x_1, $\frac{\vec{F}}{4}$ at x_2. The moment $\vec{M}$ can be replaced by equal and opposite forces pf magnitude $\frac{M}{l}$ in the direction of x_3 and reversed direction of x_2. The forces acting at x_1, x_2, x_3 are $F_1 = \frac{3F}{4}, F_2 = \frac{F}{4} - \frac{M}{l}, F_3 = \frac{M}{l}$

$$\begin{aligned} \delta x_1 &= \delta q_1 + \delta q_2 + \frac{\delta q_3}{2}. \\ \delta x_2 &= \delta q_1 - \delta q_3. \\ \delta x_3 &= \delta q_1 - \delta q_2 + \frac{\delta q_3}{2}. \end{aligned}$$

$$\begin{aligned} \delta W &= F_1 \delta x_1 + F_2 \delta x_2 + F_3 \delta x_3 \\ &= (\frac{3}{4}F)(\delta q_1 + \delta q_2 + \frac{\delta q_3}{2}) + (\frac{1}{4}F - \frac{M}{l})(\delta q_1 - \delta q_3) + (\frac{M}{l})(\delta q_1 - \delta q_2 + \frac{\delta q_3}{2}) \end{aligned}$$

$$\delta W = \left(\frac{3}{4}F + \frac{1}{4}F - \frac{M}{l} + \frac{M}{l}\right)\delta q_1 + \left(\frac{3}{4}F - \frac{M}{l}\right)\delta q_2 + \left(\frac{3}{8}F - \frac{1}{4}F + \frac{M}{l} + \frac{M}{2l}\right)\delta q_3$$
$$= F\delta q_1 + \left(\frac{3}{4}F - \frac{M}{l}\right)\delta q_2 + \left(\frac{1}{8}F + \frac{3M}{2l}\right)\delta q_3. \quad (1.42)$$

In general,

$$\delta W = Q_1\delta q_1 + Q_2\delta q_2 + Q_3\delta q_3. \quad (1.43)$$

From (1.42) and (1.43)

$$Q_1 = \vec{F}.$$
$$Q_2 = \frac{3}{4}\vec{F} - \frac{\vec{M}}{l}.$$
$$Q_3 = \frac{1}{8}\vec{F} + \frac{3\vec{M}}{2l}.$$

Let us sum up

1. We have introduced the concept of virtual displacement and virtual velocity.
2. We have derived equation for D'Alembert's principle of virtual work.
3. Also we have discussed Lagrange's modified D'Alembert's principle.
4. We have introduced the generalized force Q_i associated with the generalized co-ordinate q_i.

Check your progress

12. Define Virtual displacement.
13. What is meant by Virtual work?
14. Workless constraint.
15. State Principle of Virutal work.
16. State D'Alembert Principle.

1.4 Energy, Linear Momentum and Angular Momentum

Dear students, in this section let us discuss basic concept of potential energy, work done, kinetic energy, conservation of energy and angular momentum.

1.4.1 Potential energy

Potential energy is the energy gained by the particle by virtue of its position and therefore potential energy V can be viewed as a analytic function in position variables $(x.y.z)$.

Let us consider a single particle whose position is given by cartesian co-ordinates (x, y, z). Suppose that the total force acting on the particle has components.

$$F_x = -\frac{\partial V}{\partial x}, \quad F_y = -\frac{\partial V}{\partial y}, \quad F_z = -\frac{\partial V}{\partial z}.$$

Where V is a potential energy function $V(x, y, z)$ is a single valued function of position only and not a function of velocity or time. A force $\vec{F}$ satisfying these conditions is called a conservative force.

Problem : The work done on the particle depends up on initial and final positions, but is independent of the specific path.

Proof: Let us consider the work dW done by the conservative force $\vec{F}$ as it moves through an infinitesimal displacement $d\vec{r}$. Then

$$\begin{aligned} dW &= \vec{F} \cdot d\vec{r} \\ &= F_x dx + F_y dy + F_z dz = -\left(\frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz\right) \\ dW &= -dV \qquad (1.44) \\ dV &= \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz. \end{aligned}$$

Thus dW is an exact differentiable. Now let us consider the work W done by the force $\vec{F}$ as the particle movesfrom a point A to B.

$$W = \int_A^B F \cdot dr. \qquad (1.45)$$

Substituting (1.45) in (1.44) we get,

$$W = \int_A^B (-dV) = -\int_A^B dV = -(V_B - V_A) = V_A - V_B.$$

But potential energy V is a function of position only and hence the workdone W is independent of the specific path. Further if A and B coincide, then the work done in moving around and closed path is zero.

$$W = \oint dW = \oint \vec{F}.d\vec{r} = V_A - V_B = 0.$$

1.4.2 Principle of work and kinetic energy

Dear student, in this section let us discuss the concept of principle of work and kinetic energy.

Theorem: The increase in the kinetic energy of a particle as it moves from one arbitrary point to another is equal to the work done by the forces acting on the particle during the given interval.

Proof: The kinetic energy (T) is given by,

$$T = \frac{1}{2}mv^2,$$

where m is the mass, v is the velocity of the particle. The work done by the particle due to the total force $\vec{F}$ as the particle moves from A to B is given by,

$$W = \int_A^B dW = \int_A^B \vec{F}d\vec{r} = \int_A^B mad\vec{r} = m\int_A^B \ddot{\vec{r}}d\vec{r}.$$

Multiply and divided by dt

$$W = m\int_A^B \ddot{\vec{r}}\left(\frac{d\vec{r}}{dt}\right)dt$$

$$W = m\int_A^B (\ddot{\vec{r}}\dot{\vec{r}})dt. \tag{1.46}$$

Now $\frac{d}{dt}(\dot{\vec{r}}.\dot{\vec{r}}) = (\ddot{\vec{r}}\dot{\vec{r}}) + (\dot{\vec{r}}\ddot{\vec{r}}) = 2(\ddot{\vec{r}}\dot{\vec{r}})$

$$\frac{1}{2}\frac{d}{dt}(\dot{\vec{r}}.\dot{\vec{r}}) = \ddot{\vec{r}}.\dot{\vec{r}}. \tag{1.47}$$

Substitute (1.47) in (1.46) we get,

$$W = m\int_A^B \frac{1}{2}\frac{d}{dt}(\dot{\vec{r}}.\dot{\vec{r}})dt = \frac{1}{2}m\int_A^B \frac{d}{dt}(\dot{\vec{r}}.\dot{\vec{r}})dt$$

$$W = \frac{1}{2}m\int_A^B d(\dot{\vec{r}})^2 = \frac{1}{2}m\int_A^B d(v)^2 = \frac{1}{2}m(v_B^2 - v_A^2) = \frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2 = T_B - T_A.$$

Hence increase in kinetic energy is equal to work done.

1.4.3 Conservation of energy

If the only forces acting on the given particle are conservative,

$$W = V_A - V_B. \tag{1.48}$$

$$W = T_B - T_A. \tag{1.49}$$

From (1.51) and (1.52)

$$V_A - V_B = T_B - T_A.$$

$$V_A + T_A = T_B + V_B = E.$$

Since the points A and B are arbitrary values, the total mechanical energy E remains constant. During the motion of the particle, this is the principle of conservation of energy.

Problem : Consider a system of N particles whose configuration is specified by the cartesian co-ordinates $x_1, x_2, ..., x_{3N}$. If the only forces which do work on the system during motion are given by $F_j = -\frac{\partial v}{\partial x_j}$, where the potential energy v of $x_1, x_2, ..., x_{3N}$ is a single valued function of position only then the total energy is conserved.

Solution: Let us consider the configuration of the system of N particles is specified by $3N$ Cartesian co-ordinates and generalized co-ordinates $q_1, q_2, ..., q_n$ and the x's and q's are related by $x_j = x_{j(q_1,q_2,...,q_n)}$. The generalized force Q_i is given by,

$$Q_i = \sum_{j=1}^{3N} F_j \frac{\partial x_j}{\partial q_i}.$$

The work done by the particle is given by,

$$\begin{aligned} W &= \int_A^B dW \\ &= \int_A^B \sum_{i=1}^{n} Q_i dq_i, \end{aligned} \tag{1.50}$$

where

$$Q_i = \sum_{j=1}^{3N} F_j \frac{\partial x_j}{\partial q_i} = \sum_{j=1}^{3N} \frac{\partial v}{\partial x_j} \frac{\partial x_j}{\partial q_i}$$

$$Q_i = -\frac{\partial v}{\partial q_i}. \tag{1.51}$$

Substitute (1.57) in (1.56),

$$W = \int_A^B \sum_{i=1}^{n} -\frac{\partial v}{\partial q_i}\partial q_i = -\sum_{i=1}^{n} \partial v = v_A - v_B.$$

Where A and B are the end points of the particle. Hence W is independent of the path and hence the total energy is conserved.

1.4.4 Equilibrium and stability

Consider a system of N particles whose applied force are conservative and are obtained from a potential energy function of the form $v(x_1, x_2, ..., x_{3N})$. Now from

$$\delta W = \sum_{j=1}^{3N} F_j \delta x_j.$$

$$F_j = -\frac{\partial v}{\partial x_j},$$

we get

$$\delta W = \sum_{j=1}^{3N} -\frac{\partial v}{\partial x_j}\delta x_j = -\partial v = -\delta v.$$

By the principle of virtual work the necessary and sufficient condition for the system to be in static equilibrium is that,

$$\delta W = 0$$

$$\delta v = 0.$$

If the potential energy is expressed in terms of the generalized co-ordinates, $q_1, q_2, ..., q_n$, then

$$\delta v = \sum_{i=1}^{n} -\frac{\partial v}{\partial q_i}\delta q_i.$$

Now $\partial v = 0, \Rightarrow \frac{\partial v}{\partial q_i} = 0, \quad (i = 1, 2, .., n)$.

Let v_0 be a reference value then by Taylor's series,

$$v = v_0 + \left(\frac{\partial v}{\partial q_1}\right)_0 \delta q_1 + \left(\frac{\partial v}{\partial q_2}\right)_0 \delta q_2 + ... + \frac{1}{2}\left(\frac{\partial^2 v}{\partial q_1^2}\right)_0 \delta q_1^2 + \frac{1}{2}\left(\frac{\partial^2 v}{\partial q_1 \partial q_2}\right)_0 \delta q_1 \delta q_2 + \frac{1}{2}\left(\frac{\partial^2 v}{\partial q_2^2}\right)_0 \delta q_2^2 + ... \tag{1.52}$$

Let us assume that,

$$\left(\frac{\partial v}{\partial q_i}\right)_0 = 0. \tag{1.53}$$

Hence,

$$\Delta v = v - v_0 = \left(\frac{\partial v}{\partial q_1}\right)_0 \delta q_1 + \left(\frac{\partial v}{\partial q_2}\right)_0 \delta q_2 + ... + \frac{1}{2}\left(\frac{\partial^2 v}{\partial q_1^2}\right)_0 \delta q_1^2 + \frac{1}{2}\left(\frac{\partial^2 v}{\partial q_1 \partial q_2}\right)_0 \delta q_1 \delta q_2 + \frac{1}{2}\left(\frac{\partial^2 v}{\partial q_2^2}\right)_0 \delta q_2^2 + ...$$

where Δv denotes the change in the potential energy,

1. If $\Delta v > 0$, $\forall$ possible virtual displacement having atleast one of the δq's are non-zero. Therefore, v_0 is the minimum potential energy corresponding to the stable equilibrium.

2. If $\Delta v < 0$, then the equilibrium position is unstable.

3. If $\Delta v \geq 0$, then the equilibrium is neutral stability.

Dear students, in the next subsection we state and prove Konig's theorem for total kinectic energy.

1.4.5 Konig's theorem

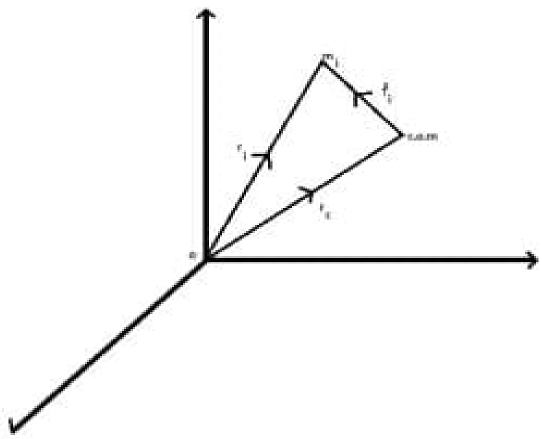
Theorem: The total kinetic energy of a system is equal to the sum of,

1. The kinetic energy due to a particle having a mass equal to the total mass of the system and moving with the velocity of the center of mass.

2. The kinetic energy due to the motion of the system relative to its center of mass

Proof: Consider a system of N particles. The total kinetic energy of the system is equal to the sum of individual kinetic energy of the particle,

$$T = \frac{1}{2}\sum_{i=1}^{N} m_i(\dot{\vec{r_i}})^2$$



$$\vec{r_i} = \vec{r_c} + \vec{\rho_i}$$

$$\dot{\vec{r_i}} = \dot{\vec{r_c}} + \dot{\vec{\rho_i}}.$$

Where $\vec{r_i}$ is the position vector of the i^{th} particle relative to the fixed point. $\vec{\rho_c}$ is the position vector of the i^{th} particle relative to the center of mass, $\vec{r_c}$ is the position vector of the center of mass relative to the fixed point o.

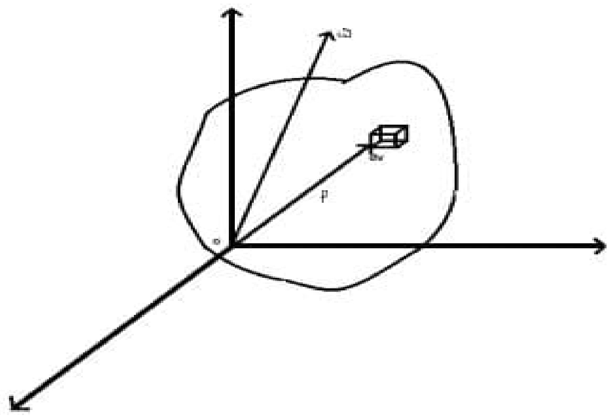
$$\begin{aligned} T &= \frac{1}{2}\sum_{i=1}^{N} m_i(\dot{\vec{r_c}} + \dot{\vec{\rho_i}})^2 = \frac{1}{2}\sum_{i=1}^{N} m_i(\dot{\vec{r_c}}^2 + 2\dot{\vec{r_c}} \cdot \dot{\vec{\rho_i}} + \dot{\vec{\rho_i}}^2) \\ &= \frac{1}{2}\dot{\vec{r_c}}^2 \sum_{i=1}^{N} m_i + \dot{\vec{r_c}} \cdot \sum_{i=1}^{N} m_i\dot{\vec{\rho_i}} + \frac{1}{2}\sum_{i=1}^{N} m_i\dot{\vec{\rho_i}}^2. \end{aligned}$$

Since, $\vec{\rho_i}$ is measured from the centre of mass, the linear momentum $\sum_{i=1}^{N} m_i\dot{\vec{\rho_i}} = 0$

$$T = \frac{1}{2}\dot{\vec{r_c}}^2 \sum_{i=1}^{N} m_i + \frac{1}{2}\sum_{i=1}^{N} m_i\dot{\vec{\rho_i}}^2.$$

Hence the proof.

2. Kinetic energy for a rotating rigid body in general motion. Let us consider a small volume element dV with density ρ. Each element of the body in general be translating and rotating.



The total kinetic energy is

$$T = \frac{1}{2}m\dot{\vec{r_c}}^2 + \frac{1}{2}\int_v \rho\dot{\vec{\rho}}^2 dV,$$

where $\vec{\rho}$ is the position of the volume element related to the mass center. (ie) The kinetic energy is the sum of the translating kinetic energy and rotational kinetic energy.

Rotational Kinetic Energy:

$$T_{rot} = \frac{1}{2}\int_v \rho\dot{\vec{\rho}}^2 dv.$$

Let o' be the center of mass and ω be the angular velocity then $\dot{\vec{\rho}} = \vec{\omega} \times \vec{\rho}$.

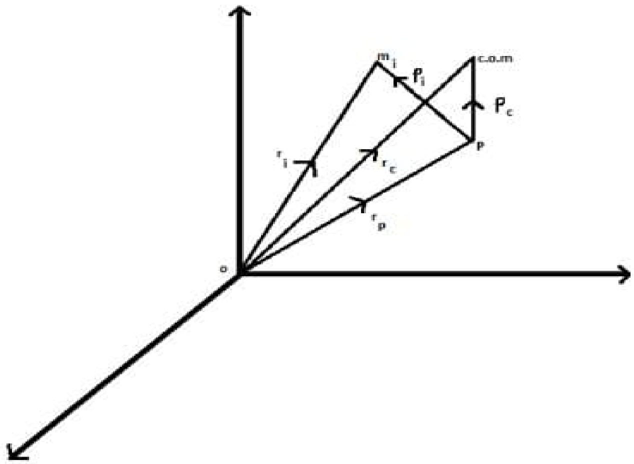
Consider

$$\dot{\vec{\rho}}^2 = \dot{\vec{\rho}}.\dot{\vec{\rho}} = \dot{\vec{\rho}}.(\vec{\omega} \times \vec{\rho}) = \vec{\omega}.(\vec{\rho} \times \dot{\vec{\rho}}) = \vec{\omega}.(\vec{\rho} \times (\vec{\omega} \times \dot{\vec{\rho}}))$$

$$\dot{\vec{\rho}}^2 = \vec{\omega}.[(\vec{\rho} \times \dot{\vec{\rho}})\vec{\omega} - (\vec{\rho}.\vec{\omega})\vec{\rho}]$$

$$\begin{aligned}\vec{\rho}^2\vec{\omega} - (\vec{\rho}\vec{\omega})\vec{\rho} &= (x^2+y^2+z^2)(\omega_x\vec{i} + \omega_y\vec{j} + \omega_z\vec{k}) - (x\omega_x + y\omega_y + z\omega_z)(x\vec{i} + y\vec{j} + z\vec{k}) \\ &= [(y^2+z^2)\omega_x - xy\omega_y - zx\omega_z]\vec{i} + [(x^2+z^2)\omega_y - xy\omega_x - zy\omega_z]\vec{j} \\ &\quad + [(x^2+z^2)\omega_z - xz\omega_x - zy\omega_y]\vec{k}.\end{aligned}$$

$$\begin{aligned}T_{rot} &= \frac{1}{2}\int_v \vec{\rho}\vec{\omega}[\vec{\rho}^2\vec{\omega} - (\vec{\rho}.\vec{\omega})\vec{\rho}]dv \\ T_{rot} &= \frac{1}{2}\int_v \vec{\rho}(\omega_x\vec{i} + \omega_y\vec{j} + \omega_z\vec{k})\{[(y^2+z^2)\omega_x - xy\omega_y - zx\omega_z]\vec{i} + \\ &\quad [(x^2+z^2)\omega_y - xy\omega_x - zy\omega_z]\vec{j} + [(x^2+z^2)\omega_z - xz\omega_x - zy\omega_y]\vec{k}\}dv \\ &= \frac{1}{2}I_{xx}\omega_x^2 + \frac{1}{2}I_{yy}\omega_y^2 + \frac{1}{2}I_{zz}\omega_z^2 + I_{xy}\omega_x\omega_y + I_{xz}\omega_x\omega_z + I_{yz}\omega_y\omega_z.\end{aligned}$$



Where

$$I_{xx} = \int_v \vec{\rho}(y^2 + z^2)dv.$$
$$I_{yy} = \int_v \vec{\rho}(x^2 + z^2)dv.$$
$$I_{zz} = \int_v \vec{\rho}(y^2 + x^2)dv.$$
$$I_{yx} = I_{xy} = -\int_v \vec{\rho}(xy)dv.$$
$$I_{zx} = I_{xz} = -\int_v \vec{\rho}(zx)dv.$$
$$I_{zy} = I_{yz} = -\int_v \vec{\rho}(yz)dv.$$

In the matrix form,

$$\vec{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \; and \quad I = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$

$$T_{rot} = \frac{1}{2}\omega^T I \omega.$$

1.4.6 Angular momemtum

Let us consider a system of N particles. The angular momentum about o is given by,

$$\vec{H} = \vec{r} \times m\vec{v}$$

$$\vec{H} = \vec{r} \times m\dot{\vec{r}}.$$

Where $\vec{r}$ is the position of the i^{th} particle with respect to the reference point o.

$$\vec{H} = \sum_{i=1}^{N} \vec{r_i} \times m_i \dot{\vec{r_i}}$$
$$\vec{r_i} = \vec{r_c} + \vec{\rho_i}$$
$$\dot{\vec{r_i}} = \dot{\vec{r_c}} + \dot{\vec{\rho_i}}$$
$$\vec{H} = \sum_{i=1}^{N} (\vec{r_c} + \vec{\rho_i}) m_i (\dot{\vec{r_c}} + \dot{\vec{\rho_i}})$$
$$= \sum_{i=1}^{N} \vec{r_c} m_i \dot{\vec{r_c}} + \sum_{i=1}^{N} \vec{r_c} m_i \dot{\vec{\rho_i}} + \sum_{i=1}^{N} \vec{\rho_i} m_i \dot{\vec{\rho_i}} + \sum_{i=1}^{N} \vec{\rho_i} m_i \dot{\vec{r_c}} \tag{1.54}$$
$$\sum_{i=1}^{N} \vec{\rho_i} m_i = 0$$
$$\sum_{i=1}^{N} \dot{\vec{\rho_i}} m_i = 0. \tag{1.55}$$

Substitute (1.55) in (1.54),

$$\vec{H} = \vec{r_c} \dot{\vec{r_c}} \sum_{i=1}^{N} m_i + \sum_{i=1}^{N} \vec{\rho_i} m_i \dot{\vec{\rho}}$$
$$\vec{H} = m \vec{r_c} \dot{\vec{r_c}} + \vec{H}$$

Where $\sum_{i=1}^{N} \vec{\rho_i} m_i \dot{\vec{\rho}}$ is the angular momentum of the center of mass.

1.4.7 Angular momentum of the rigid body

Let us consider a rigid body in an arbitrary motion w.k.t that,

$$\vec{H} = m \vec{r_c} \dot{\vec{r_c}} + \vec{H}_c.$$

Where $H_c = \sum_{i=1}^{N} \vec{\rho_i} m_i \dot{\vec{\rho}}$

$$\vec{H}_c = \int_v \rho (\vec{\rho} . \dot{\vec{\rho}}) dv$$
$$= \vec{\rho} \int_v (\vec{\rho} \times (\vec{\omega} \times \vec{\rho})) dv$$
$$= \rho \int_v \{[(y^2 + z^2)\omega_x - xy\omega_y - zx\omega_z]\vec{i} + [(x^2 + z^2)\omega_y - xy\omega_x - zy\omega_z]\vec{j}$$
$$+ [(x^2 + z^2)\omega_z - xz\omega_x - zy\omega_y]\vec{k}\} dv$$
$$\vec{H}_c = \rho \int_v (y^2 + z^2)\omega_x \vec{i} dv + \rho \int_v (x^2 + y^2)\omega_y \vec{j} dv + \rho \int_v (x^2 + y^2)\omega_z \vec{k} dv - \rho \int_v xy\omega_y \vec{i} dv$$
$$- \rho \int_v zx\omega_z \vec{i} dv - \rho \int_v xy\omega_x \vec{j} dv - \rho \int_v zy\omega_z \vec{j} dv - \rho \int_v zx\omega_x \vec{k} dv - \rho \int_v zy\omega_y \vec{k} dv$$

Where

$$I_{xx} = \int_v (y^2 + z^2) dv.$$
$$I_{yy} = \int_v (x^2 + z^2) dv.$$
$$I_{zz} = \int_v (y^2 + x^2) dv.$$
$$I_{yx} = I_{xy} = -\int_v \vec{\rho}(xy) dv.$$
$$I_{zx} = I_{xz} = -\int_v \vec{\rho}(zx) dv.$$
$$I_{zy} = I_{yz} = -\int_v \vec{\rho}(yz) dv.$$
$$\vec{H}_c = [I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z]\vec{i} + [I_{yy}\omega_y + I_{xy}\omega_x + I_{zy}\omega_z]\vec{j} + [I_{zz}\omega_z + I_{xz}\omega_x + I_{yz}\omega_y]\vec{k}.$$

In the matrix form,

$$\begin{aligned}\vec{H}_c &= I\vec{\omega} \\ &= \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}\end{aligned}$$

1.4.8 Angular momentum with respect to an arbitrary reference point

$$\vec{H} = \sum_{i=1}^{N} \vec{r_i} \times m_i \dot{\vec{r_i}}$$
$$\vec{r_i} = \vec{r_p} + \vec{\rho_i}$$
$$\vec{\rho_i} = \vec{r_i} - \vec{r_p} \tag{1.56}$$
$$\vec{r_c} = \vec{r_p} + \vec{\rho_c}$$
$$\vec{r_p} = \vec{r_c} - \vec{\rho_c}. \tag{1.57}$$

Substitute (1.56) in (1.57),

$$\vec{\rho_i} = \vec{r_i} - \vec{r_c} + \vec{\rho_c}$$
$$\dot{\vec{\rho_i}} = \dot{\vec{r_i}} - \vec{r_c} \dot{+} \vec{\rho_{ci}}$$
$$\vec{H}_p = \sum_{i=1}^{N} \vec{\rho_i} \times m_i \dot{\vec{\rho_i}}$$
$$= \sum_{i=1}^{N} (\vec{r_i} - \vec{r_c} + \vec{\rho_c}) \times m_i (\dot{\vec{r_i}} - \vec{r_c} \dot{+} \vec{\rho_{ci}}).$$
$$\dot{\vec{\rho_i}} = H - \dot{\vec{r_c}} \times m_i \vec{r_c} + \vec{\rho_c} \times m_i \dot{\vec{\rho_c}}.$$

1.4.9 Generalized momentum

Let us consider a system specified by n-generalized co-ordinate. Let the Lagrangian function $L(q, \dot{q}, t)$ is defined as, $L = T - V$.

The generalized momentum p_i assosicated with the generalized co-ordinates q_i is defined by the equation

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \quad (i - 1, 2.., n)$$
$$= \frac{\partial T}{\partial \dot{q}_i} - \frac{\partial V}{\partial \dot{q}_i}.$$

The potential energy is velocity independent,

$$\frac{\partial V}{\partial \dot{q}_i} = 0.$$

Hence $p_i = \frac{\partial T}{\partial \dot{q}_i}, \quad (i = 1, 2.., n)$.

Example:1 Consider a free particle of mass m whose position is given by the cartesian coordinates (x, y, z). The kinetic energy is

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m (\dot{x^2} + \dot{y^2} + \dot{z^2}),$$

we obtain,

$$p_x = \frac{\partial T}{\partial \dot{x}} = m\dot{x}.$$
$$p_y = \frac{\partial T}{\partial \dot{y}} = m\dot{y}.$$
$$p_z = \frac{\partial T}{\partial \dot{z}} = m\dot{z}.$$

Example:2 Consider a free particle of mass m whose position of the partial is given by the spherical co-ordinates (r, θ, ϕ). The kinetic energy is

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta^2} + r^2\dot{\phi^2}sin^2\theta),$$

we have

$$p_r = m\dot{r}.$$

$$p_\theta = mr^2\dot{\theta}.$$

$$p_\phi = mr^2\phi sin^2\theta.$$

Where p_r is the linear momentum component in the radial direction, p_θ is the horizontal momentum component of the angular momentum and p_ϕ is the vertical momentum component of the angular momentum.

Let us sum up

1. We have obtained relationship between potential energy and applied force.
2. We have discussed the principle of work and kinetic energy.
3. We have discussed the conservation of energy.
4. We have obtained the condition for equilibrium and stability.
5. We have proved the Konig's theorem for the total kinetic energy for a rotational body.
6. We have discussed the generalized momentum p_i associated with generalized coordinates q_i is discussed with illustrated example.

Check your progress

17. State rinciple of virtual work and kinetic energy.
18. What is conservation of energy.

Summary

- Derived equations of motion of a mechanical system consisting of N particles.
- Introducing the concept of inertial frame.

• Studied applied force, contact force, constraint force, body force.

• Introduced the concept of degrees of freedom.

• Define generalized co-ordinates.

• Discussed the configuration space.

• Studied the constrained motion of the particle under subject to various constraints

• Discussed following type of constraints namely non-holonomic, bilateral, unilateral, sceleronomic and Rheonomic with illustrative examples.

• Introduced the concept of virtual displacement and virtual velocity.

• Derived equation for D'Alembert's principle of virtual work.

• Discussed Lagrange's modified D'Alembert's principle.

• Introduced the generalized force Q_i associated with the generalized co-ordinate q_i.

• Obtained relationship between potential energy and applied force.

• Discussed the principle of work and kinetic energy.

• Discussed the conservation of energy.

• Obtained the condition for equilibrium and stability.

• Proved the Konig's theorem for the total kinetic energy for a rotational body.

• Discussed the generalized momentum p_i associated with generalized co-ordinates q_i is discussed with illustrated example.

Glossary

• **Holonomic System:** A system whose constraints equation are all of the holonomic constraints then the system is called holonomic system.

• **Sceleronomic system:** A mechanical system is sceleronomic, if 1. None of the constraint equation contains time explicity. 2. The transformation equation must give the x's as function of q's only,

$$x_1 = x_1(q_1, q_2, ..., q_n)$$
$$x_2 = x_2(q_1, q_2, ..., q_n)$$
$$. \quad . \quad .$$
$$. \quad . \quad .$$
$$x_{3N} = x_{3N}(q_1, q_2, ..., q_n).$$

• **Virtual work:** The concept of virtual work is fundamental in the study of analytical mechanics.

• **Workless constraint:** A workless constraint is any bilateral constraint such that, the virtual work of the corresponding constraint force is zero for any virtual displacement which is consistent with the constraints. That is,

$$\delta W_c = 0 \quad (or) \quad \sum_{i=1}^{N} \vec{R_i} \cdot \delta \vec{r_i} = 0.$$

Self-Assessment Questions

Short-Answer Questions:

1. State and prove D Alembert's Principle.

2. A Particle of mass M is suspended by a massless wire of length $r = a + b\cos\omega t; a > b > 0$ to form a spherical pendulum. Find the equation of motion.

3. State and explain the Konig's theorem for a rigid body and arbitrary points.

4. Define angular momentum of a system of particle.State and prove the principle of conservation of angular momentum.

5. Prove that with usual notation $T_{rot} = \frac{1}{2}\omega^T I \omega$.

6. Explain Holonomic constrains and give example.

Long-Answer Questions:

1. State and prove the principle of virtual work.

2. State and prove Konig's theorem.

3. Define D'Alembert's Principle and a Particle of mass M is suspended by a massless wire of length $r = a + b\cos\omega t; a > b > 0$ to form a spherical pendulum. Find the equation of motion.

4. Explain briefly Generalized Momentum.

5. Define the degree of freedom and briefly explain generalized coordinates.

6. Discuss Equilibrium and Stability.

7. Briefly explain Constrains.

8. Prove that the Total Kinetic energy $T = \frac{1}{2}m\dot{r}_p^2 + \frac{1}{2}\sum_{i=1}^{N} m_i\dot{\rho}_i^2 + \dot{r}_p m\dot{\rho}_c$.

9. Explain briefly Virtual Work.

10. State and prove Principle of conservation theorem.

11. Briefly explain Energy and Momentum.

12. With the usual notations find an expressions for the rotational kinetic energy of a rigid body.

Objectives

1. A body continuous in its state of rest or uniform motion, unless no external force is applied to it

(a) law of inertia (b) law of force (c) law of action and reaction

(d) none of the above

2. The number of degrees of freedom is equal to

(a) no. of coordinates - no. of equations (b) no. of equations - no. of coordinates

(c) no. of equations (d) no. of coordinates

3. A constraint which is expressed in the form of inequality is called

(a) Bilateral (b) Unilateral

(c) holonomic (d) Scleronomic

4. Generalized coordinates

(a) dependent on each other (b) are independent of each other

(c) are spherical coordinates (d) none of the above

5. Constraints that can be expressed as equations of coordinates and time, i.e., by an expression of the form $f(r_1, r_2, r_3.\ldots.t) = 0$, are said to be:

(a) Holonomic. (b) Nonholonomic.

(c) Scleronomous. (d) Rheonomic

6. If any of the constraint equations or the transformation equation contain time explicity

(a) Holonomic (b) Rheonomic

(c) Nonholonomic. (d) Scleronomous

7. Scleronomous constraints have:

(a) explicit time dependence. (b) no explicit time dependence.

(c) both explicit time dependence and no explicit time dependence. (d) neither explicit time dependence nor no explicit time dependence.

8. The small change δx in the configuration of the system is

(a) Virtual work (b) Principle of virtual work

(c) Virtual displacement (d) Virtual time

9. The principle of kinetic energy is

(a) $W = V_A - V_B$ (b)$W = V_B - V_A$

(c) $W = T_A - T_B$ (d) $W = T_B - T_A$

10. A constraint which is workless then

(a) $\sum R_i \delta_{ri}$ (b)$\sum R_j \delta_{xj}$

(c)$\sum F_i \delta_{ri}$ (d) $\sum F_k \delta_{xk}$

11.The second term of the equation is called $T = \frac{1}{2} m\dot{r_c}^2 + \frac{1}{2}\int\limits_{\nu} \rho\dot{\rho}d\nu$

(a) Translational kinetic energy (b) Translational potential energy

(c) Rotational kinetic energy (d) Rotational potential energy

12. Stable equilibrium is

(a) $\Delta V > 0$ (b) $\Delta V < 0$

(c) $\Delta V = 0$ (d) $\Delta V = 1$

13. Sometimes .... is consider as a form of instability.

(a) Stable equilibrium (b) Unstable

(c) Neutral stability (d) Constant

14.The existence of an inertial reference frame is a fundamental postulate of .... dynamics.

(a) Newtonian (b) Lagrangian

(c) Hamiltonian (d) Routhian

15. Equations of constraints that does not contain time as explicit variable are referred as

(a) Holonomic constraints (b) Rheonomic constraints

(c) Non- holonomic constraints (d) Scleronomic constraints

16. Non - inertial frame is

(a) non - accelerated frame of reference (b)accelerated frame of reference

(c) both (a) and (b) (d) none of the above

17. If the particles are connected by rigid rods to form a triangular body with the particles at its corners. The number of degree of freedom is ...

(a) 3 (b) 6

(c) 9 (d) 12

18. Unilateral constraints are not classed as workless constraints because allowed virtual displacement can be found in which the virtual work of constraint force is .....

(a) zero (b) one

(c) not zero (d) not one

19. Find the name of the equation $H = \sum_{i=1}^{N} r_i(m_i\dot{r}_i)$.

(a) Angular momentum (b) Generalized momentum

(c) Linear momentum (d) Non linear momentum

20. Any set of coordinates which can be express the configuration of the system is called

(a) Cartesian coordinates (b) Generalized coordinates

(c) Polar coordinates (d) Spherical coordinates

21. Non holonomic contraints which can be expressed in the form of

(a) $(x_1 - x_2)^2 + (y_1 - y_2)^2 - 1^2 = 0$ (b) $\pi_k(x_1, x_2, ..., x_n, t) = 0$

(c) $\pi_k(q_1, q_2, ..., q_n, t) = 0$ (d) $a_j idq_i + a_j idq_t = 0$

22. The process of obtaining one set of number from the other is called

(a) Jacobian transformation (b) Coordinate transformation

(c) Holonomic system (d) Non - holonomic system

23. A virtual displacement conforms to the instantaneous constraints ......

(a) any moving constraints are assumed to be stopped during the virtual displacement.

(b) any moving constraints are assumed to be stopped during the virtual work.

(c) any constraints are assumed to be stopped during the virtual displacement.

(d) any constraints are assumed to be stopped during the virtual work

24. D'Alembert's Principle is .....

(a) $F_i - R_i + m_i\ddot{r}_i = 0$ (b) $F_i - m_i\ddot{r}_i = 0$

(c)$F_i - R_i + m_i\ddot{r}_i \neq 0$ (d) $F_i + R_i - m_i\ddot{r}_i = 0$

25. Lagrangian form of D'Alembert's Principle is .....

(a) $F_i - R_i + m_i\ddot{r}_i = 0$ (b) $\sum_{i=1}^{N}(F_i - m_i\ddot{r}_i)\delta r_i = 0$

(c) $\sum_{i=1}^{N}(F_i - m_i\ddot{r}_i)\delta r_i \neq 0$ (d)$\sum_{i=1}^{N}(F_i + m_i\ddot{r}_i)\delta r_i = 0$

26. A particle is constrained to move along the inner surface of a fixed hemispherical bowl. The number of degrees of freedom of the particle is

(a) 1 (b) 2 (c) 3 (d) 4

27. The increase in the kinetic energy of a particle as it moves from one arbitary point to another is equal to the work done by the forces acting on the particle during the given interval. This statement is called ......

(a) Principle of virtual work and kinetic energy (b) Principle of work and potential energy

(c) Principle of work and kinetic energy (d) Virtual work and kinetic energy

28. The total work by all the force in an arbitrary virtual displacement is .....

(a) $\delta w = \sum_{i=1}^{N}(F_i - m_i\ddot{r}_i)\delta r_i = 0$ (b)$\delta w = \sum_{i=1}^{N}(F_i - m_i\ddot{r}_i)\delta r_i \neq 0$

(c) $\delta w = \sum_{i=1}^{N}(F_i - m_i\ddot{r}_i) = 0$ (d) $\delta w = \sum_{i=1}^{N}(F_i + m_i\ddot{r}_i) \neq 0$

29. In case of a rigid body having N particles, the number of degrees of freedom is

(a) N (b) 3N (c) 3 (d) ∞

30. Find the equation of the principle of conservation of energy .....

(a) $V_A T_A = V_B T_B = E$ (b) $V_A + T_A = V_B + T_B = E$

(c) $V_A + T_B = V_B + T_A = E$ (d) $V_A T_B = V_B T_A = E$

Answers for Check Your Progress

1. A particle is an idealized material body having its mass concentrated at a point. We shall assume that mass of each particle remains constant.

2. i. Every particle continues to move in a state of uniform motion in a straight line or remains at rest, unless acted upon by an external force.

ii. The time rate of change of linear momentum of a particle is proportional to the force acting on it and is in the direction of this force.

iii. The forces of action and reaction between two interacting bodies are equal in magnitude and opposite in direction and are collinear.

3. Body or Field forces (Constraint force $\vec{R}$) are associated with action at a distance and are represented by gravitational electrical (or) other fields.

4. A frame of reference is a rigid body in which axes of coordinates are taken.

5. Contact forces (Applied force $\vec{F}$) are transmitted to the body by an direct push or pull.

6. The number of degrees of freedom is equal to the number of co-ordinates minus the number of independent equation of constraints.

(ie.,) No. of degrees of freedom = No. of co-ordinates - No. of independent equation of constraints.

The degrees of freedom gives the minimum number of independent generalised coordinates required to describe the mechanical system completely.

7. The wide variety of possible coordinate transformations, any set of parameters which gives an unambiguous representation of the configuration of the system be serve as a system of co-ordinates in a more general sense. These parameters are known as

generalized co-ordinates.

8. The configuration system of N particles is specified by giving the values of $3N$ cartesian co-ordinates. If the system has l-independent equation of constraint, it is possible to find n independent generalized co-ordinates $q_1, q_2, ..., q_n$, where $n = 3N - l$. Here a set of n numbers namely the values of nq's are completely known, then we can specify the configuration of the system. It is convenient to think of the n numbers as the co-ordinates of a single point in an n-dimensional space is known as configuration space.

9. Constraints of the form

$$\phi_j(q_1, q_2, ..., q_n, t) = 0, (j = 1, 2, ..., k) \tag{1.58}$$

called holonomic constraints. Where $q_1, q_2, ..., q_n$ are generalized co-ordinates that there are k independent equation of constrains, t denotes the time.

Example: A particle constraints to move along any curve on a given surface is an example of Holonomic constraints. Consider the motion of two particles x, y plane are connected by a rigid rod of length l. The corresponding equation of constraint is,

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 - l^2 = 0.$$

10. A system of m constraints which are written as non-integrable, differentiable expression of the form,

$$\sum_{i=1}^{n} a_{ji} dq_i + a_{jt} dt = 0, (j = 1, 2, ..., m). \tag{1.59}$$

where a is a function of q's and t's constraints of this type is called non holonomic constraints.

11. The constraints which can be written as a inequality of the form,

$$f(q_1, q_2, ..., q_n, t) \leq 0 \tag{1.60}$$

are called unilateral constraint.

Example: Suppose that a free particle is contained within a fixed hollow sphere of radius r which is centered at the origin. Then, using (x, y, z) as the generalized coordinates of the particle, the unilaterial constraint is given by

$$x^2 + y^2 + z^2 - r^2 \leq 0. \tag{1.61}$$

12. The configuration of a system of N particles can be given by $3N$ cartesian coordinates $x_1, x_2, ..., x_{3N}$ which are measured relative to an inertial frame and subject to constraints. Further let $\delta x_1, \delta x_2, ..., \delta x_{3N}$ denote the infinitesimal displacement which are virtual or imaginary That is they are assumed to occur with out passage of time.This small change δx in the configuration of the system is called the virtual displacement.

13. The total force acting on the i^{th} particle be separated into an applied force (F_i) and a constraint force (R_i). The virtual work due to the constraint force is given by,

$$\delta W_c = \sum_{i=1}^{N} \vec{R_i} \cdot \delta \vec{r_i}.$$

14. A workless constraint is any bilateral constraint such that, the virtual work of the corresponding constraint force is zero for any virtual displacement which is consistent with the constraints. That is,

$$\delta W_c = 0 \quad (or) \quad \sum_{i=1}^{N} \vec{R_i} \cdot \delta \vec{r_i} = 0.$$

15. The necessary and sufficient condition for the static equilibrium of an initially motionless sceleronomic system which is subject to workless constraints is, that zero virtual work be done by the applied forces in moving through an arbitrary virtual displacement satisfying the constraints.

16. The sum of all forces, real and inertial acting on each particle of a system is zero.

17. The increase in the kinetic energy of a particle as it moves from one arbitrary point to another is equal to the work done by the forces acting on the particle during the given interval.

18. If the only forces acting on the given particle are conservative, $W = V_A - V_B$, $W = T_B - T_A$

$$V_A - V_B = T_B - T_A, \quad V_A + T_A = T_B + V_B = E.$$

Since the points A and B are arbitrary values, the total mechanical energy E remains constant. During the motion of the particle, this is the principle of conservation of energy.

Suggested Readings

• Greenwood. T. Donald, 1979, New Delhi: Classical Dynamics, Prentice Hall of Indian Private Limited.

• Goldstein, Herbert. 2011, New Delhi: Pearson Education India Classical Mechanics, 3rd Edition.

• Rao. Sankara. K. 2009. New Delhi: Classical Mechanics. PHI Learning Private Limited.

• Upadhyaya. J.C. 2010. New Delhi: Classical Mechanics, 2nd Edition. Himalaya Publishing House.

• Gupta. S. L. 1970. New Delhi: Classical Mechanics. Meenakshi Prakashan.

Unit 2

LAGRANGE'S EQUATIONS

Objectives

When this units are successfully finished, the students are expected

• To derive the Lagrange's equations of motion subject to holonomic and non-holonomic system.

• To obtain the differential equations of motion for spherical and double pendulum by Lagrange's method.

• To discuss the Kepler's problem by using Routhian function methods.

• To obtain the Jacobi integral or energy integral for conservative, natural and Liouville's system with illustrative examples.

2. Introduction

Dear students, in this unit we define the Lagranian for a holonomic systems with applied forces derivable from ordinary (or) generalized potenital and workless constraints. In the Lagragian formulation we are eliminating the forces of constrins form the equation of motion and acheiving this goal we have obtain many other benifits. The derivative of Lagragian equation has started from a consideration of instantaneous state of system and small virtual displacement about the instantaneous state leading to a differential principle such as D'Alembert's principle. We will study the differential type of kinetic energy T_2, T_1, T_0. Derivative of the Lagrange's equation subject to holonomic and non-holonomic system by using Lagrange's multiply method. We derived the Lagrange's equations of motion for a spherical pendulum length 'l' also we have discussed the equation of motion for a double pendulum. By using Lagrange's method we have discussed the forces of interaction between the two locks under the influence of gravity, assuming that all surfaces are frictionless. We introduced the ignorable coordinates we derive the equations of motion by using Routhian function. We also discuss the application of Routhian procedure in Kepler's problem. Jocobi integral are obtain for conservative, natural and Liouville's system.

2.1 Derivation of Lagrange's Equations of Motion

Dear students, in this section first let us derive the standard forms of Lagrange's equations for a holonomic system and Non-holonomic system.

2.1.1 Expression of kinetic energy interms of generalized co-ordinates

Let us consider a system of N particles whose positions relative to an inertial reference frame are given by the cartisian co-ordinates $x_1, x_2, ..., x_{3N}$. The total kinetic is given by,

$$T = \frac{1}{2}\sum_{k=1}^{3N} m_k \dot{x}_k^2. \tag{2.1}$$

Let $q_1, q_2, ...; q_n, t$ be the generalized co-ordinates.

Consider the transformation equation,

$$\begin{aligned} x_k &= x_k(q,t) = x_k(q_1, q_2, ..., q_n, t) \\ \dot{x_k} &= \sum_{i=1}^{n} \frac{\partial x_k}{\partial q_i}.\dot{q}_i + \frac{\partial x_k}{\partial t}. \end{aligned} \tag{2.2}$$

Substitute (2.2) in (2.1),

$$T = \frac{1}{2}\sum_{k=1}^{3N} m_k \left(\sum_{i=1}^{n} \frac{\partial x_k}{\partial q_i}.\dot{q}_i + \frac{\partial x_k}{\partial t}\right)\left(\sum_{j=1}^{n} \frac{\partial x_k}{\partial q_j}.\dot{q}_j + \frac{\partial x_k}{\partial t}\right)$$
$$T = \frac{1}{2}\sum_{k=1}^{3N} m_k \left(\sum_{i=1}^{n}\sum_{j=1}^{m} \frac{\partial x_k}{\partial q_i}\frac{\partial x_k}{\partial q_j}.\dot{q}_i\dot{q}_j + \sum_{i=1}^{n} \frac{\partial x_k}{\partial q_i}\frac{\partial x_k}{\partial t}.\dot{q}_i + \sum_{j=1}^{n} \frac{\partial x_k}{\partial q_j}\frac{\partial x_k}{\partial t}.\dot{q}_j + \frac{\partial^2 x_k}{\partial t^2}\right)$$
$$T = \frac{1}{2}\sum_{k=1}^{3N} m_k \left(\sum_{i=1}^{n}\sum_{j=1}^{m} \frac{\partial x_k}{\partial q_i}\frac{\partial x_k}{\partial q_j}.\dot{q}_i\dot{q}_j + 2\sum_{i=1}^{n} \frac{\partial x_k}{\partial q_i}\frac{\partial x_k}{\partial t}.\dot{q}_i + \frac{\partial^2 x_k}{\partial t^2}\right)$$
$$T = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \dot{q}_i\dot{q}_j + \sum_{i=1}^{n} a_i\dot{q}_i + \frac{1}{2}\sum_{k=1}^{3N} m_k\frac{\partial^2 x_k}{\partial t^2},$$

where

$$T_2 = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \dot{q}_i\dot{q}_j, \quad T_1 = \sum_{i=1}^{n} a_i\dot{q}_i, \quad T_0 = \frac{1}{2}\sum_{k=1}^{3N} m_k\frac{\partial^2 x_k}{\partial t^2}$$

$$m_{ij} = \sum_{k=1}^{3N} m_k\frac{\partial x_k}{\partial q_i}\frac{\partial x_k}{\partial q_j}, \quad a_i = \sum_{k=1}^{3N} m_k\frac{\partial x_k}{\partial q_i}\frac{\partial x_k}{\partial t}.$$

T_2- quadratic function of q's, T_1- linear function of $\dot{q}$'s

T_0- remaining terms as a function of q's and t,

m_{ij} and a_i- function of q's and t.

$T = T_2 + T_1 + T_0$.

Special case:

1. In equation (2.1)

(a) If all $m_k > 0$, then T is positive definite quadratic function of $\dot{x}$'s.

(b) If all $\dot{x}$'s are zero, then T is zero (c) If any $\dot{x}$'s are non -zero, then T is positive.

2. If T is expressed as a function of q's, $\dot{q}$'s and t

(a) $T = 0,$ if the system is motionless.

(b) $T > 0,$ for a moving system.

3. Now consider T

T_2 is the total kinetic energy if all $\frac{\partial x_k}{\partial t} = 0$. (ie.,) For a system in which any moving constraints are held fixed.

The positive definite nature of T_2 restricts the positive values of the inertia coefficients m_{ij}. The necessary and sufficient condition for T_2 to be positive is that, $m_{11} > 0$,

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} > 0, ..., \begin{bmatrix} m_{11} & m_{12}....m_{1n} \\ m_{21} & m_{22}....m2n \\ . & \\ . & \\ . & \\ mn1 & mn2....m_{nn} \end{bmatrix} > 0.$$

This $n \times n$ matrix is called generalized inertia matrix. Where $m_{ij} = \sum_{k=1}^{3N} m_k \frac{\partial x_k}{\partial q_i} \frac{\partial x_k}{\partial q_j}$. For sceleronomic system, T is a quadratic functions of q's. In this case, $T_1 = T_0 = 0$ and hence $T = T_2 + T_1 + T_0$, therefore $T = T_2$.

2.1.2 Lagrange's equation for the Holonomic system

Let us consider a system of N particles. By D'Alembert's principle, we have

$$\sum_{k=1}^{3N} (\vec{F}_k + \vec{R}_k - m_k \ddot{x}_k) \delta x_k = 0. \tag{2.3}$$

If we consider the workless constraints, then

$$\sum_{k=1}^{3N} \vec{R}_k \delta x_k = 0. \tag{2.4}$$

Substitute (2.4) in (2.3),

$$\sum_{k=1}^{3N} (\vec{F}_k - m_k \ddot{x}_k) \delta x_k = 0. \tag{2.5}$$

Where $\vec{F}_k$ is the applied force. Now $x_k = x_k(q_1, q_2, ..., q_n, t)$.

Then $\delta x_k = \sum_{i=1}^{n} \frac{\partial x_k}{\partial q_i} \delta q_i + \frac{\partial x_k}{\partial t} \delta t$.

Assume that $\delta t = 0$

$$\delta x_k = \sum_{i=1}^{n} \frac{\partial x_k}{\partial q_i} \delta q_i \tag{2.6}$$

Substitute (2.6) in (2.5)

$$\sum_{k=1}^{3N} \sum_{i=1}^{n} (\vec{F}_k - m_k \ddot{x}_k) \frac{\partial x_k}{\partial q_i} \delta q_i = \sum_{k=1}^{3N} \sum_{i=1}^{n} (\vec{F}_k \frac{\partial x_k}{\partial q_i} - m_k \frac{\partial x_k}{\partial q_i} \ddot{x}_k) \delta q_i \tag{2.7}$$

$$x_k = x_k(q_1, q_2, ..., q_n, t), \quad \dot{x_k} = \frac{\partial x_k}{\partial q_i}\dot{q}_i + \frac{\partial x_k}{\partial t}$$

$$\sum_{i=1}^{n} \frac{\partial \dot{x}_k}{\partial \dot{q}_i} = \sum_{i=1}^{n} \frac{\partial x_k}{\partial q_i}, \quad i = 1, 2, ..., n. \tag{2.8}$$

Now

$$\begin{aligned} \frac{d}{dt}\frac{\partial x_k}{\partial q_i} &= \sum_{j=1}^{n} \frac{\partial}{\partial q_j}\left(\frac{\partial x_k}{\partial q_i}\right).\dot{q}_i + \frac{\partial}{\partial t}\left(\frac{\partial x_k}{\partial q_i}\right) \\ &= \sum_{j=1}^{n} \left(\frac{\partial^2 x_k}{\partial q_j \partial q_i}\right).\dot{q}_i + \frac{\partial^2 x_k}{\partial t \partial q_i} \end{aligned} \tag{2.9}$$

Also

$$\begin{aligned} \dot{x_k}(q, \dot{q}, t) &= \sum_{i=1}^{n} \left(\frac{\partial x_k}{\partial q_i}\right).\dot{q}_i + \frac{\partial x_k}{\partial t} \\ \frac{\partial \dot{x_k}}{\partial q_i} &= \sum_{i=1}^{n} \frac{\partial}{\partial q_i}\left(\frac{\partial x_k}{\partial q_i}\right).\dot{q}_i + \frac{\partial}{\partial q_i}\left(\frac{\partial x_k}{\partial t}\right) \\ &= \sum_{i=1}^{n} \left(\frac{\partial^2 x_k}{\partial q_i \partial q_i}\right).\dot{q}_i + \left(\frac{\partial^2 x_k}{\partial t \partial q_i}\right). \end{aligned} \tag{2.10}$$

From (2.9), (2.10),

$$\frac{d}{dt}\frac{\partial x_k}{\partial q_i} = \frac{\partial \dot{x_k}}{\partial q_i}. \tag{2.11}$$

The generalized momentum is,

$$\begin{aligned} p_i &= \frac{\partial T}{\partial \dot{q}_i}, \quad T = \frac{1}{2}\sum_{k=1}^{3N} m_k \dot{x}_k^2 \\ \frac{\partial T}{\partial \dot{q}_i} &= \frac{1}{2}\sum_{k=1}^{3N} m_k 2\dot{x_k}\frac{\partial \dot{x_k}}{\partial q_i} = \sum_{k=1}^{3N} m_k \dot{x_k}\frac{\partial x_k}{\partial q_i} \\ \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} &= \sum_{k=1}^{3N} m_k \ddot{x_k}\left(\frac{\partial x_k}{\partial q_i}\right) + \sum_{k=1}^{3N} m_k \dot{x_k}\frac{d}{dt}\left(\frac{\partial x_k}{\partial q_i}\right) \\ &= \sum_{k=1}^{3N} m_k \ddot{x_k}\left(\frac{\partial x_k}{\partial q_i}\right) + \sum_{k=1}^{3N} m_k \dot{x_k}\frac{d}{dt}\left(\frac{\partial \dot{x_k}}{\partial q_i}\right) \\ \frac{\partial T}{\partial q_i} &= \frac{1}{2}\sum_{k=1}^{3N} m_k 2\dot{x_k}\frac{d}{dt}\left(\frac{\partial \dot{x_k}}{\partial q_i}\right). \end{aligned}$$

Consider,

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = \sum_{k=1}^{3N} m_k \ddot{x_k}\left(\frac{\partial x_k}{\partial q_i}\right). \tag{2.12}$$

The generalized force,

$$Q_i = \sum_{k=1}^{3N} F_k \frac{\partial x_k}{\partial q_i}. \tag{2.13}$$

Substitute (2.12)and (2.13) in (2.7)

$$\sum_{i=1}^{n} \left[Q_i - \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} \right] \delta q_i = 0$$

$$Q_i = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} + \frac{\partial T}{\partial q_i}, \quad i = 1, 2, .., n. \tag{2.14}$$

These n equations are known as Lagrange's Equation.

2.1.3 Standard form of Lagrange's equation

Assume that all the generalized force is obtained from the potential function $V(q, t)$ such that,

$$Q_i = -\frac{\partial V}{\partial q_i}. \tag{2.15}$$

Substitute (2.14) in (2.15),

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} + \frac{\partial T}{\partial q_i} = -\frac{\partial V}{\partial q_i}$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \left(\frac{\partial T}{\partial q_i} - \frac{\partial V}{\partial q_i} \right) = 0. \tag{2.16}$$

Now consider the Lagrange's function,

$$L = T - V, \quad \frac{\partial L}{\partial q_i} = \frac{\partial T}{\partial q_i} - \frac{\partial V}{\partial q_i}, \quad \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i} - \frac{\partial V}{\partial \dot{q}_i}$$

$$\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i}. \tag{2.17}$$

Substitute (2.16) and (2.17) in (2.14)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0.$$

Another form of Lagrange's equation

If Q_i are not wholly obtained from the potential energy function then,

$$Q_i = -\frac{\partial V}{\partial q_i} + Q_i'.$$

Then the equation becomes,

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} &= -\frac{\partial V}{\partial q_i} + Q_i' \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \left(\frac{\partial T}{\partial q_i} - \frac{\partial V}{\partial q_i}\right) &= Q_i' \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} &= Q_i'.
\end{aligned}$$

Form of equation of motion

We know that,

$$\begin{aligned}
&T = T_2 + T_1 + T_0 \\
&T_2 = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij}\dot{q}_i\dot{q}_j, \quad T_1 = \sum_{i=1}^{n} a_i\dot{q}_i, \quad T_0 = \frac{1}{2}\sum_{k=1}^{3N} m_k \frac{\partial^2 x_k}{\partial t^2} \\
&m_{ij} = \sum_{k=1}^{3N} m_k \frac{\partial x_k}{\partial q_i}\frac{\partial x_k}{\partial q_j}, \quad a_i == \sum_{k=1}^{3N} m_k \frac{\partial x_k}{\partial q_i}\frac{\partial x_k}{\partial t}.
\end{aligned}$$

Consider the Lagrange's equation,

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} &= 0 \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \left(\frac{\partial T}{\partial q_i} - \frac{\partial V}{\partial q_i}\right) &= 0.
\end{aligned} \tag{2.18}$$

Consider,

$$\begin{aligned}
&\frac{\partial T}{\partial \dot{q}_i} = \frac{\partial T_2}{\partial \dot{q}_i} + \frac{\partial T_1}{\partial \dot{q}_i} + \frac{\partial T_0}{\partial \dot{q}_i} \\
&\frac{\partial T_2}{\partial \dot{q}_i} = \sum_{j=1}^{n} m_{ij}\dot{q}_j, \quad \frac{\partial T_1}{\partial \dot{q}_i} = a_i, \quad \frac{\partial T_0}{\partial \dot{q}_i} = 0, \\
&p_i = \frac{\partial T}{\partial \dot{q}_i} = \sum_{j=1}^{n} m_{ij}\dot{q}_j + a_i.
\end{aligned}$$

Where p_i is a generalized momentum

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) = \sum_{j=1}^{n} m_{ij}\ddot{q}_j + \sum_{j=1}^{n} \dot{m}_{ij}\dot{q}_j + \dot{a}_i. \tag{2.19}$$

$$m_{ij} = m_{ij}(q_1, q_2, ..., q_n, t)$$

$$\dot{m}_{ij} = \sum_{l=1}^{n} \frac{\partial m_{ij}}{\partial q_l}\dot{q}_l + \frac{\partial m_{ij}}{\partial t} \tag{2.20}$$

$$a_i = a_i(q_1, q_2, .., q_n, t)$$

$$\dot{a}_i = \sum_{j=1}^{n} \frac{\partial a_i}{\partial q_j}\dot{q}_j + \frac{\partial a_i}{\partial t}. \tag{2.21}$$

Substitute (2.20) and (2.21) in (2.19)

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) = \sum_{j=1}^{n} m_{ij}\ddot{q}_j + \sum_{j=1}^{n}\left(\sum_{l=1}^{n} \frac{\partial m_{ij}}{\partial q_l}\dot{q}_l + \frac{\partial m_{ij}}{\partial t}\right)\dot{q}_j + \sum_{j=1}^{n} \frac{\partial a_i}{\partial q_j}\dot{q}_j + \frac{\partial a_i}{\partial t} \tag{2.22}$$

$$\frac{\partial T}{\partial q_i} = \frac{\partial T_2}{\partial q_i} + \frac{\partial T_1}{\partial q_i} + \frac{\partial T_0}{\partial q_i}$$

$$\frac{\partial T_2}{\partial q_i} = \sum_{j=1}^{n}\sum_{l=1}^{n} \frac{\partial m_{ij}}{\partial q_i}\dot{q}_l\dot{q}_j, \quad \frac{\partial T_1}{\partial q_i} = \sum_{j=1}^{n} \frac{\partial a_i}{\partial q_j}\dot{q}_j$$

$$\frac{\partial T}{\partial q_i} = \frac{1}{2}\sum_{j=1}^{n}\sum_{l=1}^{n} \frac{\partial m_{ij}}{\partial q_i}\dot{q}_j\dot{q}_l + \sum_{j=1}^{n} \frac{\partial a_i}{\partial q_j}\dot{q}_j + \frac{\partial T_0}{\partial \dot{q}_i}. \tag{2.23}$$

Substitute (2.22) and (2.23) in (2.18)

$$\sum_{l=1}^{n} m_{ij}\ddot{q}_j + \frac{1}{2}\sum_{j=1}^{n}\sum_{l=1}^{n}\left(\frac{\partial m_{ij}}{\partial q_l} + \frac{\partial m_{il}}{\partial q_j} + \frac{\partial m_{lj}}{\partial q_i}\right)\dot{q}_j\dot{q}_l$$
$$+ \sum_{j=1}^{n} \frac{\partial m_{il}}{\partial t}\dot{q}_j + \sum_{j=1}^{n}\left(\frac{\partial a_i}{\partial q_j} - \frac{\partial a_j}{\partial q_i}\right)\dot{q}_j + \frac{\partial a_i}{\partial t} - \frac{\partial T_0}{\partial q_i} + \frac{\partial T}{\partial q_i} = 0. \tag{2.24}$$

By introducing Christoffel symbol of first kind $[jl, i] = \frac{\partial m_{ilj}}{\partial q_l} + \frac{\partial m_{il}}{\partial q_j} - \frac{\partial m_{il}}{\partial q_i}$. Further let, $\gamma_{ij} = -\gamma_{ij} = \frac{\partial a_i}{\partial q_j} - \frac{\partial a_j}{\partial q_i}$, where γ_{ij} is an element of a skew symmetric matrix then equation (2.24) becomes,

$$\sum_{l=1}^{n} m_{ij}\ddot{q}_j + \frac{1}{2}\sum_{j=1}^{n}\sum_{l=1}^{n}[jl, i]\dot{q}_j\dot{q}_l + \sum_{j=1}^{n} \frac{\partial m_{il}}{\partial t}\dot{q}_j + \sum_{j=1}^{n} \gamma_{ij}\dot{q}_j + \frac{\partial a_i}{\partial t} - \frac{\partial T_0}{\partial q_i} + \frac{\partial T}{\partial q_i} = 0 \tag{2.25}$$

These n equations are called the equations of motion. The resulting equations of motion is given by,

$$\ddot{q}_i + f_i(q, \dot{q}, t) = 0, \quad i = 1, 2, ..., n.$$

2.1.4 Lagrangian's equation for non-holonomic system

Constraints equation for non-holonomic system is given by,

$$\sum_{i=1}^{n} a_{ji}dq_i + a_{jt}dt = 0, \quad j = 1, 2, ..., n. \tag{2.26}$$

The δq's must satisfying the equation

$$\sum_{i=1}^{n} a_{ji}\delta q_i + a_{jt}\delta t = 0.$$

Let $\delta \to 0$.

$$\sum_{i=1}^{n} a_{ji}\delta q_i = 0. \tag{2.27}$$

If the constraints are workless then

$$\sum_{i=1}^{n} c_i\delta q_i = 0. \tag{2.28}$$

Multiply eqn(2.27) by λ_j known as Lagrangian multiplier, we get

$$\lambda_j \sum_{i=1}^{n} a_{ji}\delta q_i = 0, \quad j = 1, 2..., m.$$

Interchanging the order of summation,

$$\sum_{i=1}^{m}\sum_{j=1}^{n} \lambda_j a_{ji}\delta q_i = 0. \tag{2.29}$$

(2.28)-(2.29)

$$\sum_{i=1}^{n} c_i\delta q_i - \sum_{i=1}^{m}\sum_{j=1}^{n} \lambda_j a_{ji}\delta q_i = 0$$
$$\sum_{i=1}^{n} \left(c_i - \sum_{j=1}^{m} \lambda_j a_{ji}\right)\delta q_i = 0, \quad c_i = \sum_{j=1}^{m} \lambda_j a_{ji}.$$

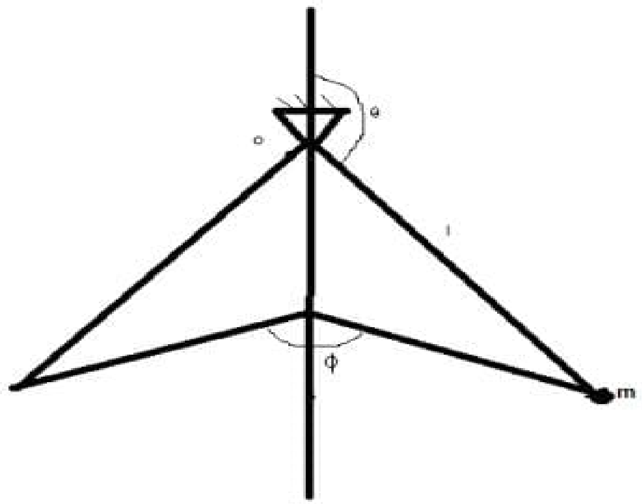
Then the coefficient of δq's are zero. Equating the generalized force Q_i's with c_i.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i' = \sum_{j=1}^{m} \lambda_j a_{ji}.$$

This is the standard form of Lagrange's equations for non holonomic system.

Problem 1 :

Find the differential equations of motion for a spherical pendulum of length 'l'.



Solution: The spherical co-ordinates are given by,

$$x = l \sin\theta \cos\phi, \quad y = l \sin\theta \sin\phi, \quad z = l \cos\theta.$$

Consider,

$$T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2). \tag{2.30}$$

$$\dot{x} = -l \sin\theta \sin\phi\dot{\phi} + l \cos\phi \cos\theta\dot{\theta}, \quad \dot{y} = l \sin\theta \cos\phi\dot{\phi} + l \sin\phi \cos\theta\dot{\theta}, \quad \dot{z} = -l \sin\theta\dot{\theta}$$

$$\begin{aligned}\dot{x}^2 + \dot{y}^2 + \dot{z}^2 &= (-l \sin\theta \sin\phi\dot{\phi} + l \cos\phi \cos\theta\dot{\theta})^2 + (l \sin\theta \cos\phi\dot{\phi} + l \sin\phi \cos\theta\dot{\theta})^2 + (-l \sin\theta\dot{\theta})^2 \\ &= l^2(\dot{\theta}^2 \sin^2\theta\dot{\phi}^2). \end{aligned} \tag{2.31}$$

Potential energy:

$$\begin{aligned} V &= mgh = -mgl \cos(\pi - \theta) = -mgl(-\cos\theta) \\ &= mgl \cos\theta. \end{aligned} \tag{2.32}$$

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2} ml^2(\dot{\theta}^2 \sin^2\theta\dot{\phi}^2) - mgl \cos\theta. \end{aligned} \tag{2.33}$$

Lagrange's equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0$$

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2}m2\dot{\theta}l^2 = ml^2\dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = ml^2\ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} = ml^2\dot{\phi}^2 sin\theta cos\theta + mglsin\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$ml^2\ddot{\theta} - ml^2\dot{\phi}^2 \sin\theta\cos\theta + mgl\sin\theta = 0 \tag{2.34}$$

$$\frac{\partial L}{\partial \dot{\phi}} = \frac{1}{2}m2\dot{\theta}l^2\sin^2\theta(2\dot{\phi})$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = ml^2 sin^2\theta\ddot{\phi} + 2ml^2\dot{\phi}sin\theta\cos\theta\dot{\phi}$$

$$\frac{\partial L}{\partial \phi} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0$$

$$ml^2\sin^2\theta\ddot{\phi} + 2ml^2\dot{\phi}\sin\theta\cos\theta\dot{\phi} = 0. \tag{2.35}$$

Equations (2.34) and (2.35) are required differential equation of motion.

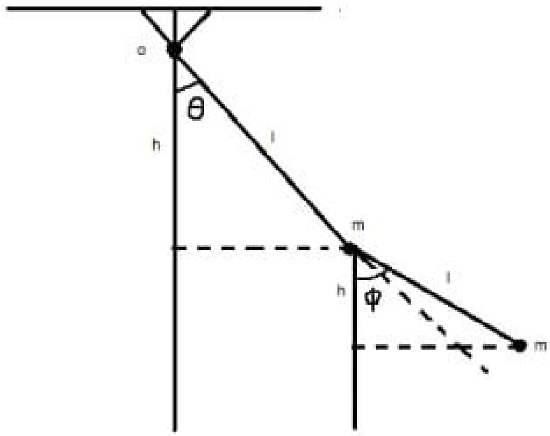
Problem 2

A double pendulum consists of two particles suspended by massless rods. Assuming that all motion takes place in a vertical plane, find the differential equation of motion. Linearize these equations assuming small motions.

Solution:kinetic energy:

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(v_1^2 + v_2^2).$$

The total velocity of the lower particle=The total velocity of the upper particle+The total velocity of the lower particle with respect to an upper particle.



$$v_1 = l\dot{\theta} \quad v_1^2 = l^2\dot{\theta}^2.$$

$$v_2 = l(\dot{\theta} + \dot{\phi})$$

$$v_2^2 = l^2\dot{\theta}^2 + l^2\dot{\phi}^2 + 2(l\dot{\theta}.l\dot{\phi}) = l^2\dot{\theta}^2 + l^2\dot{\phi}^2 + 2l^2\dot{\theta}l\dot{\phi}cos(\phi - \theta).$$

$$T = \frac{1}{2}m(l^2\dot{\theta}^2 + l^2\dot{\phi}^2 + 2l^2\dot{\theta}l\dot{\phi}\cos(\phi - \theta)).$$

Potential energy:

$$V = mgh = V_1 + V_2$$

$$= -mgl\cos\theta - mgl\cos\theta - mgl\cos\phi = -mgl(2\cos\theta + \cos\phi).$$

Consider the lagrangian function

$$L = T - V$$

$$L = \frac{1}{2}ml^2(\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}\cos(\phi - \theta)) + mgl(2\cos\theta + \cos\phi)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{\partial L}{\partial \dot{\theta}} = 2ml^2\dot{\theta} + ml^2\dot{\phi}cos(\phi - \theta).$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 2ml^2\ddot{\theta} - ml^2\dot{\phi}^2\sin(\phi - \theta) + ml^2\dot{\theta}\dot{\phi}\sin(\phi - \theta) + ml^2\cos(\phi - \theta)\ddot{\phi}$$

$$\frac{\partial L}{\partial \theta} = \frac{1}{2}ml^2 \times 2\dot{\theta}\dot{\phi}\sin(\phi - \theta) - 2mgl\sin\theta$$

$$2ml^2\ddot{\theta} - ml^2\dot{\phi}^2\sin(\phi-\theta) + ml^2\dot{\theta}\dot{\phi}\sin(\phi-\theta) + ml^2\cos(\phi-\theta)\ddot{\phi}$$
$$- ml^2\dot{\theta}\dot{\phi}\sin(\phi-\theta) + 2mgl\sin\theta = 0$$

$$2ml^2\ddot{\theta} - ml^2\dot{\phi}^2\sin(\phi-\theta) + ml^2\cos(\phi-\theta)\ddot{\phi} + 2mgl\sin\theta = 0. \quad (2.36)$$

$$\frac{\partial L}{\partial\dot{\phi}} = ml^2\dot{\phi} + ml^2\dot{\theta}\cos(\phi-\theta)$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\phi}}\right) = ml^2\ddot{\phi} - ml^2\dot{\phi}\dot{\theta}\sin(\phi-\theta) + ml^2\dot{\theta}^2\sin(\phi-\theta) + ml^2\ddot{\theta}\cos(\phi-\theta)$$
$$\frac{\partial L}{\partial\phi} = -ml^2\dot{\phi}\dot{\theta}\sin(\phi-\theta) - mgl\sin\phi$$

$$2ml^2\ddot{\theta} - ml^2\dot{\phi}^2\sin(\phi-\theta) + ml^2\cos(\phi-\theta)\ddot{\phi} + 2mgl\sin\theta = 0. \quad (2.37)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\phi}}\right) - \frac{\partial L}{\partial\phi} = 0$$
$$ml^2\ddot{\phi} - ml^2\dot{\phi}\dot{\theta}\sin(\phi-\theta) + ml^2\dot{\theta}^2\sin(\phi-\theta)$$
$$+ ml^2\ddot{\theta}\cos(\phi-\theta) + ml^2\dot{\phi}\dot{\theta}\sin(\phi-\theta) + mgl\sin\phi = 0$$

$$ml^2\ddot{\phi} + ml^2\dot{\theta}^2\sin(\phi-\theta) + ml^2\ddot{\theta}\cos(\phi-\theta + mgl\sin\phi = 0. \quad (2.38)$$

Equations (2.37) and (2.38) are the required differential equation of motion Linearizing the differential equations,

$$cos(\phi-\theta) \cong 1.$$
$$sin(\phi-\theta) \cong \theta.$$
$$sin\theta \cong \theta.$$
$$\cos\theta \cong 1.$$

Substitute these values in eqn's(2.37) and (2.38) and neglecting the higher powers, we get, from (2.37),

$$2ml^2\ddot{\theta} - ml^2\dot{\phi}^2(\phi-\theta) + ml^2\ddot{\phi} + 2mgl\theta = 0,$$
$$ml^2[\ddot{\theta} - \dot{\phi}^2(\phi-\theta) + \ddot{\phi}] + 2mgl\theta = 0,$$
$$ml^2[2\ddot{\theta} + \ddot{\phi}] + 2mgl\theta = 0. \quad (2.39)$$

From (2.38),

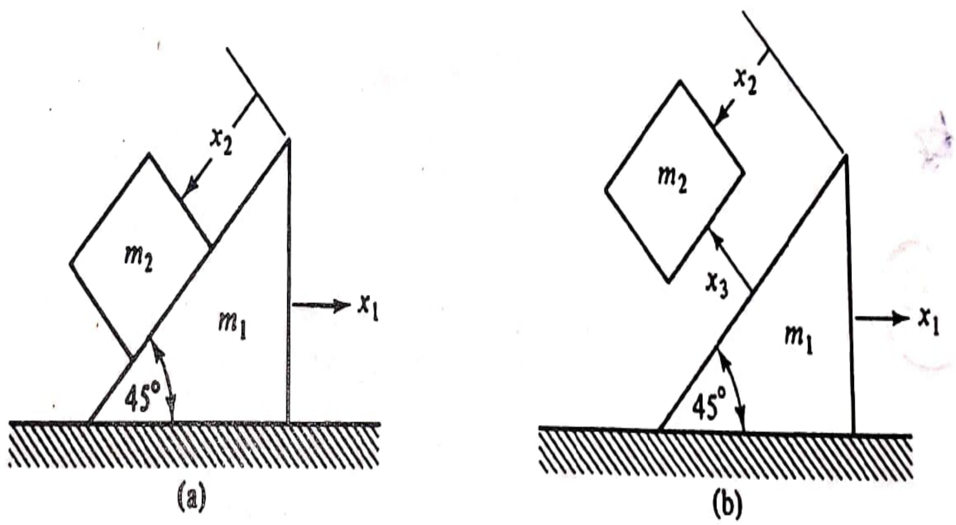
$$ml^2\ddot{\phi} - ml^2\dot{\theta}^2(\phi - \theta) + ml^2\ddot{\theta} + mgl\phi = 0,$$

$$ml^2[\ddot{\phi} - \dot{\theta}^2(\phi - \theta) + \ddot{\theta}] + mgl\phi = 0,$$

$$ml^2[\ddot{\phi} + \ddot{\theta} + mgl\phi = 0. \tag{2.40}$$

Problem 3

A block of mass m_2 can slide on another block of mass m_1 which in turn slides on a horizontal surface. Using x_1 and x_2 as co-ordinates, obtain the differential equation of motion. Solve for the acceleration of the two blocks as they move under the influence of gravity, assuming that all surfaces are frictionless. Find the force of interaction between the block.



Solution: Let x_1 is the displacement of the block m_1 and x_2 is the displacement of the block m_2 with respect to m_1.

Let v_1 be the velocity of m_1 and v_2 be the velocity of m_2 with respect to m_1.

Kinetic Energy:

$$T = \frac{1}{2}mv^2 = \frac{1}{2}[m_1v_1^2 + m_2v_2^2]$$

$$
\begin{aligned}
v_1 &= \dot{x_1} \qquad v_1^2 = \dot{x_1}^2 \\
v_2^2 &= (\dot{x_1} + \dot{x_2})^2 \\
&= \dot{x_1}^2 + \dot{x_2}^2 + 2\dot{x_1}\dot{x_2}\cos(90^\circ + 45^\circ) \\
&= \dot{x_1}^2 + \dot{x_2}^2 + 2\dot{x_1}\dot{x_2}(-\sin 45^\circ) \\
&= \dot{x_1}^2 + \dot{x_2}^2 - 2\dot{x_1}\dot{x_2}\frac{1}{\sqrt{2}} \\
v_2^2 &= \dot{x_1}^2 + \dot{x_2}^2 - \sqrt{2}\dot{x_1}\dot{x_2}.
\end{aligned}
$$

$$
T = \frac{1}{2}m_1\dot{x_1}^2 + \frac{1}{2}m_2(\dot{x_1}^2 + \dot{x_2}^2 - \sqrt{2}\dot{x_1}\dot{x_2}).
$$

Potential energy:

$$
\begin{aligned}
V &= mgh, \quad v = m_2 g\frac{x_2}{\sqrt{2}} \\
L &= T - V = \frac{1}{2}m_1\dot{x_1}^2 + \frac{1}{2}m_2(\dot{x_1}^2 + \dot{x_2}^2 - \sqrt{2}\dot{x_1}\dot{x_2}) + m_2 g\frac{x_2}{\sqrt{2}}.
\end{aligned}
$$

Lagrange's equation:

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x_1}}\right) - \frac{\partial L}{\partial x_1} = 0 \\
&\frac{\partial L}{\partial \dot{x_1}} = m_1\dot{x_1} + m_2\dot{x_1} - \frac{m_2\dot{x_2}}{\sqrt{2}} \\
&\frac{d}{dt}\frac{\partial L}{\partial \dot{x_1}} = m_1\ddot{x_1} + m_2\ddot{x_1} - \frac{m_2\ddot{x_2}}{\sqrt{2}} \\
&\frac{\partial L}{\partial x_1} = 0
\end{aligned}
$$

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x_1}}\right) - \frac{\partial L}{\partial x_1} = 0 \\
&m_1\ddot{x_1} + m_2\ddot{x_1} - \frac{m_2\ddot{x_2}}{\sqrt{2}} = 0 \qquad (2.41) \\
&\frac{\partial L}{\partial x_2} = m_2\dot{x_2} - \frac{m_2\dot{x_1}}{\sqrt{2}} \\
&\frac{d}{dt}\frac{\partial L}{\partial x_2} = m_2\ddot{x_2} - \frac{m_2\ddot{x_1}}{\sqrt{2}}. \qquad (2.42) \\
&\frac{\partial L}{\partial x_1} = \frac{m_2 g}{\sqrt{2}} \\
&\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x_2}}\right) - \frac{\partial L}{\partial x_2} = 0 \\
&m_2\dot{x_2} - \frac{m_2\dot{x_1}}{\sqrt{2}} - \frac{m_2 g}{\sqrt{2}} = 0. \qquad (2.43)
\end{aligned}
$$

To find acceleration: From (2.41)

$$\begin{aligned}\ddot{x}_1(m_1+m_2)-\frac{m_2\ddot{x}_2}{\sqrt{2}}&=0\\ \ddot{x}_1&=\frac{m_2\ddot{x}_2}{\sqrt{2}(m_1+m_2)}.\end{aligned}\tag{2.44}$$

Substitute $\ddot{x}_1$ in (2.43)

$$\begin{aligned}m_2\ddot{x}_1-\frac{m_2}{\sqrt{2}}\left(\frac{m_2\ddot{x}_2}{\sqrt{2}(m_1+m_2)}\right)-\frac{m_2g}{\sqrt{2}}&=0\\ m_2\ddot{x}_1-\frac{m_2^2\ddot{x}_2}{2(m_1+m_2)}&=\frac{m_2g}{\sqrt{2}}\\ m_2\ddot{x}_1\left(\frac{2m_1+m_2}{2(m_1+m_2)}\right)&=\frac{m_2g}{\sqrt{2}}\\ \ddot{x}_1&=\frac{\sqrt{2}g(m_1+m_2)}{(2m_1+m_2)}.\end{aligned}$$

Substitute $\ddot{x}_2$ in (2.44)

$$\begin{aligned}\ddot{x}_2&=\frac{m_2}{\sqrt{2}(m_1+m_2)}\frac{\sqrt{2}g(m_1+m_2)}{(2m_1+m_2)}\\ \ddot{x}_2&=\frac{gm_2}{(2m_1+m_2)}.\end{aligned}$$

Exercise problem

A particle mass m can slide without friction on the inside of a small tube which is bent in the form of a circle of radius r. The tube rotates about a verticle diameter with a constant angular velocity. Find the differential equations of motion.

Let us sum up

1. We have introduced the concepts of kinetic energy.

2. We have derived the derivation of standard form of Lagrange's equations for a holonomic and non-holonomic system.

3. We have introduced the integrals of the motion, and also solve the Kepler's problem.

4. We have discussed the Routhian functions.

5. We have studied the conservative, natural, Liouville's system with examples.

Check your progress

1. What are the Lagrange's equations?

2. Write the formula for the Standard form of Lagrange's equation for a holonomic and nonholonomic system.

Dear students, in the next section we will discuss about the integrals of the motion, Kepler's problem, the Routhian function. Also derive the conservative, natural and Liouville's system with examples.

2.2 Solution of Differential Equation of Motion

Any general analytic solution of the differential equation of motion contains $2n$ constants of integration which are usually evaluated from with the aid of $2n$ initial conditions. The general solution of any differential equation of motion can be obtained from the functions of the form,

$$f_j[q, \dot{q}, t] = \alpha_j, \quad j = 1, 2, ..., 2n.$$

These $2n$ functions are called constants, a integrals of motion these $2n$ equations can be used to solve q's and $2q$'s as function of α's and t.

2.2.1 Ignorable co-ordinates (or) Cyclic co-ordinates:

Consider a holonomic system described by the standard form of Lagrange's equation of the form,

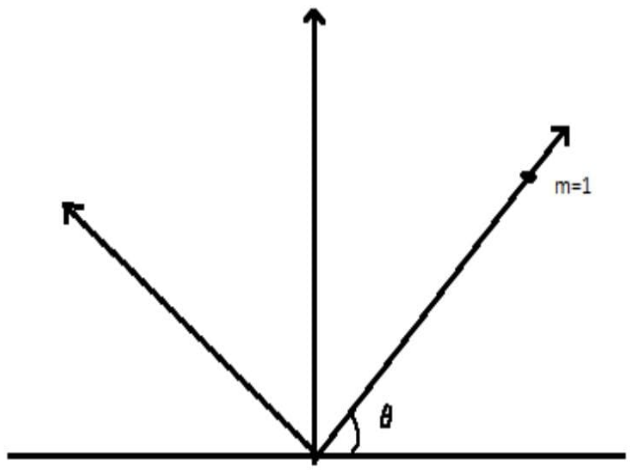
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \quad i = 1, 2, ..n$$

Suppose that, $L[q, \dot{q}, t]$ contains all $n\dot{q}$'s but some of q's say $q_1, q_2, ..., q_n$ are missing from the lagrange's equation these k co-ordinates are called ignorable co-ordinates. Since, $\frac{\partial L}{\partial q_i} = 0$ for each ignorable co-ordinates, then $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0, \quad i = 1, 2, ..., k.$ Integrating, $\int \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \beta_i, \quad i = 1, 2, ..., k.$ $p_i = \left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0, \quad i = 1, 2, ..., k$, where

β's are constant. Hence the generalised momentum corresponding to each ignorable co-ordinate constant.

2.2.2 Kepler's problem

It is the problem of motion of a particle of unit mass which is attracted by an inverse square gravitational force to a fixed point o.



Kinetic Energy: The transformation equation is,

$$x = r\cos\theta, \quad y = r\sin\theta, \quad \dot{x} = -r\sin\theta\dot{\theta} + \cos\theta\dot{r}$$

$$\dot{x}^2 = r^2\dot{\theta}^2 sin^2\theta + cos^2\theta\dot{r}^2 - 2r\dot{r}\dot{\theta}sin\theta.cos\theta, \quad \dot{y} = rcos\theta\dot{\theta} + sin\theta\dot{r}$$

$$\dot{y}^2 = r^2cos^2\theta\dot{\theta}^2 + sin^2\theta\dot{r}^2 + 2r\dot{r}\dot{\theta}sin\theta cos\theta$$

$$\dot{x}^2\dot{y}^2 = r^2\dot{\theta}^2 + \dot{r}^2$$

$$T = \frac{1}{2}mv^2 = \frac{1}{2}(r^2\dot{\theta}^2 + \dot{r}^2).$$

Potential energy: F is inversely proportional to r^2

$$F = \frac{-\mu}{r^2}, \quad W = \int dW$$

$$v = \int_{\infty}^{r} -dv = -\int_{\infty}^{r} F dr = -\mu\int_{\infty}^{r} r^{-2}dr$$

$$V = \frac{-\mu}{r}$$

$$L = T - V = \frac{1}{2}(r^2\dot{\theta}^2 + \dot{r^2}) + \frac{\mu}{r}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \ddot{r}$$

$$\frac{\partial L}{\partial r} = r\dot{\theta}^2 - \frac{\mu^2}{r}$$
$$\ddot{r} - r\dot{\theta}^2 - \frac{\mu^2}{r} = 0.$$

Since θ does not occur explicitly in Lagrangian function, it is treated as an ignorable co-ordinate. Then it's equation of motion is given by,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$
$$\frac{\partial L}{\partial \dot{\theta}} = \dot{\theta} r^2, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = r^2\ddot{\theta}, \quad \frac{\partial L}{\partial \theta} = 0$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$
$$r^2\ddot{\theta} = 0$$

Integrating on both sides, $r^2\dot{\theta} = \beta$. Where β is a constant and is equal to the angular momentum of the particle attracted towards the center o.

2.2.3 Routhian function

Suppose we consider a standard holonomic system whose configuration is given by, N generalised co-ordinates of which first k co-ordinates are ignorable.

The Lagrangian function L is defined by, $L(q_{k+1}, q_{k+2}, ..., q_n, \dot{q_1}, \dot{q_2}, ...\dot{q_n}, t)$. The Routhian function R is given by, $R(q_{k+1}, q_{k+2}, ..., q_n, \dot{q_{k+1}}, \dot{q_{k+2}}, ..., \dot{q_n}, \beta_1, \beta_2, ..., \beta_k, t)$

$$\delta R = \sum_{i=k+1}^{n} \frac{\partial R}{\partial q_i}\delta q_i + \sum_{i=k+1}^{n} \frac{\partial R}{\partial \dot{q_i}}\delta \dot{q_i} + \sum_{i=1}^{k} \frac{\partial R}{\partial \beta}\delta \beta_i + \frac{\partial R}{\partial t}\delta t. \tag{2.45}$$

Define Routhian function as, $R = L - \sum_{i=1}^{k} \beta_i q_i$.

Where

$$\beta_i = \frac{\partial L}{\partial \dot{q}_i}.$$

$$\delta R = \delta(L - \sum_{i=1}^{k} \beta_i \dot{q}_i) = \delta L - \delta \sum_{i=1}^{k} \beta_i \dot{q}_i.$$

$$\delta L = \sum_{i=k+1}^{n} \frac{\partial L}{\partial q_i}\delta q_i + \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i + \frac{\partial L}{\partial t}\delta t$$

$$\delta L = \sum_{i=k+1}^{n} \frac{\partial L}{\partial q_i}\delta q_i + \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i + \sum_{i=k+1}^{n} \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i + \frac{\partial L}{\partial t}\delta t.$$

$$\delta\left(\sum_{i=1}^{k} \beta_i \dot{q}_i\right) = \sum_{i=1}^{k} \beta_i \delta \dot{q}_i + \sum_{i=1}^{k} \delta\beta_i \dot{q}_i = \sum_{i=1}^{k} \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i + \sum_{i=1}^{k} \delta\beta_i \dot{q}_i$$

$$\begin{aligned}\delta R &= \delta L - \delta \sum_{i=1}^{k} \beta_i \dot{q}_i \\ &= \sum_{i=k+1}^{n} \frac{\partial L}{\partial q_i}\delta q_i + \sum_{i=k+1}^{n} \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i + \frac{\partial L}{\partial t}\delta t + \sum_{i=1}^{k} \delta\beta_i . \dot{q}_i. \end{aligned} \tag{2.46}$$

From (2.45) and(2.46)

$\frac{\partial L}{\partial q_i} = \frac{\partial R}{\partial q_i} \quad i = k+1, ..., n.$

$\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial R}{\partial \dot{q}_i} \quad i = k+1, ..., n.$

$\frac{\partial L}{\partial t} = \frac{\partial R}{\partial t}.$

$$\dot{q}_i = \frac{-\partial R}{\partial \beta_i} \quad i = 1, 2, ..., n. \tag{2.47}$$

Consider the Lagrange's Equation,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0. \tag{2.48}$$

Substitute (2.47) in(2.48)

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{q}_i}\right) - \frac{\partial R}{\partial q_i} = 0 \quad i = k+1, ..., n.$$

Thus, the Routhian procedure has been successful in eliminating the ignorable coordinates from the equation of motion.

Application of Routhian procedure in Kepler's problem

The Routhian function is given by,

$$R = L - \sum_{i=1}^{k} \beta_i q_i.$$

Since θ is an ignorable co-ordinate

$$R = L - \beta\dot{\theta}. \tag{2.49}$$

$$W.K.T, \quad r^2\dot{\theta} = \beta$$
$$\dot{\theta} = \frac{\beta}{r^2}. \tag{2.50}$$

Substitute (2.50) in (2.49)

$$\begin{aligned}
R &= \frac{1}{2}(\dot{r^2} + r^2\dot{\theta}^2) + \frac{\mu}{r} - \beta\left(\frac{\beta}{r^2}\right) \\
&= \frac{1}{2}\left(\dot{r^2} + r^2\left(\frac{\beta^2}{r^4}\right)\right) + \frac{\mu}{r} - \beta\left(\frac{\beta}{r^2}\right) = \frac{1}{2}\dot{r^2} + \frac{\mu}{r} - \frac{\beta^2}{2r^2} \\
&\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{r}}\right) - \frac{\partial R}{\partial r} = 0, \frac{d}{dt}\left(\frac{\partial R}{\partial \dot{r}}\right) = \ddot{r} \\
&\frac{\partial R}{\partial \dot{r}} = \frac{\beta^2}{r^3} - \frac{\mu}{r^2} \\
&\ddot{r} - \frac{\beta^2}{r^3} + \frac{\mu}{r^2} = 0.
\end{aligned}$$

Let us sum up

1. We introduce the ignorable co-ordinates.
2. We derive the Kepler's problem.
3. We have define Routhian function.
4. We have derive applications for Routhian function.

Check your progress

3. Define the cyclic co-ordinates.
4. State Kepler's problem.

2.3 Jacobi Integrals for Conservative system

A system is said to be conservative if it satisfies the following conditions,

1. The standard form of Lagrange's equation (holonomic or non-holonomic) applies.
2. The Lagrangian function L is not explicitly function of time t.
3. Any constraint equation can be expressed in the differential form as,

$$\sum_{i=1}^{n} a_{ij} dq_i + a_{jt} dt = 0$$
$$\sum_{i=1}^{n} a_{ij} dq_i = 0, \quad j = 1, 2, .., m.$$

2.3.1 Evaluate energy integral (or) Jacobi integral

To show that the three given conditions are sufficient to ensure the existence of an energy integral. Let us consider a system described by the standard form of non-holonomic form of Lagrange's equation,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \sum_{j=1}^{m} \lambda_j a_{ij}, \quad I = 1, 2, .., n \tag{2.51}$$

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \sum_{j=1}^{m} \lambda_j a_{ij}. \tag{2.52}$$

Where $L(q, \dot{q})$ is not an explicit function of time 't'. Now, $L = L(q, \dot{q})$

$$\frac{\partial L}{\partial t} = \sum_{i=1}^{n} \frac{\partial L}{\partial q_i}\dot{q}_i + \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i. \tag{2.53}$$

Substitute (2.52) in (2.53)

$$\begin{aligned}\frac{\partial L}{\partial t} &= \sum_{i=1}^{n}\left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \sum_{j=1}^{m} \lambda_j a_{ij}\right)\dot{q}_i + \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i \\ &= \sum_{i=1}^{n} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\dot{q}_i - \sum_{i=1}^{n}\sum_{j=1}^{m} \lambda_j a_{ij}\dot{q}_i + \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i \end{aligned} \tag{2.54}$$

Consider the equations of constraints in the form,

$$\sum_{i=1}^{n} a_{ij} dq_i = 0$$
$$\sum_{i=1}^{n} a_{ij}\dot{q}_i = 0. \tag{2.55}$$

Substitute (2.55) in (2.54)

$$\begin{aligned}\frac{\partial L}{\partial t} &= \sum_{i=1}^{n} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\dot{q}_i + \sum_{i=1}^{n}\frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i \\ &= \sum_{i=1}^{n}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right).\end{aligned}$$

Which when integrated gives,

$$\begin{aligned}L + h &= \sum_{i=1}^{n}\left(\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right) \\ h &= \sum_{i=1}^{n}\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L, \end{aligned} \tag{2.56}$$

where h is a constant. Thus we have obtained the constant of motion which is known as Jacobi integral (or) Energy integral. We known that,

$$\begin{aligned}&L = T - V \\ &\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i} - \frac{\partial V}{\partial \dot{q}_i} \\ &T = T_2 + T_1 + T_0 \\ &\frac{\partial T}{\partial \dot{q}_i} = \frac{\partial T_2}{\partial \dot{q}_i} + \frac{\partial T_1}{\partial \dot{q}_i} + \frac{\partial T_0}{\partial \dot{q}_i} \\ &\frac{\partial T}{\partial \dot{q}_i} = \sum_{j=1}^{m} m_{ij}\dot{q}_j + a_j \quad i = 1, 2.., n \\ &\frac{\partial V}{\partial \dot{q}_i} = 0\end{aligned}$$

$$\frac{\partial L}{\partial \dot{q}_i} = \sum_{j=1}^{m} m_{ij}\dot{q}_j + a_j. \tag{2.57}$$

Substitute (2.57) in (2.56)

$$\begin{aligned}h &= \sum_{i=1}^{n}(\sum_{j=1}^{m} m_{ij}\dot{q}_j + a_j)\dot{q}_i - L \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} m_{ij}\dot{q}_j\dot{q}_i + a_j\dot{q}_i - (T - V) \\ &= 2T_2 + T_1 - (T_2 + T_1 + T_0) + V \\ &= T_2 - T_0 + V \\ h &= T' + V'.\end{aligned}$$

Where $T' = T_2$ and $V' = V - T_0$

2.3.2 Natural system

Dear students, in this subsection we will study a natural system.

A natural system is a conservative system with some additional properties.

1. It is described by the standard holonomic form of Lagrange's equation.

2. The kinetic energy is expressed as a homogeneous quadratic function of q's.

$$T = T_2 = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m} m_{ij}\dot{q}_j\dot{q}_i \quad (T_1 = T_0 = 0),$$

where m_{ij}'s are function of q's but not of time.

Jacobi integral for the natural system:

The jacobi integral for the natural system is equal to the total energy. The kinetic equation is expressed as a homogeneous quadratic function of q's, $T_1 = T_0 0$

W.K.T, $T_2 - T_0 + V = h$

$T + V = h$

(ie.,) The total energy is conserved.

Equation of motion for natural system:

We know that,

$$T_2 = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m} m_{ij}\dot{q}_j\dot{q}_i.$$

$$T_1 = \sum_{i=1}^{n} a_j\dot{q}_i$$

$$T_0. = \frac{1}{2}\sum_{k=1}^{3N} m_k\left(\frac{\partial x_k}{\partial t}\right)^2.$$

Since $T_1 = T_0 = 0$ and T_2 is not a function of time and $a_i = 0, \frac{\partial m_{ij}}{\partial t} = 0$.

Then the equation,

$$\sum_{j=1}^{m} m_{ij}\ddot{q}_j + \frac{1}{2}\sum_{j}\sum_{l}\left[\frac{\partial m_{ij}}{\partial q_l} + \frac{\partial m_{il}}{\partial q_j} - \frac{\partial m_{jl}}{\partial q_i}\right]\dot{q}_j\dot{q}_l + \sum_{j=1}^{m}\frac{\partial m_{ij}}{\partial t}\dot{q}_j$$
$$+ \sum_{j=1}^{m}\left(\frac{\partial m_{il}}{\partial q_j} - \frac{\partial a_j}{\partial q_i}\right)\dot{q}_j + \frac{\partial a_i}{\partial t - \frac{\partial T_0}{\partial q_i}} + \frac{\partial V}{\partial q_i} = 0.$$

Hence, we get

$$\sum_{j=1}^{m} m_{ij}\ddot{q}_j + \frac{1}{2}\sum_j \sum_l \left[\frac{\partial m_{ij}}{\partial q_l} + \frac{\partial m_{il}}{\partial q_j} - \frac{\partial m_{jl}}{\partial q_i}\right]\dot{q}_j\dot{q}_l + \frac{\partial V}{\partial q_i} = 0.$$

This is the required equation of motion for the natural system.

Remark:

A holonomic conservative system with $T_1 \neq 0$ is called gyroscopic system.

Orthogonal system:(or) Show that the orthogonal system can be reduced to quadratures

Let us consider a orthogonal system. (ie.,) Natural system in which T contains only $\dot{q_i}^2$ and no cross product of $\dot{q}$'s. Suppose that

$$T = \frac{1}{2}f\sum_{i=1}^{n}\dot{q_i}^2$$
$$V = \frac{1}{f}\sum_{i=1}^{n}v_i q_i.$$

Where $f = \sum_{i=1}^{n} f_i(q_i) > 0$.

Let us consider the Lagrange's equation of motion,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = 0.$$

We know that,

$$\frac{1}{2}f\sum_{i=1}^{n}\dot{q_i}^2 = \frac{1}{2}f[\dot{q_1}^2 + \dot{q_2}^2 + ... + \dot{q_n}^2]$$
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) = \frac{d}{dt}(f\dot{q}_i)$$
$$\frac{\partial T}{\partial q_i} = \frac{1}{2}\frac{\partial f_i}{\partial q_i}\sum_{i=1}^{n}\sum_{i=1}^{n}\dot{q_i}^2$$
$$frac\partial V\partial q_i = \frac{-1}{f^2}\frac{\partial f_i}{\partial q_i}\sum_{i=1}^{n}v_i + \frac{1}{f}\frac{\partial v_i}{\partial q_i}$$
$$= \frac{-1}{f^2}\frac{\partial f_i}{\partial q_i}Vf + \frac{1}{f}\frac{\partial v_i}{\partial q_i}$$
$$= \frac{-v}{f}\frac{\partial f_i}{\partial q_i} + \frac{1}{f}\frac{\partial v_i}{\partial q_i}$$

Hence we get,

$$\frac{d}{dt}(f\dot{q}_i) - \frac{1}{2}\frac{\partial f_i}{\partial q_i}\sum_{i=1}^{n}\dot{q_i}^2 - \frac{v}{f}\frac{\partial f_i}{\partial q_i} + \frac{1}{f}\frac{\partial v_i}{\partial q_i} = 0. \tag{2.58}$$

This is a natural system

$$\begin{aligned}
&T + V = h \\
&\frac{1}{2}f\sum_{i=1}^{n}\dot{q_i}^2 + V = h \\
&\frac{1}{2}\sum_{i=1}^{n}\dot{q_i}^2 = \frac{h-V}{f}.
\end{aligned} \tag{2.59}$$

Substitute (2.59) in (2.58),

$$\begin{aligned}
&\frac{d}{dt}(f\dot{q}_i) - \frac{h-V}{f}\frac{\partial f_i}{\partial q_i} - \frac{1}{f^2}\frac{\partial f_i}{\partial q_i}\sum_{i=1}^{n}v_i + \frac{1}{f}\frac{\partial v_i}{\partial q_i} = 0 \\
&\frac{d}{dt}(f\dot{q}_i) - \frac{h}{f}\frac{\partial f_i}{\partial q_i} - \frac{V}{f}\frac{\partial f_i}{\partial q_i} - \frac{1}{f^2}\frac{\partial f_i}{\partial q_i}vf + \frac{1}{f}\frac{\partial v_i}{\partial q_i} = 0 \\
&\frac{d}{dt}(f\dot{q}_i) - \frac{h}{f}\frac{\partial f_i}{\partial q_i} + \frac{1}{f}\frac{\partial v_i}{\partial q_i} = 0.
\end{aligned} \tag{2.60}$$

Multiply $2f\dot{q}_i$

$$\begin{aligned}
&2f\dot{q}_i\frac{d}{dt}(f\dot{q}_i) - 2f\dot{q}_i\frac{h}{f}\frac{\partial f_i}{\partial q_i} + 2f\dot{q}_i\frac{1}{f}\frac{\partial v_i}{\partial q_i} = 0 \\
&\frac{d}{dt}(f\dot{q}_i)^2 - 2h\dot{q}_i\frac{\partial f_i}{\partial q_i} + 2\dot{q}_i\frac{\partial v_i}{\partial q_i} = 0 \\
&\frac{d}{dt}(f\dot{q}_i)^2 - 2h\frac{d}{dt}(f_i q_i) + 2\frac{d}{dt}(v_i q_i) = 0.
\end{aligned}$$

Integrating we get,

$$\begin{aligned}
&(f\dot{q}_i)^2 - 2h(f_i q_i) + 2(v_i q_i) = 2c_i \\
&(f\dot{q}_i)^2 = 2c_i + 2h(f_i q_i) - 2(v_i q_i) \\
&(\dot{q}_i)^2 = \frac{2(c_i + h(f_i q_i) - (v_i q_i))}{f^2} \\
&\frac{dq_i}{dt} = \frac{\sqrt{2(c_i + h(f_i) - v_i)}}{f} \\
&\frac{dq_i}{\sqrt{2(c_i + h(f_i) - v_i)}} = \frac{dt}{f} \\
&\frac{dq_1}{\sqrt{2(c_i + h(f_i) - v_i)}} = \frac{dq_2}{\sqrt{2(c_i + h(f_i) - v_i)}} = ... = \frac{dt}{f} = dr.
\end{aligned}$$

Each differential expression is a function of a single q_i. So the problem is reduced to quadratures.

2.3.3 Liouville's system

A natural system having T and V of the form

$$T = \frac{1}{2} f \sum_{i=1}^{n} M_i \dot{q_i}^2, \quad V = \frac{1}{f} \sum_{i=1}^{n} v_i.$$

Where $\sum_{i=1}^{n} v_i = Vf$

$$f = \sum_{i=1}^{n} f_i(q_i), v_i = v_i(q_i), M_i = M_i(q_i)$$

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q_i}}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = 0 \\
&\frac{\partial V}{\partial q_i} = \frac{1}{2} f M_i 2\dot{q_i} \\
&\quad = f M_i \dot{q_i}
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q_i}}\right) &= \frac{d}{dt} f M_i \dot{q_i} \\
\frac{\partial T}{\partial \dot{q_i}} &= \frac{1}{2} \sum_{i=1}^{n} \dot{q_i}^2 \left[\frac{\partial f_i}{\partial q_i} M_i + f \frac{\partial M_i}{\partial q_i}\right] \\
&= \frac{1}{2} \sum_{i=1}^{n} \dot{q_i}^2 \frac{\partial f_i}{\partial q_i} M_i + \frac{1}{2} \sum_{i=1}^{n} \dot{q_i}^2 f \frac{\partial M_i}{\partial q_i}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial V}{\partial q_i} &= \frac{1}{f} \frac{\partial v_i}{\partial q_i} + \sum_{i=1}^{n} v_i - \frac{1}{f^2} \frac{\partial f_i}{\partial q_i} \\
&= \frac{1}{f} \frac{\partial v_i}{\partial q_i} - \frac{V}{f} \frac{\partial f_i}{\partial q_i}
\end{aligned}$$

$$\frac{d}{dt}(f M_i \dot{q_i}) - \frac{1}{2} \sum_{i=1}^{n} \dot{q_i}^2 \frac{\partial f_i}{\partial q_i} M_i - \frac{1}{2} \sum_{i=1}^{n} \dot{q_i}^2 f \frac{\partial M_i}{\partial q_i} + \frac{1}{f} \frac{\partial v_i}{\partial q_i} - \frac{V}{f} \frac{\partial f_i}{\partial q_i} = 0. \tag{2.61}$$

Since it is a natural system,

$$\begin{aligned}
&h = T + V \\
&\frac{h - V}{f} = \frac{f}{2} \sum_{i=1}^{n} M_i \dot{q_i}^2
\end{aligned} \tag{2.62}$$

substitute (2.62) in (2.61)

$$\frac{d}{dt}(fM_i\dot{q}_i) - \frac{h-V}{f}\frac{\partial f_i}{\partial q_i}M_i - \frac{1}{2}\sum_{i=1}^{n}\dot{q}_i{}^2 f\frac{\partial M_i}{\partial q_i} + \frac{1}{f}\frac{\partial v_i}{\partial q_i} - \frac{V}{f}\frac{\partial f_i}{\partial q_i} = 0$$

$$\frac{d}{dt}(fM_i\dot{q}_i) - \frac{h}{f}\frac{\partial f_i}{\partial q_i}M_i - \frac{1}{2}\sum_{i=1}^{n}\dot{q}_i{}^2 f\frac{\partial M_i}{\partial q_i} + \frac{1}{f}\frac{\partial v_i}{\partial q_i} = 0$$

$$\frac{df}{dt}(M_i\dot{q}_i) + \frac{dM_i}{dt}(f\dot{q}_i) + \frac{d\dot{q}_i}{dt}(M_i f) - \frac{h}{f}\frac{\partial f_i}{\partial q_i}M_i - \frac{1}{2}\sum_{i=1}^{n}\dot{q}_i{}^2 f\frac{\partial M_i}{\partial q_i} + \frac{1}{f}\frac{\partial v_i}{\partial q_i} = 0. \quad (2.63)$$

Multiply (2.63) by $2f\dot{q}_i$

$$\frac{df}{dt}(2fM_i\dot{q}_i{}^2) + \frac{dM_i}{dt}(2f^2\dot{q}_i{}^2) + \frac{d\dot{q}_i}{dt}(2M_i f^2\dot{q}_i) - \frac{2f\dot{q}_i h}{f}\frac{\partial f_i}{\partial q_i}M_i - \frac{2f\dot{q}_i}{2}\sum_{i=1}^{n}\dot{q}_i{}^2 f\frac{\partial M_i}{\partial q_i} + \frac{2f\dot{q}_i}{f}\frac{\partial v_i}{\partial q_i} = 0$$

$$\frac{df}{dt}(2fM_i\dot{q}_i{}^2) + \frac{dM_i}{dt}(2f^2\dot{q}_i{}^2) + \frac{d\dot{q}_i}{dt}(2M_i f^2\dot{q}_i) - 2\dot{q}_i h\frac{\partial f_i}{\partial q_i}M_i - f^2\dot{q}_i\sum_{i=1}^{n}\dot{q}_i{}^2\frac{\partial M_i}{\partial q_i} + 2\dot{q}_i\frac{\partial v_i}{\partial q_i} = 0$$

$$\frac{df}{dt}(2fM_i\dot{q}_i{}^2) + \frac{dM_i}{dt}(f^2\dot{q}_i{}^2) + \frac{d\dot{q}_i}{dt}(2M_i f^2\dot{q}_i) - 2h\frac{\partial f_i}{\partial t} + 2\frac{\partial v_i}{\partial t} = 0$$

$$\frac{d}{dt}(f^2M_i\dot{q}_i{}^2) - 2h\frac{\partial f_i}{\partial t} + 2\frac{\partial v_i}{\partial t} = 0$$

Integrating on both sides,

$$\int\frac{d}{dt}(f^2M_i\dot{q}_i{}^2) - \int 2h\frac{\partial f_i}{\partial t} + \int 2\frac{\partial v_i}{\partial t} = 0$$

$$\int(f^2M_i\dot{q}_i{}^2) - \int 2hf_i + \int 2v_i = 2c_i$$

$$\dot{q}_i{}^2 = \frac{2(c_i + hf_i - v_i)}{f^2M_i}$$

$$\dot{q}_i{}^2 = \frac{\phi_i(q_i)}{f^2}.$$

Where $\phi_i(q_i) = \frac{2(c_i+hf_i-v_i)}{M_i}$

$$\dot{q}_i = \sqrt{\frac{\phi_i(q_i)}{f^2}}$$
$$\frac{dq_i}{dt} = \frac{\sqrt{\phi_i(q_i)}}{f}$$
$$\frac{dq_i}{\sqrt{\phi_i(q_i)}} = \frac{dt}{f}$$
$$\frac{dq_1}{\sqrt{\phi_1(q_1)}} = \frac{dq_2}{\sqrt{\phi_2(q_2)}} = .. = \frac{dt}{f} = dr$$
$$\sum_{i=1}^{n} \frac{dq_i}{\sqrt{\phi_i(q_i)}} = \frac{dt}{\sum_{i=1}^{n} f_i}$$
$$\sum_{i=1}^{n} \frac{f_i dq_i}{\sqrt{\phi_i(q_i)}} = dt$$

Integrating on both sides

$$\int \sum_{i=1}^{n} \frac{f_i dq_i}{\sqrt{\phi_i(q_i)}} = \int dt$$
$$= t + \beta_i.$$

Thus the n-constants $(\beta' s)$ along with other n-independent constant c_i and h gives an 2n independent constants of motion.

Problem 1: Suppose a mass spring system is attached to a frame which is translating with a uniform velocity v_0. Let l_0 be the unstressed spring length and use the elongation x as the generalized co-ordinates. Find the Jacobi integral.

Solution:Kinetic eneergy:

$$T = \frac{1}{2}m(v_0 + \dot{x})^2$$
$$= \frac{1}{2}m(v_0^2 + 2v_0\dot{x} + \dot{x}^2)$$
$$= \frac{1}{2}m\dot{x}^2 + v_0^2 + mv_0\dot{x} + \frac{1}{2}mv_0^2$$
$$= T_2 + T_1 + T_0.$$

Potential energy:

$$\begin{aligned} V &= \int dV = -\int F.dx \\ &= \int kxdx \\ &= k\frac{x^2}{2}. \end{aligned}$$

The Jocobi integral:

$$\begin{aligned} h &= T_2 - T_0 + V \\ &= \frac{1}{2}m\dot{x}^2 - \frac{1}{2}mv_0^2 + k\frac{x^2}{2} \\ &= Constant. \end{aligned}$$

(ie.,) The total energy is conserved.

Problem 2: For spherical pendulum, obtain the integrals of motion and reduce the problem to quadratures.

Solution:

$$\begin{aligned} T &= \frac{1}{2}ml^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) \\ V &= mgl\cos\theta \\ T &= \frac{1}{2}ml^2\sin^2\theta\left(\frac{\dot{\theta}^2}{\sin^2\theta} + \dot{\phi}^2\right) \\ &= \frac{1}{2}f(M_\theta\dot{\theta}^2 + M_\phi\dot{\phi}^2), \end{aligned}$$

where $f = ml^2 \sin^2\theta$

$$\begin{aligned}
M_\theta &= \frac{1}{\sin^2\theta} \\
M_\phi &= 1 \\
f &= f_\theta + f_\phi \\
V &= \frac{1}{f}(V_\theta + V_\phi) \\
V &= mgl\cos\theta \\
Vf &= V_\theta \\
V_\theta &= mgl\cos\theta \times ml^2\sin^2\theta \\
V_\theta &= m^2gl^3 sin^2\theta\cos\theta \\
V_\phi &= 0 \\
f_\theta &= ml^2\sin^2\theta \\
f_\phi &= 0
\end{aligned}$$

$$\begin{aligned}
\phi_i(q_i) &= \frac{2(hf_i - v_i + c_i)}{M_i} \\
\phi_\theta &= \frac{2}{M\theta}(hf_\theta - v_\theta + c_\theta) \\
&= 2\sin^2\theta[hml^2\sin^2\theta - m^2gl^3 sin^2\theta\cos\theta + c_\theta] \\
\phi_\theta &= \frac{2}{1}(c_\theta)
\end{aligned}$$

$$\begin{aligned}
\frac{d\theta}{\sqrt{\phi_\theta}} &= \frac{d\phi}{\sqrt{\phi_\phi}} = \frac{dt}{f} \\
\frac{d\theta}{\sqrt{\phi_\theta}} &= \frac{dt}{f} \\
\int_{\theta_0}^{\theta} \frac{f d\theta}{\sqrt{\phi_\theta}} &= \int_{\theta_0}^{\theta} dt
\end{aligned}$$

$$\int_{\theta_0}^{\theta} \frac{f d\theta}{\sqrt{\phi_\theta}} = t - t_0. \tag{2.64}$$

$$\frac{d\phi}{\sqrt{\phi_\phi}} = \frac{d\phi}{\sqrt{\phi_\phi}} = \frac{dt}{f}$$

$$\frac{d\phi}{\sqrt{\phi_\phi}} = \frac{dt}{f}$$

$$\int_{\phi_0}^{\phi} \frac{f d\phi}{\sqrt{\phi_\phi}} = \int_{\phi_0}^{\phi} dt$$

$$\int_{\phi_0}^{\phi} \frac{f d\phi}{\sqrt{\phi_\phi}} = t - t_0. \tag{2.65}$$

From (2.64) and (2.65),

$$\int_{\theta_0}^{\theta} \frac{f d\theta}{\sqrt{\phi_\theta}} = \int_{\phi_0}^{\phi} \frac{f d\phi}{\sqrt{2c_\phi}}$$

$$\int_{\theta_0}^{\theta} \frac{d\theta}{\sqrt{\phi_\theta}} = \int_{\phi_0}^{\phi} \frac{d\phi}{\sqrt{2c_\phi}}.$$

Thus the problem is reduced to quadratures.

Let us sum up

1. We introduce Jacobi integral.
2. We have derive Jacobi integral for natural system.
3. We have discussed Liouville's system.

Check your progress

5. Explain Jacobi integral.
6. What is mean by natural system?
7. What is orthogonal system?

Summary

- Introduced the concepts of kinetic energy.
- Derived the derivation of standard form of Lagrange's equations for a holonomic and non-holonomic system.
- Introduced the integrals of the motion, and also solve the Kepler problem.

- Discussed the Routhian functions.
- Studied the conservative, natural, Liouville's system with examples.
- Introduce the ignorable co-ordinates.
- Derive the Kepler's problem.
- Define Routhian function.
- Derive applications for Routhian function.
- Introduce Jacobi integral.
- Derive Jacobi integral for natural system.
- Discussed Liouville's system.

Glossary

• **Lagrange multiplier:** Lagrange multiplier are the scalars to obtain constrains forces.

•**Ignorable co-ordinates:** $L(q'\dot{q},t)$ contains all $n\dot{q}$'s but some of the q's say $q_1, q_2, ..., q_k$ are missing form the Lagrangian. These k co-ordinates are called ignorable coordinates.

•**Quadratures :** In terms of known elementry functions or indefinite integrals of such functions is called quadratures.

•**Routhian function:** The Routhian function R is given by,
$R(q_{k+1}, q_{k+2}, ..., q_n, \dot{q_{k+1}}, \dot{q_{k+2}}, ..., \dot{q_n}, \beta_1, \beta_2, ..., \beta_k, t)$ is defined by $R = L - \sum_{i=1}^{k} \beta_i q_i$.

•**Conservative system:** If no other forces do work on the system, the total mechanical energy is conserved, hence the system is called consevative sytem.

•**Jacobi integral:** The equation $\frac{\partial L}{\partial t} = \sum_{i=1}^{n} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right)$, which can be integrated to gives, $h = \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L$, where h is a constant. Thus we have obtained the constant of motion which is known as Jacobi integral.

•**Gyroscopic system:** In general a holonomic conservative system with $T_1 \neq 0$ is called gyroscopic system.

Self-Assessment Questions

Short-Answer Questions:

1) Show that Jacobi integral has the unit of energy.

2) Discuss the integrals of motion.

3) Derive the Lagrangian form of D'Almbert's principle in term of generalized co-ordinates.

4) Discuss the Kepler's problem using ignorable coordinates.

5) Find the differential equation of motion for a spherical pendulum of length 'l'.

6) Illustrate a Routhian method for Kepler's problem.

Long-Answer Questions:

1) Derive the standard form of Lagrange's equation for a non-holonomic system.

2) Explain the Liouville system.

3) Derive the standard form of Lagrange's equation for a holonomic system.

4) Routhian function and prove that $\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{q_i}}\right) - \frac{\partial R}{\partial q_i} = 0$.

5) Define the Lagrangian equation and derive the Lagrangian form of D'Almbert's principle in term of generalized co-ordinates.

6) Briefly explain conservative systems.

7) Derive the Lagrangian form of D'Almbert's principle in term of generalized co-ordinates

8) Define a Routhian function and explain the procedure for eliminating the ignorable co-ordinates from the equations of motion using Routhian function.

9) Discuss briefly derivatives of Lagrange's equations.

10) Solve the differential equation of motion for a spherical pendulum of length 'l' for the motion.

11) Differentiate holonomic and non-holonomic Systems.

12) Find the differential equation of motion for a double pendulum.

Objective Questions:

1. ……. dynamics is based on a direct application of Newton's law of motiom.

a) Analytical b) Vectorial c) Classical d) Newtonian

2. Find the name of equation $\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{q}_i}\right) - \frac{\partial R}{\partial q_i} = Q_i$

a) Lagrange's equation b) Hamiltonian equation

c) Routhian equation d) Jacobi equation

3. Find the name of the equation $\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{q}_i}\right) - \frac{\partial R}{\partial q_i} = 0,$

a) The standard form of Lagrange's equation for a non-holonomic system.

b) The standard form of Lagrange's equation for a holonomic system.

c) Lagrange's equation d) Hamiltonian equation

4. A …… consists of two particles suspended by massless rods.

a) Spherical pendulum b) Double pendulum

c) Single pendulum d) Compound pendulum

5. Suppose that $L(q, \dot{q}, t)$ contains all nq's but some of the q's say, $q_1, q_2, ..., q_k$ are missing from the Lagrangian. These k coordinates are called …….

a) Cartesian coordinates b) Generalized coordinates

c) Ignorable coordinates d) Spherical coordinates

6. Find the name of the equation $R = L - \sum_{i=1}^{k} \beta_i \dot{q}_i$

a) Routhian function b) Lagrangian function

c) Hamiltonian function d) Jacobi function

7. Jacobi integral is

a) constant of time b) constant of energy

c) constant of displacement d) constant of motion

8) The ……. is particularly simple for a natural system; it is equal to the total energy.

a) Energy b) Total energy c) Mass d) Total mass

9) In general, a holonomic conservative system with $T_1 \neq 0$ is called …..

a) Natural system b) Conservative system

c) Rheonomic system d) Gyroscopic system

10) A natural system having T and V of the form is called …..

a) Natural system b) Liouville system

c) Rheonomic system d) Conservative system

11) The existence of the Jacobi integral implies that the total kinetic energy is.....

a) Constant b) Zero c) One d) Two

12) The Lagrangian L is not an explicit function of time, even though the system is rheonomic. Hence , the system is...

a) Natural system b) Liouville system

c) Rheonomic system d) Conservative system

Answers for Check Your Progress

1. The following set of s number of second order differential equations satisfied by the Lagrangian system are called the Lagrange's equations of motion.
$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q_k}}\right) - \frac{\partial L}{\partial q_k} = 0 \quad k = 1, 2, .., s.$

2. The Standard form of Lagrange's Equation for a holonomic system is $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$ and the Standard form of Lagrange's Equation for a nonholonomic system is $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \sum_{j=1}^{m} \lambda_j a_{ji}$.

3. $L(q'\dot{q}, t)$ contains all $n\dot{q}$'s but some of the q's say $q_1, q_2, ..., q_k$ are missing form the Lagrangian. These k coordinates are called icyclic coordinates.

4. It is the problem of motion of a particle of unit mass which is attracted by an inverse square gravitational force to a fixed point o.

5. The equation $\frac{\partial L}{\partial t} = \sum_{i=1}^{n} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right)$ Which can be integrated to gives, $h = \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L$, where h is a constant. Thus we have obtained the constant of motion which is known as Jacobi integral.

6. A natural systemis a conservative system with some additional properties.

i. It is described by the standard holonomic form of Lagrange's equation.

ii. The kinetic energy is expressed as a homogeneous quadratic function of q's.

7. Natural system in which T contains only $\dot{q_i}^2$ and no cross product of $\dot{q}$'s. Suppose that $T = \frac{1}{2} f \sum_{i=1}^{n} \dot{q_i}^2, \quad V = \frac{1}{f} \sum_{i=1}^{n} v_i q_i$, where $f = \sum_{i=1}^{n} f_i(q_i) > 0$.

Suggested Readings

- Greenwood. T. Donald, Classical Dynamics, 1979, New Delhi: Prentice Hall of Indian Private Limited.
- Goldstein, Herbert. 2011. New Delhi: Classical Mechanics, 3rd Edition. Pearson Education India.
- Rao, Sankara. K. 2009. New Delhi: Classical Mechanics. PHI Learning Private Limited.
- Upadhyaya. J.C. 2010. New Delhi: Classical Mechanics, 2nd Edition. Himalaya Publishing House.
- Gupta. S. L. 1970. New Delhi: Classical Mechanics. Meenakshi Prakashan.

Unit 3

HAMILTON'S EQUATIONS

Objectives

After the successful completion of this unit; the students are expected

• To recall the basic concepts of the stationary values of the functions several variables.

• To gain the knowledge about the Lagranges multiplir method and Euler Lagrange equation method with illustrated examples.

• To understand the concepts of Hamilton's principle.

• To derive the Hamilton's equation with holonomic and non-holonomic system.

• To discuss about the mass-spring system and Kepler's problems by using Hamilton procedure.

• To develope the concepts of the modified Hamilton's principle.

• To analyse the method of principle of least action.

• To solve the problem related to Jacobi's form of principle of least action.

3. Introduction

Dear students, in the last two units we have developed Lagrange's formulation of mechanics. In this present chapter, we will resume the formal development of mechanics, turning our attention to an alternative statement of the structure of the theory known as the Hamiltonian formulation. In this formulation the variation principle is used as the basics for the discribution of dynamic system. In this approach the motion is consider as the hole and involues finiding the path in configurence space which yields stationary value for certain definite integral The variational principle of most importance in dynamics is Hamilton's principle which was first announced in 1834.

3.1 Stationary value of a function of several variables

Dear students, in this section we will introduce the stationary value of a function of several variables. Also we will discuss Lagrange multiplier method, stationary value of a definite integral, Brachistochrone problem, Geodesic problem and Hamilton's principle.

3.1.1 The necessary and sufficient condition for stationary values

Consider a function $f(q_1, q_2, ..., q_n)$. The first variation of f at the reference point q_0 is given by,

$$\delta f = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i}\right)_0 \delta q_i,$$

where δq's are virtual displacement.

The necessary and sufficient condition that f have a stationary value at q_0 is that $\delta f = 0$, for all δq's . Now,

$$\delta q = q - q_0$$

$$\Rightarrow q = q_0 + \delta q.$$

For the case in which δq's are independent and reversible.
We have

$$\left(\frac{\partial f}{\partial q_i}\right)_0 = 0, \ (i = 1, 2, ..., n).$$

.

Consider the second variation of f ,

$$\delta^2 f = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{\partial^2 f}{\partial q_i \partial q_j}\right)_0 \delta q_i \delta q_j.$$

Using the notation $k_{ij} = \left(\frac{\partial^2 f}{\partial q_i \partial q_j}\right)_0$, where k's are the elements of a symmetric matrix of order $n \times n$.

Conditions for maximum and minimum

1. The sufficient condition that q_0 is a local minimum is that **k** must be positive definite.
2. If **k** is negative definite, the point q_0 is a local maximum.
3. If **k** is indefinite, the point q_0 is a saddle point.

3.1.2 Lagrange multiplier method

Consider the free variations of an augmented function $F(q_1, q_2, ..., q_n;\ \ \lambda_1, \lambda_2, ..., \lambda_m)$ and F defined as $F = f + \sum_{j=1}^{m} \lambda_j \phi_j$ The necessary and sufficient conditions for δF to be stationary is

$$\delta F = 0$$
$$\sum_{i=1}^{n} \left(\frac{\partial F}{\partial q_i}\right)_0 \delta q_i + \sum_{j=1}^{m} \left(\frac{\partial F}{\partial \lambda_j}\right)_0 \delta \lambda_j = 0. \quad Then$$
$$\left(\frac{\partial F}{\partial q_i}\right)_0 = 0, \quad (i = 1, 2, ..., n) \quad and \quad \left(\frac{\partial F}{\partial \lambda_j}\right)_0 = 0, \quad (j = 1, 2, ..., m)$$

Examples:

Find the stationary values of the function $f = z$, subject to the constraints

$\phi_1 = x^2 + y^2 + z^2 - 4$ =0 and ϕ_2= $xy - 1$ =0.

Solution :

Let us consider the augmented function F

$$F = f + \sum_{j=1}^{m} \lambda_j \phi_j$$
$$F = z + \lambda_1 \phi_1 + \lambda_2 \phi_2$$
$$= z + \lambda_1 (x^2 + y^2 + z^2 - 4) + \lambda_2 (xy - 1). \tag{3.1}$$

The necessary and sufficient condition for F to be stationary is

$$\left(\frac{\partial F}{\partial q_i}\right)_0 = 0, \ (i = 1, 2, ..., n),$$

we have

$$\frac{\partial F}{\partial x} = 2\lambda_1 x + \lambda_2 y = 0.$$
$$\frac{\partial F}{\partial y} = 2\lambda_1 y + \lambda_2 x = 0.$$
$$\frac{\partial F}{\partial z} = 1 + 2\lambda_1 z = 0.$$

$$\frac{\partial F}{\partial x} = 0, \Rightarrow 2x\lambda_1 = -\lambda_2 y, \Rightarrow \frac{\lambda_1}{\lambda_2} = \frac{-y}{2x}. \tag{3.2}$$
$$\frac{\partial F}{\partial y} = 0, \Rightarrow 2y\lambda_1 = -\lambda_2 x, \Rightarrow \frac{\lambda_1}{\lambda_2} = \frac{-x}{2y}. \tag{3.3}$$
$$\frac{\partial F}{\partial z} = 0, \Rightarrow 1 = -2z\lambda_1, \Rightarrow \lambda_1 = \frac{-1}{2z}. \tag{3.4}$$
$$\frac{\partial F}{\partial \lambda_1} = 0, \Rightarrow x^2 + y^2 + z^2 = 4. \tag{3.5}$$
$$\frac{\partial F}{\partial \lambda_2} = 0, \Rightarrow xy = 1. \tag{3.6}$$

From equations (3.2)and(3.3) of L.H.S are equal. Then R.H.S, implies

$$\frac{-y}{2x} = \frac{-x}{2y}, \Rightarrow x^2 = y^2, \Rightarrow x = \pm y.$$
$$x = \pm y \quad in \quad (3.6), \Rightarrow x^2 = 1, \Rightarrow x = \pm 1, \Rightarrow y = \pm 1.$$
$$If \quad x = 1, y = 1 \quad in \quad (3.5), \Rightarrow 1 + 1 + z^2 = 4, \Rightarrow z^2 = 2, \Rightarrow z = \pm\sqrt{2}.$$
$$If \quad x = -1, y = -1 \quad in \quad (3.5), \Rightarrow 1 + 1 + z^2 = 4, \Rightarrow z^2 = 2, \Rightarrow z = \pm\sqrt{2}.$$

$\therefore$ The stationary points are $(1, 1, \sqrt{2})$,$(-1, -1, -\sqrt{2})$,$(-1, -1, \sqrt{2})$ and $(1, 1, -\sqrt{2})$.
From (3.4)

$$1 + 2\lambda_1 z = 0.$$

At the point $(1, 1, \sqrt{2})$

$$1 + 2\lambda_1\sqrt{2} = 0$$
$$\lambda_1 = \pm\frac{1}{2\sqrt{2}}.$$

Substitute λ_1 in (3.2)

$$\begin{aligned} 2\lambda_1 x + \lambda_2 y &= 0 \\ -\frac{1}{2\sqrt{2}} + \lambda_2 &= 0 \\ \lambda_2 &= \pm\frac{-1}{\sqrt{2}}. \end{aligned}$$

$\therefore$ The Lagrangian multiplier are

$$\lambda_1 = \pm\frac{1}{2\sqrt{2}}, \quad \lambda_2 = \pm\frac{-1}{\sqrt{2}}.$$

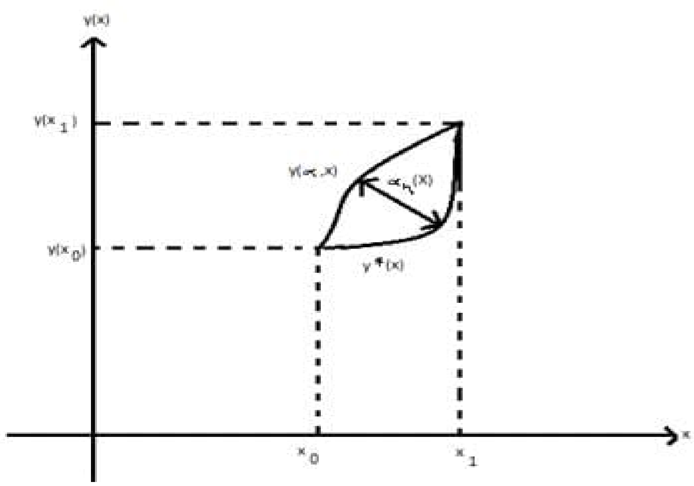
3.1.3 Stationary value of a definite integral

The necessary conditions for a stationary value of the definite integral

$$I = \int_{x_0}^{x} f\left(y(x), y'(x), x\right) dx, \tag{3.7}$$

where I is a functional and $y'(x) = \dfrac{dy}{dx}$. The limits x_0 and x_1 are fixed. Let us consider the curve y to be a function of α and x

$$(ie)., y(\alpha, x) = y^*(x) + \alpha\eta(x). \tag{3.8}$$



Here $\alpha\ \eta(x)$ denotes the variation in y ($\alpha\ \eta(x) = \delta y$) with $\eta(x)$ is an arbitrary value and α is a small parameter, which does not depend on x. Clearly, I is a function of α only.

The necessary condition for the stationary value of I is that its variation must be zero.(ie).,

$$\begin{aligned} \delta I =& 0 \\ \frac{dI}{d\alpha}\delta\alpha =& 0. \end{aligned}$$

Consider

$$\begin{aligned} \frac{dI}{d\alpha} &= \int_{x_0}^{x_1} \left[\frac{\partial f}{\partial y}.\frac{\partial y}{\partial \alpha} + \frac{\partial F}{\partial y'}.\frac{\partial y'}{\partial \alpha} + \frac{\partial f}{\partial x}.\frac{\partial x}{\partial \alpha}\right] dx \\ &= \int_{x_0}^{x_1} \left[\frac{\partial f}{\partial y}.\frac{\partial y}{\partial \alpha} + \frac{\partial F}{\partial y'}.\frac{\partial y'}{\partial \alpha}\right] dx. \end{aligned} \tag{3.9}$$

From (3.8),

$$\begin{aligned} \frac{\partial y}{\partial \alpha} =& \eta(x) \\ y'(\alpha, x) =& y'^{*}(x) + \alpha\eta'(x) \\ \frac{\partial y'}{\partial \alpha} =& \eta'(x). \end{aligned}$$

Substitute these values in (3.9)

$$\begin{aligned} \frac{dI}{d\alpha} &= \int_{x_0}^{x_1} \left[\frac{\partial f}{\partial y}.\eta(x) + \frac{\partial F}{\partial y'}.\eta'(x)\right] dx \\ &= \int_{x_0}^{x_1} \frac{\partial f}{\partial y}.\eta(x)dx + \int_{x_0}^{x_1} \frac{\partial F}{\partial y'}.\eta'(x)dx \\ &= \int_{x_0}^{x_1} \frac{\partial f}{\partial y}.\eta(x)dx + \int_{x_0}^{x_1} \frac{\partial F}{\partial y'}.d\left(\eta(x)\right) dx. \end{aligned}$$

Let

$$u = \frac{\partial f}{\partial y'}, \int dv = \int d(\eta(x))$$
$$du = \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) dx, v = \eta(x)$$
$$\frac{dI}{d\alpha} = \frac{\partial f}{\partial y}.\eta(x)dx + \left\{ \left[\frac{\partial F}{\partial y'}.\eta(x)\right]_{x_0}^{x_1} - \int\limits_{x_0}^{x_1} \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\eta(x)dx \right\}$$
$$= \int\limits_{x_0}^{x_1} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\eta(x)dx$$

$$\therefore \int\limits_{x_0}^{x_1} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\eta(x)dx = 0$$
$$\therefore \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0.$$

3.1.4 Brachistochrone problem

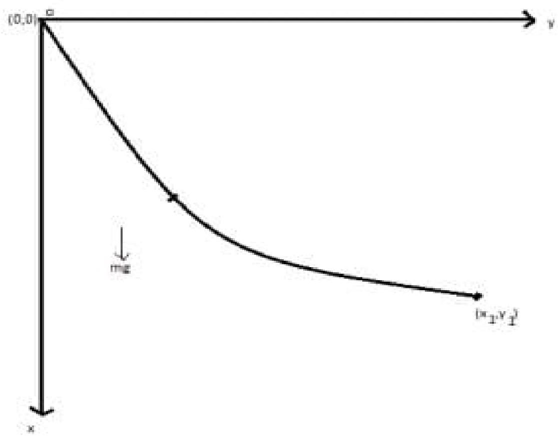
To find a curve $y(x)$ between the origin O and the point (x_1, y_1), such that a particle is starting from reset at O and sliding down the curve with out friction under the influence of a uniform gravitational field, will reach the end of the curve in a minimum time.

Let t be the time required by the particle to reach the point (x_1, y_1)

$$v = \frac{ds}{dt}$$
$$dt = \frac{ds}{v}$$
$$\int dt = \int\limits_0^{s_1} \frac{ds}{v}$$
$$t = \int\limits_0^{s_1} \frac{ds}{v}. \tag{3.10}$$

The infinitesimal path element is given by

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$
$$= \sqrt{1 + y'^2} dx. \tag{3.11}$$



By the principle of conservation of energy

Total kinetic energy = Total potential energy

$$\frac{1}{2}mv^2 = mgx$$
$$v = \sqrt{2gx}. \tag{3.12}$$

Substitute (3.11) and (3.12) in (3.10)

$$t = \int_0^{x_1} \sqrt{\frac{1+y^{'2}}{2gx}} dx.$$

Comparing the above equation with

$$I = \int_0^{x_1} f\left(y, y^{'}, x\right) dx$$
$$f\left(y, y^{'}, x\right) = \sqrt{\frac{1+y^{'2}}{2gx}}$$
$$f\left(y, y^{'}, x\right) = \left(\frac{1+y^{'2}}{2gx}\right)^{\frac{1}{2}}.$$

Euler-Lagrange equation,

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

$$\frac{\partial f}{\partial y} = 0$$

$$\frac{\partial f}{\partial y'} = \left(\frac{y^{'}}{2gx}\right)\frac{1}{\left(\frac{1+y^{'2}}{2gx}\right)^{\frac{1}{2}}}$$

$$= \frac{y^{'}}{\sqrt{2gx(1+y^{'2})}}$$

$$\therefore -\frac{d}{dx}\left(\frac{y^{'}}{\sqrt{2gx(1+y^{'2})}}\right) = 0.$$

Integrating on both sides,

$$\frac{y^{'}}{\sqrt{2gx(1+y^{'2})}} = c \quad (constant)$$

$$y^{'} = c\sqrt{2gx(1+y^{'2})}$$

$$y^{'2} = 2gxc^2 + 2gxc^2y^{'2}$$

$$y^{'2} - 2gxc^2y^{'2} = 2gxc^2$$

$$y^{'2} = \frac{2gxc^2}{(1-2gxc^2)}$$

$$y^{'} = \sqrt{\frac{2gxc^2}{(1-2gxc^2)}}$$

$$dy = \sqrt{\frac{2gxc^2}{(1-2gxc^2)}}dx.$$

The transformation equation of a curve

$$
\begin{aligned}
x &= a(1-\cos\theta)\\
dx &= \frac{1}{4gc^2}(\sin\theta)d\theta, where \quad a=\frac{1}{4gc^2}\\
dy &= \sqrt{\frac{2g(1-\cos\theta)c^2\times\frac{1}{4gc^2}}{(1-2g(1-cos\theta)c^2)\times\frac{1}{4gc^2}}}\frac{1}{4gc^2}(sin\theta)d\theta\\
&= \frac{1}{4gc^2}\sqrt{\frac{(1-\cos\theta)}{(1+\cos\theta)}}\sin\theta d\theta\\
&= \frac{1}{4gc^2}\sqrt{\frac{(1-\cos\theta)}{(1+\cos\theta)}\times\frac{(1-\cos\theta)}{(1-\cos\theta)}}\sin\theta d\theta\\
&= \frac{1}{4gc^2}\sqrt{\frac{(1-\cos\theta)^2}{1^2-\cos^2\theta}}\sin\theta d\theta\\
dy &= \frac{1}{4gc^2}(1-\cos\theta)d\theta
\end{aligned}
$$

Integrating,

$$
\begin{aligned}
\int dy &= \frac{1}{4gc^2}\int(1-\cos\theta)d\theta\\
y &= \frac{1}{4gc^2}(\theta-\sin\theta)+k. \qquad (3.13)
\end{aligned}
$$

Using the initial conditions ,

$$
x=0, y=0 \quad and \quad \theta=0 \quad in \quad (3.13)
$$

$$
\Rightarrow k=0. \qquad (3.14)
$$

Substitute (3.14) in (3.13) we get,

$$
y=a(\theta-\sin\theta),\ where \quad a=\frac{1}{4gc^2}
$$

$\therefore$ The equations $x=a(1-\cos\theta)$ and $y=a(\theta-\sin\theta)$ are parametric equations of a cycloid. The cycloid path leads to a stationary value of t. It is actually the path of a minimum time.

3.1.5 Geodesic problem

The problem of finding the shortest path between two points in the space.

Solution:

Let us consider the problem of finding the path of minimum length between two given points on the 2-dimensional surface of a sphere of radius r. Let us use the spherical co-ordinate (θ, ϕ) as variable and r is a constant.

We know that

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta.$$

The differential element of length ds is given by,

$$ds^2 = r^2 d\theta^2 + r^2\sin^2\theta d\phi^2$$
$$ds = \pm\sqrt{r^2 d\theta^2 + r^2\sin^2\theta d\phi^2}$$

Consider

$$ds = \sqrt{r^2 d\theta^2 + r^2\sin^2\theta d\phi^2}$$
$$ds = r\sqrt{1 + \sin^2\theta\left(\frac{d\phi}{d\theta}\right)^2}d\theta$$
$$s = r\int_{\theta_0}^{\theta_1}\sqrt{1 + \phi'^2\sin^2\theta}d\theta.$$

Comparing the above equation with

$$I = \int_0^{x_1} f\left(y, y', x\right)dx$$
$$f\left(y, y', x\right) = \sqrt{1 + \phi'^2\sin^2\theta} = (1 + \phi'^2\sin^2\theta)^{\frac{1}{2}}.$$

Euler-Lagrange's equation,

$$\begin{aligned}
\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) &= 0\\
\frac{\partial f}{\partial \phi} - \frac{d}{d\theta}\left(\frac{\partial f}{\partial \phi'}\right) &= 0\\
\frac{\partial f}{\partial \phi} &= 0\\
\frac{\partial f}{\partial \phi'} &= \frac{1}{2}(1+\phi'^2 sin^2\theta)^{\frac{-1}{2}}.2\phi'\sin^2\theta\\
\frac{\partial f}{\partial \phi'} &= \frac{\phi'\sin^2\theta}{\sqrt{1+\phi'^2\sin^2\theta)}}\\
\therefore \frac{d}{d\theta}\left(\frac{\phi'\sin^2\theta}{\sqrt{1+\phi'^2\sin^2\theta)}}\right) &= 0.
\end{aligned}$$

Integrating on both sides,

$$\begin{aligned}
\frac{\phi'\sin^2\theta}{\sqrt{1+\phi'^2\sin^2\theta)}} &= c \quad (constant)\\
\phi' &= c\frac{\sqrt{1+\phi'^2\sin^2\theta)}}{\sin^2\theta}\\
\phi'^2 &= \frac{c^2\left(1+\phi'^2\sin^2\theta)\right)}{\sin^4\theta}\\
\phi'^2\sin^4\theta - c^2\phi'^2\sin^2\theta = c^2 \Rightarrow \phi'^2 &= \frac{c^2}{\sin^2\theta(\sin^2\theta - c^2)}\\
\phi' &= \frac{c}{\sin\theta\sqrt{\sin^2\theta - c^2}}\\
d\phi &= \frac{cd\theta}{\sin\theta\sqrt{\sin^2\theta - c^2}}
\end{aligned}$$

Integrating,

$$\begin{aligned}
\phi &= \int\frac{cd\theta}{\sin\theta\sqrt{\sin^2\theta - c^2}}\\
&= \int\frac{cd\theta}{\sin^2\theta\sqrt{1-\dfrac{c^2}{\sin^2\theta}}}\\
&= \int\frac{cd\theta}{\sin^2\theta\sqrt{1-c^2\csc^2\theta}}\\
&= \int\frac{cd\theta}{\sin^2\theta\sqrt{1-c^2(1+\cot^2\theta)}}\\
\phi &= \int\frac{cd\theta}{\sin^2\theta\sqrt{\sqrt{(1-c^2)^2}-c^2\cot^2\theta}}. \qquad (3.15)
\end{aligned}$$

Let $\quad y = c\cot\theta = c\dfrac{\cos\theta}{\sin\theta}$

$$
\begin{aligned}
dy &= c\left(\frac{-\sin^2\theta - \cos^2\theta}{\sin^2\theta}\right)d\theta \\
&= \frac{-cd\theta}{\sin^2\theta}.
\end{aligned}
$$

From (3.15),

$$
\begin{aligned}
\therefore \phi &= \int \frac{dy}{\sqrt{\sqrt{1-c^2}-y^2}} \\
&= -\sin^{-1}\left(\frac{y}{\sqrt{1-c^2}}\right) + k \\
&= \cos^{-1}\left(\frac{y}{\sqrt{1-c^2}}\right) - \frac{\pi}{2} + k \\
&= \cos^{-1}\left(\frac{y}{\sqrt{1-c^2}}\right) + \phi_0 \\
\phi &= \cos^{-1}\left(\frac{c\cot\theta}{\sqrt{1-c^2}}\right) + \phi_0 \\
\phi - \phi_0 &= \cos^{-1}\left(\frac{c\cot\theta}{\sqrt{1-c^2}}\right) \\
\Rrightarrow \cos(\phi - \phi_0) &= \frac{c}{\sqrt{1-c^2}}\cot\theta
\end{aligned}
$$

$$
\begin{aligned}
\cos\phi\cos\phi_0 + \sin\phi\sin\phi_0 &= \frac{c}{\sqrt{1-c^2}}\cot\theta \\
\cos\phi\cos\phi_0 + \sin\phi\sin\phi_0 - \frac{c}{\sqrt{1-c^2}} \times \frac{\cos\theta}{\sin\theta} &= 0 \\
\sin\theta\cos\phi\cos\phi_0 + \sin\theta\sin\phi\sin\phi_0 - \frac{c\cos\theta}{\sqrt{1-c^2}} &= 0.
\end{aligned}
$$

The transformation equations are

$$
x = r\sin\theta\cos\phi.
$$

$$
y = r\sin\theta\cos\phi.
$$

$$
z = r\cos\theta.
$$

$$
\begin{aligned}
\frac{x}{r}\cos\phi_0 + \frac{y}{r}\sin\phi_0 - \frac{c}{\sqrt{1-c^2}} \times \frac{z}{r} &= 0. \\
x\cos\phi_0 + y\sin\phi_0 - \frac{c}{\sqrt{1-c^2}}z &= 0.
\end{aligned}
$$

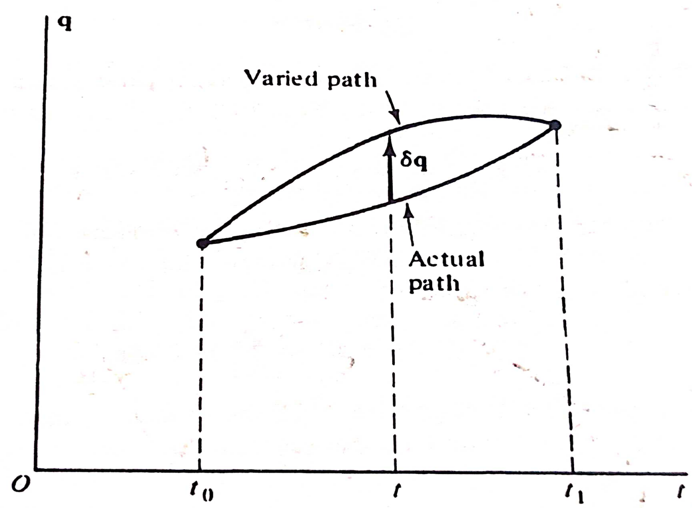
This is the equation of the plane passing through the origin. This plane intersects the sphere in a great circle which is geodesic.

3.1.6 Hamilton's principle

The actual path in configuration space followed by a holonomic dynamical system during the fixed interval t_0 and t_1 such that the integral,

$$I = \int_{t_0}^{t_1} L dt$$

is stationary with respect to the path variations which vanishes at the end points.



Let us consider a system of N particles whose position vectors are given by $r_1, r_2, ..., r_N$. The Lagrangian form of D'Alembert's principle is

$$\sum_{i=1}^{N} \left(\overrightarrow{F_i} - m_i \ddot{\overrightarrow{r_i}} \right) \delta \overrightarrow{r_i} = 0$$

$$\sum_{i=1}^{N} \overrightarrow{F_i} \delta \overrightarrow{r_i} = \sum_{i=1}^{N} m_i \ddot{\overrightarrow{r_i}} \delta \overrightarrow{r_i}, \tag{3.16}$$

where $\overrightarrow{F_i}$ is the applied force and $\delta \overrightarrow{r_i}$ is the virtual displacement.

Now the kinetic energy is given by,

$$T = \frac{1}{2} \sum_{i=1}^{N} m_i \dot{\overrightarrow{r_i}}^2$$

$$\delta T = \frac{1}{2} \sum_{i=1}^{N} m_i 2 \dot{\overrightarrow{r_i}} \delta \dot{\overrightarrow{r_i}}$$

$$= \sum_{i=1}^{N} m_i \dot{\overrightarrow{r_i}} \delta \dot{\overrightarrow{r_i}} \tag{3.17}$$

$$Now \quad \frac{d}{dt} \left(\sum_{i=1}^{N} m_i \dot{\overrightarrow{r_i}} \delta \overrightarrow{r_i} \right) = \sum_{i=1}^{N} m_i \ddot{\overrightarrow{r_i}} \delta \overrightarrow{r_i} + \sum_{i=1}^{N} m_i \dot{\overrightarrow{r_i}} \delta \dot{\overrightarrow{r_i}}. \tag{3.18}$$

Substitude (3.16) and (3.17) in (3.18),

$$\begin{aligned}\frac{d}{dt}\left(\sum_{i=1}^{N} m_i \dot{\overrightarrow{r_i}} \delta \overrightarrow{r_i}\right) &= \sum_{i=1}^{N} \overrightarrow{F_i} \delta \overrightarrow{r_i} + \delta T \\ &= \delta \left(W + T\right).\end{aligned}$$

Integrating the above equation with respect to the fixed points t_0 and t_1

$$\int_{t_0}^{t_1} \delta \left(W + T\right) dt = 0. \tag{3.19}$$

Case (i):

Let us consider the transformation to generalized co-ordinates $q_1, q_2, ..., q_n$ then the virtual work is given by

$$\delta W = \sum_{i=1}^{n} Q_i \delta q_i. \tag{3.20}$$

Where Q_i is the generalized force. Substitute (3.20) in (3.19),

$$\begin{aligned}\int_{t_0}^{t_1} \left(\delta W + \delta T\right) dt &= 0 \\ \int_{t_0}^{t_1} \left(\sum_{i=1}^{n} Q_i \delta q_i + \delta T\right) dt &= 0.\end{aligned}$$

Case (ii):

If the applied force are derived from a potential function then $\delta W = -\delta V$

$$\begin{aligned}\int_{t_0}^{t_1} \left(\delta W + \delta T\right) dt = 0 \\ \int_{t_0}^{t_1} \delta \left(T - V\right) dt = 0 \\ \int_{t_0}^{t_1} \delta L dt = 0 \\ \delta \int_{t_0}^{t_1} L dt = 0 \\ \delta I = 0.\end{aligned}$$

Hence I is stationary.

Remark:

Hamilton's principle and Lagrangian principle are equivalent. Since $L(q, \dot{q}, t)$ corresponds to $f(y, y^{'}, x)$.

Let us sum up

1. We have introduced basic concepts to obtained the stationary values of a function by using Lagrangian multiplier method and Euler - Lagrange equation.

2. In Brachistochrome problem, we have find the path of the curve which particle sliding down from rest under gravitational force from one point to another point in minimum time.

3. In Geodesic problem, we have obtained the shortest path between two points in a given space.

4. We have derived the Hamilton's principle to find the possible paths of the dynamical system.

Check your progress

1.Write the necessary and sufficient condition for stationary values by using Lagrangian multiplier method.

2. State the principle of conservation energy.

3. Write Euler - Lagrange equation.

4. write the parametric equations of cycloid.

5. Define Hamilton's principle.

3.2 HAMILTON'S CANONICAL EQUATIONS

Dear students, in this section we will derive the Hamilton's equations. We will discuss the mass-spring system and also Kepler's problem using the Hamilton canonical equation of motion. Finialy, we introduce the Legendre transformation.

3.2.1 Derivation of Hamilton's equations

Let us consider a holonomic system described by the standard form of Lagrangian equation as,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0,\ i = 1, 2, ..., n. \tag{3.21}$$

We know that

$$p_i = \frac{\partial L}{\partial \dot{q}_i}. \tag{3.22}$$

Then (3.21) can be written as,

$$\dot{p}_i = \frac{\partial L}{\partial q_i}.$$

Define the Hamiltonian function $H(q, p, t)$ as follows,

$$H(q, p, t) = \sum_{i=1}^{n} p_i \dot{q}_i - L(q, \dot{q}, t). \tag{3.23}$$

Now $H(q, p, t)$

$$\delta H = \sum_{i=1}^{n} \frac{\partial H}{\partial q_i}\delta q_i + \sum_{i=1}^{n} \frac{\partial H}{\partial p_i}\delta p_i + \frac{\partial H}{\partial t}\delta t. \tag{3.24}$$

Similarly giving the arbitrary variation for(3.23)

$$\begin{aligned}
H(q, p, t) &= \sum_{i=1}^{n} \phi \dot{q}_i - L(q, \dot{q}, t) \\
\delta H &= \sum_{i=1}^{n} p_i \delta \dot{q}_i + \sum_{i=1}^{n} \dot{q}_i \delta p_i - \left(\sum_{i=1}^{n} \frac{\partial L}{\partial q_i}\delta q_i + \sum_{i=1}^{n} \frac{\partial L}{\partial \delta \dot{q}_i}\delta \dot{q}_i + \frac{\partial L}{\partial t}\delta t\right) \\
&= \sum_{i=1}^{n} p_i \delta \dot{q}_i + \sum_{i=1}^{n} \dot{q}_i \delta p_i - \sum_{i=1}^{n} \frac{\partial L}{\partial q_i}\delta q_i - \sum_{i=1}^{n} \frac{\partial L}{\partial \delta \dot{q}_i}\delta \dot{q}_i - \frac{\partial L}{\partial t}\delta t \\
\delta H &= \sum_{i=1}^{n} \dot{q}_i \delta p_i - \sum_{i=1}^{n} \dot{p}_i \delta q_i - \frac{\partial L}{\partial t}\delta t.
\end{aligned} \tag{3.25}$$

Equating the coefficient of δq_i, δp_i and δt from eqn (3.24) and (3.25) we get,

$$\begin{aligned}
\frac{\partial H}{\partial q_i} &= -\dot{p}_i,\ i = 1, 2, ..., n \\
\frac{\partial H}{\partial p_i} &= \dot{q}_i \\
\frac{\partial H}{\partial t}\delta t &= -\frac{\partial L}{\partial t}\delta t.
\end{aligned} \tag{3.26}$$

The $2n$ first order equations given in equation (3.26) are known as Hamilton's canonical equations of motion.

Specialcase :

Let us consider a Lagrange's equation of the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i^{'}.$$

(Where the generalized force are not all derived from potential equation). For holonomic sysem, the Lagrange's equation is,

$$\dot{p}_i = \frac{\partial L}{\partial q_i} + Q_i^{'}, (i = 1, 2, ..., n)$$

The Hamilton's equation are,

$$\dot{q}_i = \frac{\partial H}{\partial p_i}$$
$$\dot{p}_i = \frac{\partial H}{\partial q_i} + Q_i^{'}, (i = 1, 2, ..., n)$$

For non-holonomic system, The Lagrange's equation are given by,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \sum_{j=1}^{m} \lambda_j a_{ji} + Q_i^{'}, (i = 1, 2, ..., n)$$

Then the Hamilton's equation are,

$$\dot{q}_i = \frac{\partial H}{\partial p_i}$$
$$\dot{p}_i = -\frac{\partial H}{\partial q_i} + \sum_{j=1}^{m} \lambda_j a_{ji} + Q_i^{'}, \ (i = 1, 2, ..., n). \tag{3.27}$$

Where the constraint equations are,

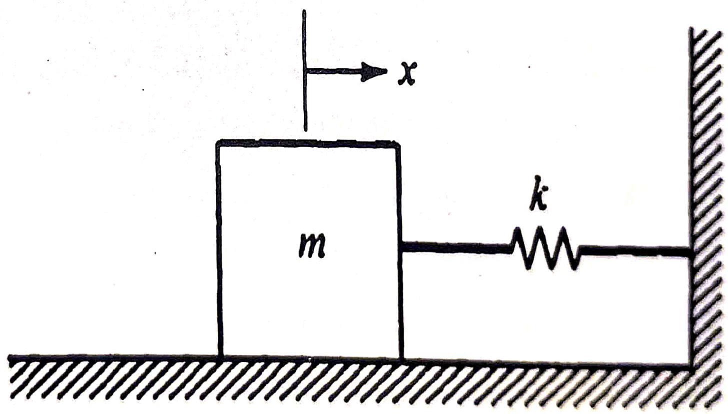
$$\sum_{i=1}^{n} a_{ji}\dot{q}_i + a_{jt} = 0, (j = 1, 2, ..., m). \tag{3.28}$$

From (3.27) and (3.28) we can solve for nq's, np's and $m\lambda$'s as function of time.

3.2.2 Discussion of mass-spring system using the Hamilton procedure

Given a mass-spring system consisting of a mass m and a linear spring of stiffness k. Find the equations of motion using the Hamiltonian procedure.

Solution: Assume that the displacement x is measured from the unstressed position of the spring. First let us find the kinetic and potential energies in the usual form.



We obtain $T = \frac{1}{2}m\dot{x}^2$ and $V = \frac{1}{2}kx^2$ which results in

$$L = T - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2.$$

The linear momentum is

$$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \Rightarrow \dot{x} = \frac{p}{m}$$

Hence we can write the kinetic energy in the form $T = \dfrac{p^2}{2m}$, and the Hamiltonian function is found to be

$$\begin{aligned} H(x,p) = p\dot{x} - L &= p(\frac{p}{m}) - \left(\frac{1}{2}m(\frac{p}{m})^2 - \frac{1}{2}kx^2\right) \\ &= \frac{p^2}{2m} + \frac{1}{2}kx^2. \end{aligned} \tag{3.29}$$

Since this is a natural system, the Hamiltonian H is equal to the total energy $T + V$ and is constant.

To obtain the equations of motion, by using

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, (i = 1, 2, ..., n).$$
$$\dot{p}_i = -\frac{\partial H}{\partial q_i}.$$

Here use $q_i = x$ and $p_i = p$ in above equations in (3.29), we get

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m}.$$
$$p = m\dot{x} \Rightarrow \dot{p} = m\ddot{x}. \tag{3.30}$$
$$\dot{p} = -\frac{\partial H}{\partial x} = -kx. \tag{3.31}$$

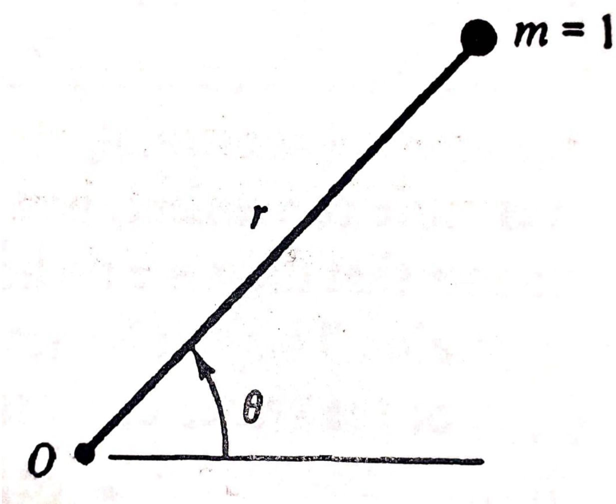
Two first order equations (3.30) and (3.31) are equivalent to the single second order equation

$$m\ddot{x} + kx = 0. \tag{3.32}$$

This equation can be obtained by using Newton's laws of motion or Lagrange's equation.

3.2.3 Discussion of Kepler's problem using Hamilton procedure

A particle of mass m is attracted to a fixed point O by an inverse square force. Find the equation of motion.



Solution : Given a particle of mass m is attracted to a fixed point O by an inverse square force

$$(\text{ie.,})F_r = \frac{-\mu m}{r^2}.$$

Where μ is a gravitational constant

$$\begin{aligned}
T &= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right). \\
V &= \frac{-\mu m}{r}. \\
L &= T - V \\
&= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{\mu m}{r}.
\end{aligned}$$

The Hamiltonian equation is given by,

$$\begin{aligned}
H = T + V &= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{\mu m}{r} \\
&= \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} - \frac{\mu m}{r}. \qquad (3.33)
\end{aligned}$$

From (3.33)

$$\begin{aligned}
\dot{r} &= \frac{\partial H}{\partial p_r} = \frac{p_r}{m} \\
\dot{\theta} &= \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{r^2 m} \\
\dot{p}_r &= -\frac{\partial H}{\partial r} = \frac{p_\theta^2}{mr^3} - \frac{\mu m}{r^2} \\
\dot{p}_\theta &= -\frac{\partial H}{\partial \theta} = 0 \\
p_\theta &= \beta \quad (constant)
\end{aligned}$$

$$\begin{aligned}
\dot{r} &= \frac{p_r}{m} \\
\ddot{r} &= \frac{\dot{p}_r}{m} \\
\ddot{r} &= \frac{1}{m}\left[\frac{p_\theta^2}{mr^3} - \frac{\mu m}{r^2}\right] \\
\ddot{r} &= \frac{\beta^2}{m^2r^3} - \frac{\mu}{r^2} \\
\Rightarrow m\ddot{r} - \frac{\beta^2}{m^2r^3} + \frac{\mu}{r^2} &= 0.
\end{aligned}$$

Next, another method of obtaining Hamilton's equations from Lagrange's equations is by means of Legendre transformation.

3.2.4 The Legendre transformation

Consider a function $F(u_1, u_2, ..., u_n, w_1, w_2, ..., w_m, t)$, where u' s are active variables and w 's and t are passive variables. Let us define a new set of active variables as,

$$v_i = \frac{\partial F}{\partial u_i}, (i = 1, 2, ..., n) \tag{3.34}$$

Now,

$$\left| \frac{\partial^2 F}{\partial u_i \partial u_j} \right| = \left| \frac{\partial v_i}{\partial u_j} \right| \neq 0.$$

Define a new function $F(v_1, v_2, ..., v_n, w_1, w_2, ..., w_m, t)$ as

$$G = \sum_{i=1}^{n} u_i v_i - F$$

$$\delta G = \sum_{i=1}^{n} \frac{\partial G}{\partial v_i} \delta v_i. \tag{3.35}$$

Differentiate with respect to active variables,

$$\begin{aligned} \delta G &= \delta \left[\sum_{i=1}^{n} u_i v_i - F \right] \\ &= \sum_{i=1}^{n} \delta(u_i v_i) - \delta F \\ &= \sum_{i=1}^{n} u_i \delta v_i + \sum_{i=1}^{n} v_i \delta u_i - \delta F \\ &= \sum_{i=1}^{n} u_i \delta v_i + \sum_{i=1}^{n} v_i \delta u_i - \sum_{i=1}^{n} \frac{\partial F}{\partial u_i} \delta u_i \\ &= \sum_{i=1}^{n} u_i \delta v_i + \sum_{i=1}^{n} \left(v_i - \frac{\partial F}{\partial u_i} \right) \delta u_i. \end{aligned} \tag{3.36}$$

From (3.35) and (3.36)

$$\sum_{i=1}^{n} \frac{\partial G}{\partial v_i} \delta v_i = \sum_{i=1}^{n} u_i \delta v_i + \sum_{i=1}^{n} \left(v_i - \frac{\partial F}{\partial u_i} \right) \delta u_i.$$

Comparing the coefficients of δv_i

$$u_i = \frac{\partial G}{\partial v_i} \tag{3.37}$$

$$v_i - \frac{\partial F}{\partial u_i} = 0$$

$$v_i = \frac{\partial F}{\partial u_i}.$$

To obtain Hamilton canonical form by using Legendre transformation : Consider the Hamilton function

$$H(p,q,t) = \sum_{i=1}^{n} p_i\dot{q}_i \text{ - } L(q,\dot{q},t)$$

Varying H w.r.t active variables

$$\delta H = \sum_{i=1}^{n} \frac{\partial H}{\partial p_i}\delta p_i. \tag{3.38}$$

$$\begin{aligned}
\delta H &= \delta\left(\sum_{i=1}^{n} p_i\dot{q}_i\right) - \sum_{i=1}^{n}\frac{\partial L}{\partial \dot{q}_i}\delta\dot{q}_i \\
&= \sum_{i=1}^{n} p_i\delta\dot{q}_i + \sum_{i=1}^{n}\delta p_i\dot{q}_i - \sum_{i=1}^{n}\frac{\partial L}{\partial \dot{q}_i}\delta\dot{q}_i \\
&= \sum_{i=1}^{n}\delta p_i\dot{q}_i + \sum_{i=1}^{n}\left(p_i - \frac{\partial L}{\partial \dot{q}_i}\right)\delta\dot{q}_i.
\end{aligned} \tag{3.39}$$

From (3.38) and (3.39)

$$\sum_{i=1}^{n}\frac{\partial H}{\partial p_i}\delta p_i = \sum_{i=1}^{n}\delta p_i\dot{q}_i + \sum_{i=1}^{n}\left(p_i - \frac{\partial L}{\partial \dot{q}_i}\right)\delta\dot{q}_i$$

$$\frac{\partial H}{\partial p_i} = \dot{q}_i \quad and \quad \frac{\partial L}{\partial \dot{q}_i} = p_i.$$

Varying H w.r.t passive variables

$$\delta H = \sum_{i=1}^{n}\frac{\partial H}{\partial q_i} + \frac{\partial H}{\partial t}\delta t. \tag{3.40}$$

$$\delta H = -\sum_{i=1}^{n}\frac{\partial L}{\partial q_i} - \frac{\partial L}{\partial t}\delta t. \tag{3.41}$$

From(3.40) and (3.41)

$$\sum_{i=1}^{n}\frac{\partial H}{\partial q_i} + \frac{\partial H}{\partial t}\delta t = -\sum_{i=1}^{n}\frac{\partial L}{\partial q_i} - \frac{\partial L}{\partial t}\delta t$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}.$$

Hence the equation of motion are,

$$\dot{q}_i = \frac{\partial H}{\partial p_i}.$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}.$$

Let us sum up

1. We have derived the Hamilton's equations.

2. We have discussed the mass- spring system and Kepler's problem by using Hamilton equations.

3. We have obtained the Hamilton canonical form by using Legendre transformation.

Check your progress

6. Define Hamilonian function.

7. Write the Hamilton's canonical equation for holonomic system.

8. Write the Hamilton's canonical equation for non-holonomic system.

9. Write the equations for generalized momenta.

3.3 Some other Variational Principles

Dear students, in this section we will derive the modified Hamilton's principles, Principle of least action and also discuss of Kepler'S problem by using Jacobi form.

3.3.1 Modified Hamilton's principle

Let us consider a holonomic system. The usual form of Hamilton's principle is given by,

$$\delta \int_{t_0}^{t_1} L dt = 0.$$

Using

$$H = \sum_{i=1}^{n} p_i \dot{q}_i - L, \quad L = \sum_{i=1}^{n} p_i \dot{q}_i - H.$$

The modified Hamilton's principle is

$$\delta \int_{t_0}^{t_1} \left(\sum_{i=1}^{n} p_i \dot{q}_i - H \right) dt = 0. \tag{3.42}$$

$$\int_{t_0}^{t_1} \sum_{i=1}^{n} \left(\delta p_i \dot{q}_i + p_i \delta \dot{q}_i - \frac{\partial H}{\partial q_i} \delta q_i - \frac{\partial H}{\partial p_i} \delta p_i \right) dt = 0. \tag{3.43}$$

Consider

$$\int_{t_0}^{t_1} \sum_{i=1}^{n} p_i \delta \dot{q}_i dt = \int_{t_0}^{t_1} \sum_{i=1}^{n} p_i \frac{d}{dt} (\delta \dot{q}_i) \, dt$$
$$= -\sum_{i=1}^{n} \int_{t_0}^{t_1} \dot{p}_i \delta q_i dt$$

Now (3.43) becomes

$$\int_{t_0}^{t_1} \sum_{i=1}^{n} \left(\delta p_i \dot{q}_i - \delta q_i \dot{p}_i - \frac{\partial H}{\partial q_i} \delta q_i - \frac{\partial H}{\partial p_i} \delta p_i \right) dt = 0$$

$$\int_{t_0}^{t_1} \sum_{i=1}^{n} \left[\left(\dot{q}_i - \frac{\partial H}{\partial p_i} \right) \delta p_i - \left(\dot{p}_i + \frac{\partial H}{\partial q_i} \right) \delta q_i \right] dt = 0, \tag{3.44}$$

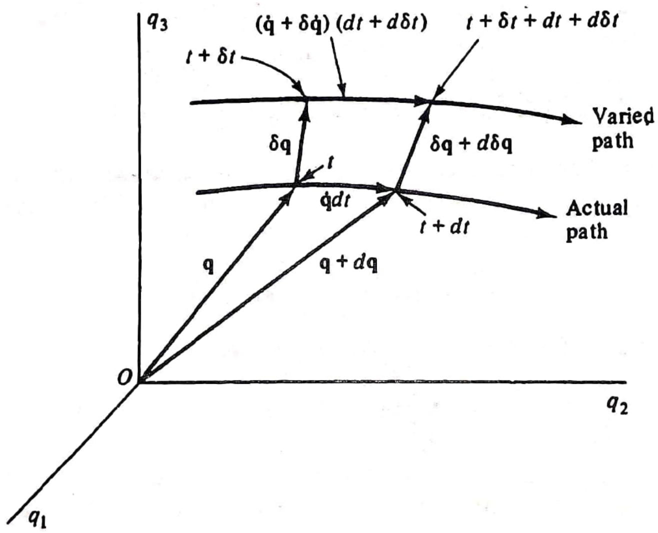
where

$$\dot{q}_i = \frac{\partial H}{\partial p_i}.$$
$$\dot{p}_i = -\frac{\partial H}{\partial q_i}.$$

Because $\delta q's$ are independent in equation (3.44) which implies that the co-efficient must be zero. The modified Hamilton's principle states that the actual part is such that the integral of equation (3.42) is stationary.

3.3.2 Principle of least action

The actual path of a conservative holonomic system such that the action is stationary with respect to varied paths having the same energy integral h and the same end points in q-space.



Consider the quadrilateral $ABCD$,

$$AB + BC = CD + DA$$

$$\dot{q}dt + \delta q + d\delta q = \delta q + (\dot{q} + \delta \dot{q})\,(dt + d\delta t)$$

$$d\delta q = \dot{q}d\delta t + \delta \dot{q}dt + \delta \dot{q}d\delta t$$

$$d\delta q = \dot{q}d\delta t + \delta \dot{q}dt$$

$$\delta \dot{q}dt = -\dot{q}d\delta t + d\delta q$$

$$\delta \dot{q} = \frac{-\dot{q}d\delta t + d\delta q}{dt}. \tag{3.45}$$

In terms of components,

$$\delta \dot{q}_i = -\dot{q}_i \frac{d}{dt}\delta t + \frac{d}{dt}\delta q_i, (i = 1, 2, ..., n). \tag{3.46}$$

Now consider

$$I = \int\limits_{t_0}^{t_1} L dt$$

$$\delta I = \frac{d}{dt}\left(\delta I\right) - I \frac{d}{dt}\left(\delta t\right)$$

$$\delta \left[\frac{d}{dt}\left(I\right)\right] = \frac{d}{dt}\left(\delta I\right) - \frac{d}{dt}\left(I\right)\frac{d}{dt}\left(\delta t\right)$$

$$\delta \left[\frac{d}{dt}\left(\int\limits_{t_0}^{t_1} L dt\right)\right] = \frac{d}{dt}\left(\delta I\right) - \frac{d}{dt}\left(\int\limits_{t_0}^{t_1} L dt\right)\frac{d}{dt}\left(\delta t\right)$$

$$\delta L = \frac{d}{dt}\left[\delta I - \int\limits_{t_0}^{t_1} L \frac{d}{dt}\left(\delta t\right) dt\right]$$

Integrating

$$\int_{t_0}^{t_1} \delta L = \delta I - \int_{t_0}^{t_1} L \frac{d}{dt}(\delta t)\, dt$$

$$\delta I = \int_{t_0}^{t_1} \left[\delta L + L\frac{d}{dt}(\delta t)\right] dt. \tag{3.47}$$

Now

$$\begin{aligned}\delta L &= \sum_{i=1}^{n}\left(\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i\right) + \frac{\partial L}{\partial t}\delta t\\ &= \sum_{i=1}^{n}\left(\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\frac{d(\delta q_i)}{dt} - \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\frac{d(\delta t}{dt}\right) + \frac{\partial L}{\partial t}\delta t. \end{aligned}\tag{3.48}$$

Consider

$$\begin{aligned}\frac{d}{dt}\left(\sum_{i=1}^{n}\frac{\partial L}{\partial \dot{q}_i}\delta q_i\right) &= \sum_{i=1}^{n}\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}(\delta q_i) + \sum_{i=1}^{n}\frac{\partial L}{\partial \dot{q}_i}\frac{d}{dt}(\delta q_i)\\ \sum_{i=1}^{n}\frac{\partial L}{\partial \dot{q}_i}\frac{d}{dt}(\delta q_i) &= \frac{d}{dt}\left(\sum_{i=1}^{n}\frac{\partial L}{\partial \dot{q}_i}\delta q_i\right) - \sum_{i=1}^{n}\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}(\delta q_i).\end{aligned}\tag{3.49}$$

$$\delta L = \sum_{i=1}^{n}\left[\frac{\partial L}{\partial q_i}\delta q_i + \frac{d}{dt}\left(\sum_{i=1}^{n}\frac{\partial L}{\partial \dot{q}_i}\delta q_i\right) - \sum_{i=1}^{n}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\delta q_i - \sum_{i=1}^{n}\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\frac{d\delta t}{dt} + \frac{\partial L}{\partial t}\delta t\right]. \tag{3.50}$$

Substitute (3.49) in (3.48)

$$\delta I = \int_{t_0}^{t_1}\left[\left(\sum_{i=1}^{n}\frac{\partial L}{\partial q_i}\delta q_i + \frac{d}{dt}\left(\sum_{i=1}^{n}\frac{\partial L}{\partial \dot{q}_i}\delta q_i\right) - \sum_{i=1}^{n}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\delta q_i - \sum_{i=1}^{n}\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\frac{d\delta t}{dt} + \frac{\partial L}{\partial t}\delta t\right) + L\frac{d}{dt}(\delta t)\right] dt$$

$$\begin{aligned}&= \int_{t_0}^{t_1}\sum_{i=1}^{n}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\delta q_i\right) dt + \int_{t_0}^{t_1}\left[\frac{\partial L}{\partial t}\delta t - \left(\sum_{i=1}^{n}\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L\right)\frac{d}{dt}(\delta t)\right] dt\\ &\quad - \int_{t_0}^{t_1}\sum_{i=1}^{n}\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i}\right]\delta q_i dt.\end{aligned}\tag{3.51}$$

Let us consider a holonomic system, By setting $\delta t = 0$ in second integral of (3.51) vanishes. By assuming the standard form of Lagrange's equation in (3.51) the second integral vanishes

$$\begin{aligned}\delta I &= \int_{t_0}^{t_1} \frac{d}{dt}\left(\sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i}\delta q_i\right) dt \\ &= \left[\sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i}\delta q_i\right]_{t_0}^{t_1} \\ \Rightarrow \delta I &= 0.\end{aligned}$$

Thus we obtain the Hamilton's principle.

Now let us consider a conservative holonomic system. The path at the end points so the first and third integral vanishes.

$$\delta I = \int_{t_0}^{t_1} \left[L\frac{d}{dt}(\delta t) - \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i \frac{d}{dt}(\delta t)\right] dt.$$

Since $\dfrac{\partial L}{\partial t} = 0$ for a conservative system

$$\begin{aligned}\delta I &= -\int_{t_0}^{t_1} \left[\sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i \frac{d}{dt}(\delta t) - L\frac{d}{dt}(\delta t)\right] dt \\ &= -\int_{t_0}^{t_1} \left[\sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L\right] \frac{d}{dt}(\delta t) dt.\end{aligned}$$

Now $\sum\limits_{i=1}^{n} \dfrac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L = h$ is an energy integral.

$$\begin{aligned}\therefore \delta I &= -\int_{t_0}^{t_1} h\frac{d}{dt}(\delta t) dt \\ \delta I &= -h\left[\delta t\right]_{t_0}^{t_1} = -h\left(\delta t_1 - \delta t_0\right).\end{aligned}$$

Define the action as an integral,

$$A = \int_{t_0}^{t_1} \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i dt = \int_{t_0}^{t_1} \sum_{i=1}^{n} p_i \dot{q}_i dt \quad (\because \frac{\partial L}{\partial \dot{q}_i} = p_i).$$

Now

$$\delta A = \delta \int_{t_0}^{t_1} (L+h)dt$$
$$= \delta I + \int_{t_0}^{t_1} \delta(hdt)$$
$$= \delta I + \int_{t_0}^{t_1} \delta h dt + h \int_{t_0}^{t_1} \delta dt$$
$$= -h(\delta t_1 - \delta t_0) + \delta h(t_1 - t_0) + h(\delta t_1 - \delta t_0) \quad (\because \delta I = -h\,(\delta t_1 - \delta t_0)$$
$$\Rightarrow \delta A = \delta h(t_1 - t_0).$$

Restricting the varied paths to those for which h has the same value as the actual path then $\delta h = 0$. Therefore $\delta A = 0$.

3.3.3 Jacobi's form of the principle of least action

Let us consider a natural system, $\sum_{i=1}^{n} p_i \dot{q}_i = 2T_2 + T_1$. For, $T_1 = 0$ and the principle of least action becomes,

$$\delta A = \delta \int_{t_0}^{t_1} 2T dt$$
$$= \delta \int_{t_0}^{t_1} p_i \dot{q}_i dt = 0$$
$$\delta A = \delta \int_{t_0}^{t_1} 2\sqrt{T(h-V)} dt = 0.$$

If ds is defined as,

$$ds^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} m_{ij} \dot{q}_i \dot{q}_j dt^2 = 2T^2$$
$$ds = \sqrt{2T} dt$$
$$\delta A = \delta \int_{t_0}^{t_1} 2\sqrt{T(h-V)} dt$$

$$\delta A = \delta \int_{t_0}^{t_1} \sqrt{2.2T(h-V)}dt$$

$$\delta A = \delta \int_{t_0}^{t_1} \sqrt{2(h-V)}ds = 0.$$

This is the Jacobi's form of least action.

3.3.4 Discuss of Kepler problem by using Jacobi form of principle of least action

Use the Jacobi form of the principle of least action. Obtaining orbit for the Kepler's problem.

Solution: Let a particalof mass m attracted to a fixed point 'O' by an inverse square force $F_r = \frac{-\mu m}{r^2}$. The K.E and P.E is $T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$ and $V = \frac{-\mu m}{r}$.
Consider the natural system having the total energy is

$$h = T + V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{\mu m}{r} \tag{3.52}$$

Using the Jacobi form of the principle of least action

$$\delta A = \delta \int \sqrt{2\,(h-V)}ds = 0 \tag{3.53}$$

From (3.52) $\quad \delta \int \sqrt{2(h + \frac{\mu m}{r})}ds = 0, where$

$$ds^2 = m(dr^2 + r^2 d\theta^2) \tag{3.54}$$

$$\Rightarrow ds = \delta \int_{\theta_0}^{\theta_1} \sqrt{2(h + \frac{\mu m}{r})(r^2 + r'^2)}d\theta = 0 \quad (\text{Here} \quad r' = \frac{dr}{d\theta}).$$

Now let us choose θ as the independent variable and $r' = \dfrac{dr}{d\theta}$. Then equation (3.53) becomes

$$\delta \int_{\theta_0}^{\theta_1} f(r, r')d\theta = 0 \tag{3.55}$$

$$\delta \int_{\theta_0}^{\theta_1} \sqrt{2m\left(h + \frac{\mu m}{r}\right)(r^2 + r'^2)}d\theta = 0$$

where $f(r,r^{'}) = \sqrt{2m\left(h+\frac{\mu m}{r}\right)(r^2+r'^2)}$.
By Euler-Lagrange equation

$$\frac{d}{d\theta}\left(\frac{\partial f}{\partial r^{'}}\right) - \frac{\partial f}{\partial r} = 0. \tag{3.56}$$

From, we know that,

$$\frac{\partial f}{\partial r^{'}}r^{'} - f = C \quad (constant).$$

$$\begin{aligned}
f(r,r^{'}) &= \sqrt{2m(r'^2+r^2)\left(h+\frac{\mu m}{r}\right)} \\
\frac{\partial f}{\partial r^{'}} &= \frac{2m\left(h+\frac{\mu m}{r}\right)2r^{'}}{2\sqrt{2m(r'^2+r^2)\left(h+\frac{\mu m}{r}\right)}} = \frac{2m\left(h+\frac{\mu m}{r}\right)r^{'}}{\sqrt{2m(r'^2+r^2)\left(h+\frac{\mu m}{r}\right)}} \\
&= \sqrt{\frac{2m\left(h+\frac{\mu m}{r}\right)}{(r'^2+r^2)}}r'
\end{aligned}$$

$$\begin{aligned}
\frac{\partial f}{\partial r^{'}}r^{'} - f &= C \\
\sqrt{\frac{2m\left(h+\frac{\mu m}{r}\right)}{(r'^2+r^2)}}r^{'}r^{'} - \sqrt{2m(r'^2+r^2)\left(h+\frac{\mu m}{r}\right)} &= C \\
\sqrt{2m\left(h+\frac{\mu m}{r}\right)}\left[\frac{r'^2}{\sqrt{(r'^2+r^2)}} - \sqrt{(r'^2+r^2)}\right] &= C \\
\sqrt{2m\left(h+\frac{\mu m}{r}\right)}\left[\frac{r'^2-r'^2-r^2}{\sqrt{(r'^2+r^2)}}\right] &= C \\
-\frac{\sqrt{2m\left(h+\frac{\mu m}{r}\right)}}{\sqrt{(r'^2+r^2)}}r^2 &= C
\end{aligned} \tag{3.57}$$

Now

$$\begin{aligned}
h &= T+V \\
&= \frac{1}{2}m\left(\dot{r}^2+r^2\dot{\theta}^2\right) - \frac{\mu m}{r} \\
m\left(\dot{r}^2+r^2\dot{\theta}^2\right) &= 2\left(h+\frac{\mu m}{r}\right) \Rightarrow m^2\left(\dot{r}^2+r^2\dot{\theta}^2\right) = 2m\left(h+\frac{\mu m}{r}\right)
\end{aligned}$$

$$\dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = r'\dot{\theta} \Rightarrow \dot{\theta} = \frac{\dot{r}}{r'}$$

Values of $\dot{r} = r'\dot{\theta}$ and $\dot{\theta} = \frac{\dot{r}}{r'}$ in above equation.

$$m^2\left(r'^2\dot{\theta}^2 + r^2\dot{\theta}^2\right) = 2m\left(h + \frac{\mu m}{r}\right)$$
$$m^2\dot{\theta}^2\left(r'^2 + r^2\right) = 2m\left(h + \frac{\mu m}{r}\right). \tag{3.58}$$

Equation (3.58) in(3.57),

$$-\sqrt{\frac{m^2\dot{\theta}^2\left(r'^2 + r^2\right)}{\left(r'^2 + r^2\right)}}r^2 = C$$
$$-\sqrt{m^2\dot{\theta}^2}r^2 = C, \Rightarrow m\dot{\theta}r^2 = C$$
$$\dot{\theta} = \frac{-C}{mr^2}.$$

Angular momentum is constant

$$\dot{\theta}^2 = \frac{-C^2}{m^2r^4}$$

To find the equation of orbit,in (3.57) squaring and rearranging

$$2m\left(h + \frac{\mu m}{r}\right)r^4 = C^2\left(r'^2 + r^2\right) = C^2r'^2 + C^2r^2$$
$$C^2r'^2 = 2m\left(h + \frac{\mu m}{r}\right)r^4 - C^2r^2 = 2mr^2\left(hr^2 + \mu mr\right) - \frac{2mC^2r^2}{2m}$$
$$= \frac{2mr^2}{C^2}\left[hr^2 + \mu mr - \frac{C^2}{2m}\right]$$
$$\left(\frac{dr}{d\theta}\right)^2 = \frac{2mr^2}{C^2}\left[hr^2 + \mu mr - \frac{C^2}{2m}\right]$$
$$\frac{dr}{d\theta} = \sqrt{\frac{2mr^2}{C^2}\left[hr^2 + \mu mr - \frac{C^2}{2m}\right]}$$
$$\frac{dr}{\sqrt{\frac{2mr^2}{C^2}\left[hr^2 + \mu mr - \frac{C^2}{2m}\right]}} = d\theta$$
$$\int d\theta = \frac{C}{\sqrt{2m}}\int\frac{dr}{\sqrt{hr^4 + \mu mr^2 - C^2\frac{r^2}{2m}}}$$
$$\theta = \frac{C}{\sqrt{2m}}\int_{r_0}^{r}\frac{dr}{r^2\sqrt{h + \frac{\mu m}{r} - \frac{C^2}{2m}}}$$
$$= \int_{r_0}^{r}\frac{d\left(\frac{\mu m^2}{C^2} - \frac{1}{r}\right)}{\sqrt{\frac{\mu^2m^4}{C^4} + \frac{2mh}{C^2} - \left(\frac{\mu m^2}{C^2} - \frac{1}{r}\right)}}$$

$$\theta = sin^{-1}\left[\frac{\left(\frac{\mu m^2}{C^2} - \frac{1}{r}\right)}{\sqrt{\frac{\mu^2 m^4}{C^4} + \frac{2mh}{C^2}}}\right]_{r_0}^{r}$$

$$= sin^{-1}\left[\frac{\left(\frac{\mu m^2}{C^2} - \frac{1}{r}\right)}{\sqrt{\frac{\mu^2 m^4}{C^4} + \frac{2mh}{C^2}}} - \frac{\pi}{2}\right]$$

$$\Rightarrow \sin(\theta + \frac{\pi}{2}) = \frac{\left(\frac{\mu m^2}{C^2} - \frac{1}{r}\right)}{\sqrt{\frac{\mu^2 m^4}{C^4} + \frac{2mh}{C^2}}}$$

$$\Rightarrow \cos\theta\sqrt{\frac{\mu^2 m^4}{C^4} + \frac{2mh}{C^2}} = \frac{\mu m^2}{C^2} - \frac{1}{r}$$

$$\Rightarrow \frac{1}{r} = \frac{\mu m^2}{C^2} - \sqrt{\frac{\mu^2 m^4}{C^4} + \frac{2mh}{C^2}}\cos\theta.$$

Multiplying by $\frac{C^2}{\mu m^2}$ we get

$$\frac{C}{\mu m^2/r} = 1 - \sqrt{1 + \frac{2hC^2}{\mu^2 m^3}}\cos\theta.$$

This is a conic with eccentricity

$$i.e., \sqrt{1 + \frac{2hC^2}{\mu^2 m^3}} \quad \left(\frac{l}{r} = 1 + e\cos\theta\right)$$

To find h:

$$h = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{\mu m}{r}$$

$$Atr = r_0 = r_{min}, \theta = \theta_0 = 0$$

$$\frac{m}{2}\left[0 + r^2\dot{\theta_0}^2\right] = h + \frac{\mu m}{r_0}.$$

Also

$$mr_0^2\theta_0 = -C \Rightarrow \theta_0 = \frac{-C}{mr_0^2}$$

$$\Rightarrow \frac{m}{2}\left[r_0^2\frac{C^2}{m^2 r_0^4}\right] = h + \frac{\mu m}{r_0} \Rightarrow \frac{C^2}{2mr_0^2} = h + \frac{\mu m}{r_0}$$

$$\Rightarrow h = \frac{C^2}{2mr_0^2} - \frac{\mu m}{r_0}.$$

Let us sum up

1. We have derived modified Hamilon's from Hamilton's function.

2. We have obtained optimum path of a conservative holonomic system by using the principle of least action.

3.Also we have find the actual path of a natural system by using Jacobi's form of the principle of least action.

4. We have obtain the optimum path for the Kepler's problem by using Jacobi's form of the principle of least action.

Check your progress

10. State modified Hamilton's principle.

11. Define principle of least action.

12. Write the equation of natural system.

13. Write the equation of Jacobi's form of least action method.

Summary

• Introduced basic concepts to obtained the stationary values of a function by using Lagrangian multiplier method and Euler- Lagrange equation.

• In Brachistochrome problem, we have find the path of the curve which particle sliding down from rest under gravitational force from one point to another point in minimum time.

• In Geodesic problem, we have obtained the shortest path between two points in a given space.

• Derived the Hamilton's principle to find the possible paths of the dynamical system.

• Derived the Hamilton's equations.

• Discussed the mass- spring system and Kepler's problem by using Hamilton equations.

• Obtained the Hamilton canonical form by using Legendre transformation.

• Derived modified Hamilon's from Hamilton's function.

• Obtained optimum path of a conservative holonomic system by using the principle of least action.

• Find the actual path of a natural system by using Jacobi's form of the principle of least action.

• Obtain the optimum path for the Kepler problem by using Jacobi's form of the principle of least action.

Glossary

•**Hamiltonian:** The Hamiltonian of a system is defined to be the sum of the kinetic and potential energies expressed oF a function of positions and their conjugate momenta.

•**Legendre transformation:** It refers to the mathematical method for changing the basis of the dscription of a system from one set of independent variables to another set of independent variables.

•**Multiplier rule:** A standard method for the analysis of these problems is the multiplier rule.

Self-Assessment Questions

Short-Answer Questions:

1) Derive the Euler-Lagrange equation.

2) Find the stationary values of the function $f = z$, subject to the constraints

$\phi_1 = x^2 + y^2 + z^2 - 4$ =0 and ϕ_2= $xy - 1$ =0.

3) With usual derive $q_i = \frac{\partial H}{\partial p_i}, \quad p_i = \frac{\partial H}{\partial q_i}, \quad \frac{\partial L}{\partial t} = \frac{\partial H}{\partial t}$.

4) Discuss Geodesic problem.

5) Explain the Brachistochrone problem.

6) Explain Hamilton's equation of motion.

Long-Answer Questions:

1) Derive the Lagrange's Equation of motion in the standard form for a holonomic system.

2) Solve the differential equations of motion for a spherical pendulum of length 'l' for the integrals of the motion.

3) Derive the Euler-Lagrange Equation and Explain the Brachistochrone problem.

4) State and prove the principle of least action.

5) Derive the Hamilton's equation of motion.

6) A particle of mass m attached to a fixed point O by an inverse square force i.e, $F_r = -\frac{\mu m}{r^2}$, where μ is the gravitational coefficient. Using the polar co-ordinates (r, θ) to describes the position of the particle and find the equation of motion.

7) Derive Hamilton's canonical equation of motion.

8) Deduce the Jacobi's form of the principle of least action.

9) Derive the Euler-Lagrange Equation and prove the Geodesic problem.

10) State and prove the Hamilton's principle.

11) Deduce the Jacobi's form of the principle of least action to obtain for the Kepler problem. Show that Jacobi integral has the unit of energy.

Objective Questions:

1. In which principle it is viewed, the motion as a whole and involves a search for the path in configuration space which yields a stationary value for a certain integra

a) Hamiltonian's principle   b) Lagrange's principle

c) Jacobi principle   d) Principle of least square

2. The necessary and sufficient condition that a function $f(q_1, q_2, ..., q_n)$ have a stationary value q_0 is

a) $\partial f = 1$ b)$\partial f = 0$ c) $\partial f \neq 1$ d) $\partial f \neq 0$

3. The notation $k_{ij} \equiv \left(\frac{\partial^2 f}{\partial q_i q_j}\right)_0$ at the stationary point q_0, are the elements of the $n \times n$ matrix k. Then k is .... matrix.

a) Skew   b) Non symmetry   c) Diagonal   d) Symmetry

4. The notation $k_{ij} \equiv \left(\frac{\partial^2 f}{\partial q_i q_j}\right)_0$ then the sufficient condition that q_0, be a local minim is that the matrix k be. . . . . .

a) Positive definite b) Negative definite c) Positive semi definite d) Indefinite

5. If $k = (k_{ij})_{n\times n}$, where $k_{ij} \equiv \left(\frac{\partial^2 f}{\partial q_i q_j}\right)_0$ at the stationary point q_0, and k is negative definite then the point q_0 is called . . . .

a) Saddle point b) Local maximum c) Local minimum d) Absolute maximum

6. Which method is applied to the problems involving constrained minima or maxima

a) Lagrange multipler method b) Zero derivative principle

c) D'Alembertz principle d) Principle of virtual work.

7. If $k = (k_{ij})_{n\times n}$, where $k_{ij} \equiv \left(\frac{\partial^2 f}{\partial q_i q_j}\right)_0$ at the stationary point q_0, and k is indefinite then the point q_0 is termed as . . . ..

a) Local maximum b) Local minimum c) Focus d) Saddle point

8. If q_0 ia an interior point, $f(q_1, q_2, .., q_n)$ takes on a minimum or a maximum value only it is a . . . . . .

a) Saddle point b) Stationary c) Point of inflexion d) Not a stationary

9. The solution of the Brachistochrone problem is. . . . . . .

a) Circular path b) Elliptic path c) Gaodisc path d) Cycloidal path

10. The problem of finding the shortest path between 2 given points in given space is known as . . . . . .

a) Geodisc problem b) Hamilton's problem c) Brachistochrone problem d) Minimal surface problem

11) Which principle is popularly known as an integrated form of the D'Alembert's principle

a) Hamiltonian's principle b) Lagrange's principle c) Jacobi principle d) Principle of least square

12) $\delta \int_{t_0}^{t_1} L dt = 0$ is known as. . . . . .

a) Lagrange's principle b) Hamiltonian's principle

c) Jacobi principle d) Principle of least square

13) The Hamiltonian function of a scleronomic system is equal to . . . .

a) $H = T + V$ b) $H = T - V$ c) $H = T_2 \check{} T_0 + V$ d) $H = T_0 - T_2 + V$

Answers for Check Your Progress

1. The necessary and sufficient conditions for δF to be stationary is $\delta F = 0$, $\sum\limits_{i=1}^{n} \left(\dfrac{\partial F}{\partial q_i}\right)_0 \delta q_i +$ $\sum\limits_{j=1}^{m} \left(\dfrac{\partial F}{\partial \lambda_j}\right)_0 \delta \lambda_j = 0$. Then $\left(\dfrac{\partial F}{\partial q_i}\right)_0 = 0, \quad (i = 1, 2, ..., n)$ and
$\left(\dfrac{\partial F}{\partial \lambda_j}\right)_0 = 0, \quad (j = 1, 2, ..., m)$.

2. Total kinetic energy = Total potential energy

3. $\dfrac{\partial f}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial f}{\partial y'}\right) = 0$

4. The differential element of length ds is given by, $ds = \pm\sqrt{r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2}$.

5. The actual path in configuration space followed by a holonomic dynamical system during the fixed interval t_0 and t_1 is such that the integral, $I = \int\limits_{t_0}^{t_1} L dt$ is stationary with respect to the path variations which vanishes at the end points.

6. Define the Hamiltonian function H(q, p, t) as follows, $H(q, p, t) = \sum\limits_{i=1}^{n} p_i \dot{q}_i$- $L(q, \dot{q}, t)$.

7. For holonomic sysem, the Lagrange's equation is, $\dot{p}_i = \dfrac{\partial L}{\partial q_i} + Q_i^{'}, (i = 1, 2, ..., n)$.

8. For non-holonomic system ,The lagrange's equation are given by,
$\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}_i}\right) - \dfrac{\partial L}{\partial q_i} = \sum\limits_{j=1}^{m} \lambda_j a_{ji} + Q_i^{'}, (i = 1, 2, ..., n)$

9. The generalized momenta are given by, $p_r = \dfrac{\partial L}{\partial \dot{r}} = m\dot{r}, p_\theta = \dfrac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$.

10. The modified Hamilton's principle states that the actual part is such that the integral of equation $\delta \int\limits_{t_0}^{t_1} \left(\sum\limits_{i=1}^{n} p_i \dot{q}_i - H\right) dt = 0$ is stationary.

11. The actual path of a conservative holonomic system such that the action is stationary with respect to varied paths having the same energy integral h and the same end points in q-space.

12. $\sum\limits_{i=1}^{n} p_i \dot{q}_i = 2T_2 + T_1$.

13. $\delta A = \delta \int\limits_{t_0}^{t_1} \sqrt{2(h - V)} ds = 0$, which is the Jacobi's form of least action.

Suggested Readings

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• Gupta. S. L. 1970. New Delhi: Classical Mechanics. Meenakshi Prakashan.

Unit 4

HAMILTON's- JACOBI THEORY

Objectives

After the successful completion of this unit; the students are expected

• To understand the concepts of Hamilton's principle function and Pfaffian differential form.

• To gain the knowledge Hamilton-Jacobi equation.

• To develope the method of modified Hamilton-Jacobi equation with illustrated example.

• To gain the knowledge of Lioville's system.

• To understand the Stackel's theorem.

4. Introduction

Dear students, Hamiltonian is conserved then a solution could be obtained by the transforming to new canonical coordinates that are all cyclic, since the integration of the new equations of motion becomes trivial. An alternative techique is to seek a canonical transformation from the coordinates and momenta, (q, p), at the time t, to a new set of constant quantities, $2n$ initial values (q_0, p_0) at $t = 0$. Now the important question? how to find the transformation from the old coordintes to new coordinates. This is the fundamental problem. In this present chapter we shall approach proposal problem by studying the generality function which is associated with the required canonical transformation. This generality technique is the solution of the partial differential equation known as Hamilton Jacobi equation. The Jacobi Hamilton equation is named after William Rowthan Hamilton and Carl cursav Jacob Jacobi. The Hamilton canonical equation is first order partial nonlinear differential equation applicable in understudy the conserved quantities for mechanical systems. In this unit, we will study the charateristic function and Hamiltonian Jacobi equation.

4.1 Hamilton's Principle Function

Dear students, in this section we will discuss the canonical integral and also Pfaffian differential form.

4.1.1 Canonical integral

Now consider the canonical integral of the form $s(q_0, q_1, t_n, t_1) = \int_{t_0}^{t_1} L dt$. The function s is a twice differential in all its arguments and is known as Hamilton's principle function.

For a holonomic system

$$\delta I = \int_{t_0}^{t_1} \frac{d}{dt}\left(\sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i}\right)\delta q_i dt + \int_{t_0}^{t_1} \frac{\partial L}{\partial t}\delta t + L\frac{d}{dt}\delta t - \sum_{i=1}^{n} \frac{\partial L}{\partial q_i} q_i \frac{d}{dt}\delta t dt$$

$$= \int_{t_0}^{t_1} \frac{d}{dt}\left(\sum_{i=1}^{n} \frac{\partial L}{\partial q_i}\right)\delta q_i dt + \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial t}\delta t - (\sum_{i=1}^{n} p_i q_i - L)\frac{d}{dt}\delta t\right) dt$$

we know that $H(p,q,t)$

$$\begin{aligned}
\dot{H} &= \sum_{i=1}^{n}(\frac{\partial H}{\partial p_i}\dot{p}_i + \frac{\partial H}{\partial q_i}\dot{q}_i) + \frac{\partial H}{\partial t} \\
&= \sum_{i=1}^{n}(\dot{q}_i\dot{p}_i - \dot{p}_i\dot{q}_i) + \frac{\partial H}{\partial t} \\
\frac{\partial H}{\partial t} &= -\frac{\partial L}{\partial t} \\
\frac{\partial L}{\partial t} &= -\dot{H} \\
\delta I &= \int_{t_0}^{t_1}\sum_{i=1}^{n}\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i})\delta q_i dt + \int_{t_0}^{t_1}(-\dot{H}\delta t - (\sum_{i=1}^{n}\dot{p}_i\dot{q}_i - L)\frac{\partial L}{\partial t}\delta t)dt \\
\delta I &= \int_{t_0}^{t_1}\sum_{i=1}^{n}\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i})\delta q_i dt - \int_{t_0}^{t_1}(\dot{H}\delta t + (\sum_{i=1}^{n}\dot{p}_i\dot{q}_i - L)\frac{\partial L}{\partial t}\delta t)dt \\
\delta I &= \int_{t_0}^{t_1}\sum_{i=1}^{n}\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i})\delta q_i dt - \int_{t_0}^{t_1}\frac{d}{dt}(H\delta t)dt \\
\delta I &= \int_{t_0}^{t_1}\frac{d}{dt}\left(\sum_{i=1}^{n}\frac{\partial L}{\partial \dot{q}_i}\delta q_i - H\delta t\right)dt.
\end{aligned}$$

The principle function s is obtained from the canonical integral I,

$$\begin{aligned}
\delta s &= \left[\sum_{i=1}^{n}\frac{\partial L}{\partial \dot{q}_i}\delta q_i - H\delta t\right]_{t_0}^{t_1} \\
\delta s &= \sum_{i=1}^{n}\frac{\partial L}{\partial \dot{q_{i1}}}\delta q_{i1} - H_1\delta t_1 - \sum_{i=1}^{n}\frac{\partial L}{\partial \dot{q_{i0}}}\delta q_{i0} + H_0\delta t_0
\end{aligned}$$

Writing in differential form,

$$ds = \sum_{i=1}^{n} p_{i1}dq_{i1} - H_1 dt_1 - \frac{\partial L}{\partial \dot{q_{i0}}}dq_{i0} + H_0 dt_0. \tag{4.1}$$

Differentiating $s(q_0, q_1, t_0, t_1)$ we get,

$$ds = \sum_{i=1}^{n}\frac{\partial s}{\partial q_{i0}}dq_{i0} + \sum_{i=1}^{n}\frac{\partial s}{\partial q_{i1}}dq_{i1} + \frac{\partial s}{\partial t_0}dt_0 + \frac{\partial s}{\partial t_1}dt_1. \tag{4.2}$$

Equating the corresponding coefficients,

$$\begin{aligned}
p_{i0} &= -\frac{\partial s}{\partial q_{i0}}, \qquad & p_{i1} &= -\frac{\partial s}{\partial q_{i1}} \\
H_1 &= -\frac{\partial s}{\partial t_1}, \qquad & H_0 &= -\frac{\partial s}{\partial t_0}.
\end{aligned}$$

The equation $p_{i0} = -\frac{\partial s}{\partial q_{i0}}(i = 1, 2, ..., n)$ gives p_{i0} as function of $(q_{i0}, q_{i1}, t_0, t_1)$. Assuming that $\left|\frac{\partial^2 s}{\partial q_{i0} q_{i1}}\right| \neq 0$, we can solve for q_{i1} and is given by $q_{i1} = q_{i1}(q_0, q_1, t_0, t_1)$. It is the solution of lagrange's problem substituting q_{i1} in $p_{i1} = \frac{\partial s}{\partial q_{i1}}$, we get $p_{i1} = p_{i1}(q_0, q_1, t_0, t_1)$. Hence we get the complete solution of the Hamilton problem.

4.1.2 Pfaffian differential form

A pfaffian form is defined by $\Omega = \sum_{i=1}^{m} X_i(x)dx_i$, for arbitrary displacement and $\omega = \sum_{j=1}^{m} X_j(x)\delta x_j$

$$\begin{aligned}
\delta\omega - \delta\Omega &= \sum_{i=1}^{m}(\partial X_i dx_i + X_i \delta dx_i) - \sum_{j=1}^{m}(dX_j + X_j d\delta x_j) \\
&= \sum_{i=1}^{m}(\sum_{j=1}^{m}\frac{\partial X_j}{\partial x_j}\delta x_j dx_i) - \sum_{j=1}^{m}(\sum_{i=1}^{m}\frac{\partial X_j}{\partial x_i}dx_i \delta x_j)) \\
&= \sum_{i=1}^{m}\sum_{j=1}^{m}(\frac{\partial X_i}{\partial x_j} - \frac{\partial X_j}{\partial x_i})\delta x_j dx_i \\
&= \sum_{i=1}^{m}\sum_{j=1}^{m} c_{ij}\delta x_j dx_i
\end{aligned}$$

$$where \qquad \frac{\partial X_i}{\partial x_j} - \frac{\partial X_j}{\partial x_i} = c_{ij}.$$

$$\delta\omega - \delta\Omega = \sum_{i=1}^{m}\sum_{j=1}^{m} c_{ij}\delta x_j dx_i \tag{4.3}$$

is called the bilinear covariant associated with Ω.

Further if m is odd, we get m another aspect of pfaffian differential form.

(ie) we get m differential equations known as first pfaffian systems.

The equation taken the form

$$\sum_{i=1}^{m} c_{ij} dx_i = 0, \qquad (j = 1, 2, ..., m).$$

These are obtained by setting the coefficients of $\delta x's$ to zero in equation (4.3). Expressing ds as the different of two pfaffian form

$$ds = (\sum_{i=1}^{n} p_{i1} dq_{i1} - H_1 dt_1) - (\sum_{i=1}^{m} p_{i0} dq_{i0} - H_0 dt_0).$$

(ie.,) It is the form of $\sum_{i=1}^{n} p_i dq_i - Hdt$

$$\begin{aligned}
Let \qquad \Omega &= \sum_{j=1}^{n} P_j dq_j - Hdt \\
\omega &= \sum_{j=1}^{n} P_j \delta q_j - H\delta t \\
\partial\Omega &= \sum_{j=1}^{n} (\delta P_j dq_j + P_j \delta q_j) - \delta H dt - H\delta dt \\
d\omega &= \sum_{j=1}^{n} (dP_j \delta q_j + P_j d\delta q_j) - dH\delta t - Hd\delta t. \\
\delta\Omega - d\omega &= \sum_{j=1}^{n} \delta p_j dq_j - \delta H dt - \sum_{j=1}^{n} dP_j \delta q_j + dH\delta t \\
&= \sum_{j=1}^{n} \delta P_j dq_j - \left(\sum_{j=1}^{n} (\frac{\partial H}{\partial q_j}\delta q_j + \frac{\partial H}{\partial P_j}\delta P_j) + \frac{\partial H}{\partial t}\delta t \right) dt \\
&\qquad - \sum_{j=1}^{n} dP_j \delta q_j - \left(\sum_{j=1}^{n} (\frac{\partial H}{\partial q_j} dq_j + \frac{\partial H}{\partial P_j} dP_j) + \frac{\partial H}{\partial t} dt \right) \delta t \\
&= \sum_{j=1}^{n} (dq_j - \frac{\partial H}{\partial P_j} dt)\delta P_j + \sum_{j=1}^{n} (-dP_j - \frac{\partial H}{\partial q_j} dt)\delta q_j + \sum_{j=1}^{n} (\frac{\partial H}{\partial q_j} dq_j + \frac{\partial H}{\partial P_j} dP_j)\delta t.
\end{aligned}$$

Applying the idea of first pfaffian system we get,

$$dq_i = \frac{\partial H}{\partial p_j} dt = 0. \tag{4.4}$$

$$-dp_i = \frac{\partial H}{\partial q_j} dt = 0. \tag{4.5}$$

$$\frac{\partial H}{\partial q_j} dq_j + \frac{\partial H}{\partial p_j} dp_j = 0$$

(4.4) becomes

$$\begin{aligned}
dq_j &= \frac{\partial H}{\partial p_j} dt \\
(ie.,) \qquad \dot{q_j} &= \frac{\partial H}{\partial p_j} \qquad (4.6)
\end{aligned}$$

(4.5) becomes

$$\begin{aligned}
dp_j &= \frac{\partial H}{\partial q_j} dt \\
(ie.,) \qquad \dot{p_j} &= \frac{\partial H}{\partial q_j}, \qquad (j = 1, 2, ..., n). \qquad (4.7)
\end{aligned}$$

$(4.6), (4.7)$ are Hamilton's canonical equations,

Now

$$\begin{aligned}\dot{H} =& \sum_{j=1}^{n} (\frac{\partial H}{\partial q_j}\dot{P}_j + \frac{\partial H}{\partial p_j}\dot{q}_j) + \frac{\partial H}{\partial t} \\ =& \sum_{j=1}^{n} (\frac{\partial H}{\partial P_j}\frac{-\partial H}{\partial q_j} + \frac{\partial H}{\partial q_j}\frac{\partial H}{\partial P_j}) + \frac{\partial H}{\partial t} \\ \dot{H} =& \frac{\partial H}{\partial t}.\end{aligned}$$

We find that ds is equal to the difference between two pfaffian differential forms at initial and final positions.

Hence s is considered as a generating function for the canonical transformation

Case (i): Consider the transformation

$q_{i0} = q_{i0}(\gamma_1, \gamma_2, ...\gamma_{2n}), \qquad p_{i0} = p_{i0}(\gamma_1, \gamma_2, ...\gamma_{2n}),$

where the Jacobian of the transformation is given by, $\frac{\partial(q_{10}, ...p_{n0})}{\partial(\gamma_1, ...\gamma_{2n})} \neq 0.$

Then

$$\begin{aligned}\sum_{i=1}^{n} p_{i0} dq_{i0} =& \sum_{i=1}^{n} p_{i0} \sum_{j=1}^{2n} \frac{\partial q_{i0}}{\partial \gamma_i} d\gamma_i \\ =& \sum_{j=1}^{2n} \Gamma_j(\gamma) d\gamma_j, \qquad (4.8) \\ \text{where} \qquad \Gamma_j(\gamma) =& \sum_{i=1}^{n} p_{i0} \frac{\partial q_{i0}}{\partial \gamma_i}.\end{aligned}$$

Case (ii): $\gamma_1, \gamma_2, ..., \gamma_{2n}$ can be replaced $n\alpha's$ and $n\beta's$.

Let $\alpha_i = \alpha_i(\gamma_1, \gamma_2...\gamma_{2n}), \quad \beta_i = \beta_i(\gamma_1, \gamma_2...\gamma_{2n})$

such that $\quad \sum_{i=1}^{n} \beta_i d\alpha_i = \sum_{i=1}^{2n} \Gamma_j(\gamma) d\gamma_j$, where $\quad \Gamma_j(\gamma) = \sum_{i=1}^{n} \beta_i \frac{\partial \alpha_j}{\partial \gamma_j}$.

Therefore $\sum_{i=1}^{n} p_{i0} dq_i = \sum_{i=1}^{n} \beta_i d\alpha_i \quad$ from case (i).

Let us sum up

1. We introduce the canonical integral.
2. We have derive Pfaffian differential form.

Check your progress

1. What is Hamilton's principle function?
2. Write the Pfaffian differential form.

Dear students, in this section we will discuss about the Hamilton Jacobi's equation, modified Hamilton Jacobi's equation.

4.2 Hamilton Jacobi's equation

Consider a holonomic system giving $2n$ independent initial conditions at time t_0 as $q_0 \neq p_0$.

Now we have differential equation

Assume that

$$ds = \sum_{i=1}^{n} p_{i1} dq_{i1} - \sum_{i=1}^{n} p_{i0} dq_{i0} - H_1 dt_1 + H_0 dt_0, \tag{4.9}$$

where s is the Hamilton's principle function. It is associated cononical transformation relating the initial and final point of a path in a phase space.

Let the initial conditions are specified by

$\alpha_i = \alpha_i(q_{10}, q_{20} ... q_{n0}, p_{10}, p_{10}, ... p_{n0})$.

$\beta_i = \beta_i(q_{10}, q_{20} ... q_{n0}, p_{10}, p_{10}, ... p_{n0}), \qquad (i = 1, 2, ..., n),$

which satisfied

$$\sum_{i=1}^{n} p_{i0} dq_{i0} = \sum_{i=1}^{n} \beta_i d\alpha_i$$

$$ds = \sum_{i=1}^{n} p_{i1} dq_{i1} - \sum_{i=1}^{n} \beta_i d\alpha_i - H_1 dt_1 + H_0 dt_0. \tag{4.10}$$

Now we consider s number can be associated as a function of

$s = s(q_{i1}, q_{i0}, t_1, t_0)$

$$ds = \sum_{i=1}^{n} \frac{\partial s}{\partial q_{i1}} dq_{i1} + \sum_{i=1}^{n} \frac{\partial s}{\partial \alpha_i} d\alpha_i + \frac{\partial s}{\partial t_1} dt_1 + \frac{\partial s}{\partial t_0} dt_0. \tag{4.11}$$

Hence $q's$ and $\alpha's$ are independent equality variable.

Assume that $|\frac{\partial^2 s}{\partial q_{i1}\partial \alpha_j}| \neq 0$

$\implies \alpha's$ can be solved in terms of $\frac{\partial s}{\partial q_{i1}}$

comparing (4.10) and (4.11) we get

$$p_{i1} = \frac{\partial s}{\partial q_{i1}}, \qquad \frac{\partial s}{\partial t_1} = H_1$$
$$\frac{\partial s}{\partial \alpha_i} = -\beta_i, \qquad \frac{\partial s}{\partial t_0} = H_0.$$

Equation (4.10) can be further simplified by setting the initial time $t_0 = 0$

(ie.,) $dt_0 = 0$

Equation (4.10) becomes,

$$ds = \sum_{i=1}^{n} p_{i1} dq_{i1} - \sum_{i=1}^{n} \beta_i d\alpha_i - H_1 dt_1$$
$$ds = \sum_{i=1}^{n} p_i dq_i - \sum_{i=1}^{n} \beta_i d\alpha_i - H dt. \tag{4.12}$$

From (4.12), it is clear that the principle function is of the form $s(q, \alpha, t)$

$$\therefore ds = \sum_{i=1}^{n} \frac{\partial s}{\partial q_i} dq_i + \sum_{i=1}^{n} \frac{\partial s}{\partial \alpha_i} d\alpha_i + \frac{\partial s}{\partial t} dt. \tag{4.13}$$

Equating the coefficients in (4.12) and (4.13) we get,

$$p_i = \frac{\partial s}{\partial q_i}, \qquad i = 1, 2, ..., n. \tag{4.14}$$
$$-\beta_i = \frac{\partial s}{\partial \alpha_i} \qquad i = 1, 2, ..., n. \tag{4.15}$$
$$-H = \frac{\partial s}{\partial t}. \tag{4.16}$$

From (4.15) we can get $q's$ as function (α, β, t) using the equation (4.14). We csn find p as function of (α, β, t). Hence we have the solution for Hamilton's Principle. H is usuallt as a function of (q, p, t).

Using (4.14) and (4.16) we have

$$\frac{\partial s}{\partial t} + H(q, p, t) = 0$$
$$\frac{\partial s}{\partial t} + H(q, \frac{\partial s}{\partial t}, t) = 0.$$

The first order partial differential equation is known as Hamilton's Jacobi equation.

4.2.1 Jacobi's theorem

If $s(q, \alpha, t)$ is any complete solution of the Hamilton Jacobi equation

$$\begin{aligned}\frac{\partial s}{\partial t} + H(q, \frac{\partial s}{\partial t}, t) =&0 \\ -\beta_i =&\frac{\partial s}{\partial \alpha_i} \\ p_i =&\frac{\partial s}{\partial q_i},\end{aligned} \tag{4.17}$$

where $\beta's$ are arbitrary constant are used to solve for $q_i(\alpha, \beta, t)$ and $p_i(\alpha, \beta, t)$.

Then these expressions provide the general solution of the canonical equation associated with Hamiltonian $H(q, p, t)$.

Proof:

$$\frac{\partial s}{\partial t} + H(q, \frac{\partial s}{\partial t}, t) =0 \tag{4.18}$$

$$p_i =\frac{\partial s}{\partial q_i}. \tag{4.19}$$

Which is a function of (q, α, t). Differentiating W.r.to α_i

$$\frac{\partial^2 s}{\partial \alpha_i \partial t} + \frac{\partial^2}{\partial \alpha_i}(H(q, \frac{\partial s}{\partial t}, t)) =0$$

$$\frac{\partial^2 s}{\partial \alpha_i \partial t} + (\sum_{i=1}^{n} \frac{\partial H}{\partial q_j}\frac{\partial q_j}{\partial \alpha_i}) + \sum_{j=1}^{n} \frac{\partial H}{\partial p_j}\frac{\partial p_j}{\partial \alpha_j} + \frac{\partial H}{\partial t}\frac{\partial t}{\partial \alpha_i} =0$$

$$\frac{\partial^2 s}{\partial \alpha_i \partial t} + \sum_{i=1}^{n} \frac{\partial H}{\partial p_j}\frac{\partial p_j}{\partial \alpha_i} =0. \tag{4.20}$$

$$\frac{\partial^2 s}{\partial \alpha_i \partial t} + \sum_{i=1}^{n} \frac{\partial H}{\partial p_j}(\frac{\partial^2 s}{\partial q_j \partial \alpha_j}) = 0, \tag{4.21}$$

where p_j is considered as a function of (q, α, t) in (4.20), $\frac{\partial s}{\partial \alpha_i}$ is a function of (q, α, t) and $\alpha's$ and $\beta's$ are constants. In (4.18), taking the total time derivative of this w.r.t. t

$$-\frac{d}{dt}(\beta_i) =\frac{d}{dt}(\frac{\partial s}{\partial \alpha_i}) \tag{4.22}$$

$$0 = \sum_{i=1}^{n} \frac{\partial}{\partial q_j}(\frac{\partial s}{\partial \alpha_i})\dot{q_j} + \sum_{j=1}^{n} \frac{\partial}{\partial \alpha_j}(\frac{\partial s}{\partial \alpha_i})\dot{\alpha_j} + \frac{\partial}{\partial t}(\frac{\partial s}{\partial \alpha_i})$$

$$0 = \sum_{j=1}^{n} \frac{\partial^2 s}{\partial \alpha_i \partial q_j} q_j + \frac{\partial^2 s}{\partial \alpha_i \partial t} \tag{4.23}$$

(4.22) and $(4.21) \Longrightarrow$ Using $(4.18), (4.19)$ and (4.20), we have

$$\sum_{i=1}^{n}(\dot{q}_j - \frac{\partial H}{\partial q_j \alpha_i})\frac{\partial^2 s}{\partial \alpha_i \partial q_j} = 0 \qquad i = 1, 2, ...n \tag{4.24}$$

Since $|\frac{\partial^2}{\partial \alpha_i \partial q_j}| \neq 0$, We get

$$\dot{q}_j = \frac{\partial H}{\partial p_j}, \qquad (j = 1, 2, ..., n). \tag{4.25}$$

Differentiating (4.19) partially with respect to q_j

$$\frac{\partial^2 s}{\partial t \partial q_j} + \frac{\partial H}{\partial q_j} + \sum_{i=1}^{n} \frac{\partial H}{\partial p_j}\frac{\partial p_i}{\partial q_j} = 0. \tag{4.26}$$

Differentiating (4.20) partially with respect to t

$$\frac{d}{dt}(p_j) = \frac{d}{dt}(\frac{\partial s}{\partial q_j})$$

$$\dot{p}_j = \sum_{i=1}^{n} \frac{\partial}{\partial q_i}(\frac{\partial s}{\partial q_j})\dot{q}_i + \frac{\partial}{\partial t}\frac{\partial s}{\partial q_j}$$

$$\dot{p}_j - \sum_{i=1}^{n} \frac{\partial}{\partial q_i}(\frac{\partial s}{\partial q_j})\dot{q}_i - \frac{\partial}{\partial t}\frac{\partial s}{\partial q_j} = 0. \tag{4.27}$$

Adding equations (4.24) and (4.25)

$$\frac{\partial H}{\partial q_j} + \dot{p}_j + \sum_{i=1}^{n}(\frac{\partial H}{\partial p_i} - \dot{q}_i)\frac{\partial^2 s}{\partial q_i \partial q_j} = 0,$$

using equation (4.19) we get

$$\dot{p}_j = -\frac{\partial H}{\partial q_j}. \tag{4.28}$$

Equation (4.20) and (4.23) are Hamilton's canonical equation.

4.2.2 Conservative system and ignrable co-ordinates (or) modified Hamilton Jacobi equation

(i) Let us consider a conservative system (holonomic) described by n independent $q's$. The Hamiltonian function for this system is not a function of time. The principle function of this system is given by

$$\frac{\partial s}{\partial t} = - H = -\alpha_n \tag{4.29}$$

$$s = - \alpha_n t + \omega(q, \alpha) \tag{4.30}$$

$$s(q, \alpha, t) = - \alpha_n t + \omega(q, \alpha). \tag{4.31}$$

The function $\omega(q_1, ..., q_n, \alpha_1, ..., \alpha_n)$ is called the characteristic function,

$$\frac{\partial s}{\partial \alpha_i} = \frac{\partial \omega}{\partial \alpha_i} \qquad (i = 1, 2, ..., n-1) \tag{4.32}$$

$$\frac{\partial s}{\partial \alpha_n} = -t + \frac{\partial s}{\partial \alpha_n} \tag{4.33}$$

$$\frac{\partial s}{\partial q_i} = \frac{\partial \omega}{\partial q_i}, \qquad (i = 1, 2, ..., n-1). \tag{4.34}$$

From (4.27) and (4.32) we get

$$H = \alpha_n$$
$$H(q, \frac{\partial s}{\partial q_i}) = \alpha_n.$$

(ie.,) $H(q, \frac{\partial s}{\partial q_i}) = \alpha_n$ is the modified - Hamilton Jacobi equation, we know that

$$-\beta_i = \frac{\partial s}{\partial \alpha_i} \qquad (i = 1, 2, ..., n)$$
$$p_i = \frac{\partial s}{\partial q_i}.$$

Comparing (4.30), (4.31) and (4.32) with these two equations, We get

$$-\beta_i = \frac{\partial \omega}{\partial \alpha_i} \qquad (i = 1, 2, ..., n)$$
$$-\beta_n = -t + \frac{\partial \omega}{\partial \alpha_n}$$
$$-\beta_n + t = \frac{\partial \omega}{\partial \alpha_n}$$
$$p_i = \frac{\partial \omega}{\partial q_i}.$$

(ii) Now let us consider a system having ignorable co-ordinates $q_1, q_2, ..., q_k$. Further assume that the system is not conservative

Let us assume the principle function as

$$s(q, \alpha, t) = \sum_{i=1}^{k} \alpha_i q_i + s'(q_{k+1}, ..., q_n, \alpha_1, ..., \alpha_n)$$
$$p_i = \alpha_i, \qquad i = 1, 2, ..., k.$$

The Hamilton Jacobi's equation is given by

$$\frac{\partial s'}{\partial t} + H(q_{k+1}, ..., q_n, \alpha_1, ..., \alpha_k, \frac{\partial s'}{\partial q_{k+1}}, ..., \frac{\partial s'}{\partial q_n}, t) = 0.$$

The solution is obtained from

$$\begin{aligned}-\beta_i =&\frac{\partial s}{\partial \alpha_i}\\ =&q_i + \frac{\partial s'}{\partial \alpha_i},\ (i = 1, 2, ..., k)\\ -\beta_i =&\frac{\partial s'}{\partial \alpha_i}, \qquad (i = k+1, ..., n).\\ p_i =&\frac{\partial s}{\partial q_i}\\ =&\alpha_i, \qquad (i = 1, 2, ..., k)\\ p_i =&\frac{\partial s'}{\partial q_i},\ (i = k+1, ..., n).\end{aligned}$$

(iii) Now let us consider a conservative system with ignorable co-ordinates $q_1, q_2, ..., q_k$. The principle function is given by,

$$s(q, \alpha, t) = \sum_{i=1}^{n} \alpha_i q_i - \alpha_n t + \omega'(q_{k+1}, ..., q_n, \alpha_1, ..., \alpha_n).$$

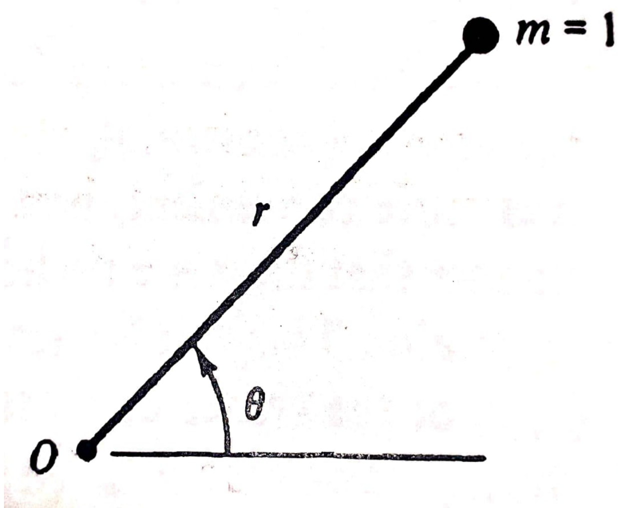
The modified Hamilton Jacobi's equation is given by

$$H(q_{k+1}, ..., q_n, \alpha_1, ..., \alpha_k, \frac{\partial \omega'}{\partial q_{k+1}}, ..., \frac{\partial \omega'}{\partial q_n}, t) = \alpha_n$$

The solution is obtained from,

$$\begin{aligned}-\beta_i =&\frac{\partial s}{\partial \alpha_i}\\ =&q_i + \frac{\partial \omega'}{\partial \alpha_i}, \qquad (i = 1, 2, ...k)\\ -\beta_i =&\frac{\partial \omega'}{\partial \alpha_i}, \qquad (i = k+1, ..., n-1).\\ -\beta_n =&-t + \frac{\partial \omega'}{\partial \alpha_n}\\ -\beta_n + t =&\frac{\partial \omega'}{\partial \alpha_n}\\ p_i =&\frac{\partial s}{\partial q_i}, \qquad (i = 1, 2, ..., k)\\ =&\alpha_i\\ p_i =&\frac{\partial \omega'}{\partial q_i}, \qquad (i = k+1, ..., n).\end{aligned}$$

Kepler's problem: Use Hamilton-Jacobi method to analyze the Kepler's problem or modified Jacobi method.



Solution: Suppose a particle of unit mass attached by an inverse square gravitational force at a fixed point $'O'$. The position of a given problem is given in terms of the polar coordinates (r, θ) in the plane of the orbits.

The K.E and P.E are $T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta^2}), \quad V = \frac{-\mu}{r}$, where μ is a gravitation coefficient.

$$L = T - V$$

$$L = \frac{1}{2}(r^2 + r^2(\dot{\theta}_i)^2) + \frac{\mu}{r}$$

$$p_r = \frac{\partial L}{\partial \dot{r}} = \dot{r}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}}$$

$$\dot{\theta} = \frac{p_\theta}{r^2} \qquad H = T + V = constant$$

$$H = \frac{1}{2}((p_r^2) + \frac{(p_\theta)^2}{r^4}r^2) - \frac{\mu}{r} = \alpha_t. \tag{4.35}$$

Since θ does not in H it is considered as an ignorable co-ordinate.

Therefore $p_\theta = \alpha_\theta \qquad (constant)$

The modified Hamilton-Jacobi equation is,

$$H(q, \frac{\partial \omega}{\partial q}) = \alpha_t$$

$$p_i = \frac{\partial \omega_t}{\partial q_i}$$

$$p_r = \frac{\partial \omega}{\partial r}.$$

Therefore (4.44) becomes

$$\begin{aligned}
\frac{1}{2}(\frac{\partial \omega'}{\partial r})^2 + \frac{1}{2}(\frac{\alpha\theta^2}{r^2}) - \frac{\mu}{r} =& \alpha_t \\
\frac{1}{2}(\frac{\partial \omega'}{\partial r})^2 =& 2\alpha_t + \frac{2\mu}{r} - \frac{\alpha(\theta)^2}{r^2} \\
\omega' =& \int_{r_0}^{r} \sqrt{2\alpha_t + \frac{2\mu}{r} - \frac{\alpha(\theta)^2}{r^2}} dr. \\
\text{W.K.T} \qquad t - \beta_n =& \frac{\partial \omega'}{\partial \alpha_n} \\
t - \beta_t =& \frac{\partial \omega'}{\partial \alpha_t} \\
\frac{\partial \omega'}{\partial \alpha_t} =& \int_{r_0}^{r} \frac{2dr}{2\sqrt{2\alpha_t + \frac{2\mu}{r} - \frac{\alpha(\theta)^2}{r^2}}} \\
t - \beta_n = \frac{\partial \omega'}{\partial \alpha_t} =& \int_{r_0}^{r} \frac{dr}{\sqrt{2\alpha_t + \frac{2\mu}{r} - \frac{\alpha(\theta)^2}{r^2}}} \\
\text{set} \qquad \beta_t =& t_0 \\
t - t_0 =& \frac{\partial \omega'}{\partial \alpha_t} = \int_{r_0}^{r} \frac{dr}{\sqrt{2\alpha_t + \frac{2\mu}{r} - \frac{\alpha(\theta)^2}{r^2}}}.
\end{aligned}$$

$$\begin{aligned}
\text{W.K.T} \qquad -\beta_i =& q_i + \frac{\partial \omega'}{\partial \alpha_t} \text{ igonarble co-ordinate} \\
-\beta_\theta =& \theta + \frac{\partial \omega'}{\partial \alpha_\theta} \\
\frac{\partial \omega'}{\partial \alpha_\theta} =& \int_{r_0}^{r} \frac{-\alpha\theta}{r^2\sqrt{2\alpha_t + \frac{2\mu}{r} - \frac{\alpha(\theta)^2}{r^2}}} dr \\
\frac{\partial \omega'}{\partial \alpha_\theta} =& \int_{r_0}^{r} \frac{-\alpha\theta}{r\sqrt{2\alpha_t r^2 + 2\mu r - (\alpha_\theta)^2}} dr,
\end{aligned}$$

choose

$$\begin{aligned}
-\beta_\theta =& \theta_0 \\
\theta_0 =& \theta + \frac{\partial \omega'}{\partial \alpha_\theta} \\
\theta - \theta_0 =& \frac{\partial \omega'}{\partial \alpha_\theta}
\end{aligned}$$

If $r_0 = r_{min}$ then $\theta_0 = 0$. Therefore

$$\theta = \int_{r_0}^{r} \frac{-\alpha\theta}{r\sqrt{2\alpha_t r^2 + 2\mu r - (\alpha_\theta)^2}} dr$$
$$= \int_{r_0}^{r} \frac{-\alpha\theta}{r\frac{r^2}{\alpha_\theta^2}\frac{\alpha_\theta^2}{r^2}\sqrt{2\alpha_t r^2 + 2\mu r - (\alpha_\theta)^2}} dr$$
$$= \int_{r_0}^{r} \frac{\alpha_\theta^2 dr}{r^2\sqrt{2\alpha_t\alpha_\theta^2 + \frac{2\mu\alpha_\theta^2}{r} - \frac{\alpha_\theta^4}{r^2} + \mu^2 - \mu^2}}$$
$$= \int_{r_0}^{r} \frac{\alpha_\theta^2 dr}{r^2\sqrt{(2\alpha_t\alpha_\theta^2 + \mu^2) - (\frac{\alpha_\theta^2}{r} - \mu)^2}}$$
$$= \int_{r_0}^{r} \frac{d(\frac{\alpha_\theta^2}{r} - \mu)}{r^2\sqrt{(2\alpha_t\alpha_\theta^2 + \mu^2) - (\frac{\alpha_\theta^2}{r} - \mu)^2}}$$
$$\theta = -\sin^{-1}\left(\frac{(\frac{\alpha_\theta^2}{r} - \mu)}{\sqrt{\mu^2 + 2\alpha_t\alpha_\theta^2}}\right) + \frac{\pi}{2}$$
$$= \cos^{-1}\left(\frac{(\frac{\alpha_\theta^2}{r} - \mu)}{\sqrt{\mu^2 + 2\alpha_t\alpha_\theta^2}}\right)$$
$$\cos\theta = \left(\frac{(\frac{\alpha_\theta^2}{r} - \mu)}{\sqrt{\mu^2 + 2\alpha_t\alpha_\theta^2}}\right)$$
$$1 + \cos\theta\sqrt{1 + \frac{2\alpha_t\alpha_\theta^2}{\mu^2}} = \frac{\alpha_\theta^2}{r}.$$

This is of the form $\frac{l}{r} = 1 + e\cos\theta$,

where $e = \sqrt{1 + \frac{2\alpha_t\alpha_\theta^2}{\mu^2}}$

To find r

$$\frac{1}{1 + \cos\theta\sqrt{1 + \frac{2\alpha_t\alpha_\theta^2}{\mu^2}}} = \frac{\mu r}{\alpha_\theta^2}$$
$$r = \frac{\frac{\alpha_\theta^2}{\mu}}{1 + \sqrt{1 + \frac{2\alpha_t\alpha_\theta^2}{\mu^2}}} \cos\theta.$$

Let us sum up

1. We have discuss the Hamilton Jacobi's equation.
2. We have derive the Jacobi's theorem.
3. We have derive the modified Hamilton Jacobi's equation.

Check your progress

3. Define the Hamilton Jacobi's equation.

4. Write the modified Hamilton Jacobi's equation.

4.3 Separability

Dear students, in this section we will discuss about the Lioville's system and Stackel's theorem. The term separability implies that a characteristic function for the system has the form $\omega = \sum_{i=1}^{n} \omega_i(q_i)$

(ie.,) It consists of the sum of n functions. Where each function ω_i contains only one of the equations

4.3.1 Lioville's system

It is an orthogonal system which has kinetic and potential energy of the form

$$T = \frac{1}{2}\sum_{i=1}^{n} f_i(q_i)(\sum_{i=1}^{n} \frac{\dot{q_i}^2}{R_i(q_i)}).$$

and

$$V = \sum_{i=1}^{n} \frac{\nu_i(q_i)}{f_i(q_i)},$$

where f_i, q_i and ν_i are each function of q_i. We assume that $\sum_{i=1}^{n} f_i(q_i) > 0$ and$R_i(q_i) > 0$. T is modified as follows, W.K.T

$$P_i = \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i}$$

$$\frac{\partial T}{\partial \dot{q}_i} \sum_{i=1}^{n} f_i(q_i) \frac{2\dot{q}_i}{R_i(q_i)}$$

$$p_i = \frac{f\dot{q}_i}{R_i(q_i)}, \qquad \dot{q}_i = \frac{p_i R_i(q_i)}{f}.$$

Therefore

$$
\begin{aligned}
T =& \frac{1}{2}\sum_{i=1}^{n} f_i \sum_{i=1}^{n} \frac{p_i^2 R_i^2(q_i)}{f^2 R_i(q_i)} \\
T =& \frac{1}{2}\frac{\sum_{i=1}^{n} p_i^2 R_i(q_i)}{\sum_{i=1}^{n} f_i^2} \\
\text{(ie.,) } T =& \frac{p_1^2 R_1 + ...p_1^2 R_n}{2(f_1 + ... + f_n)} \\
V =& \frac{\nu_1 + ... + \nu_n}{f_1 + ...f_n}.
\end{aligned}
$$

To find that the system is separable: The modified Hamilton-Jacobi's equation for this system can be written as

$$
\begin{aligned}
H(q, \frac{\partial \omega}{\partial q}) =& h \\
\frac{1}{2}\left(\frac{\sum_{i=1}^{n}(\frac{\partial \omega}{\partial q_i})^2 R_i q_i}{\sum_{i=1}^{n} f_i}\right) + \frac{\sum_{i=1}^{n}\nu_i}{\sum_{i=1}^{n} f_i} =& h. \\
\sum_{i=1}^{n}\left(\frac{1}{2}R_i(\frac{\partial \omega}{\partial q_i})^2 + \nu_i\right) =& h\sum_{i=1}^{n} f_i \qquad (4.36) \\
\sum_{i=1}^{n}\left(\frac{1}{2}R_i(\frac{\partial \omega}{\partial q_i})^2 + \nu_i - hf_i\right) =& 0.
\end{aligned}
$$

Let $\alpha_1, \alpha_2, ..., \alpha_n$ be the separation constants such that $\sum \alpha_i = 0$. Setting each term to the corresponding α_i we get

$$
\begin{aligned}
\frac{1}{2}R_i(\frac{\partial \omega_i}{\partial q_i})^2 + \nu_i =& h\sum_{i=1}^{n} f_i \\
\sum_{i=1}^{n}\frac{1}{2}R_i(\frac{\partial \omega_i}{\partial q_i})^2 + \nu_i - hf_i =& \alpha_i. \qquad (4.37)
\end{aligned}
$$

Let

$$
\begin{aligned}
\phi_i(q_i) =& 2R_i(hf_i - \nu_i + \alpha_i) \\
\frac{\partial \omega_i}{\partial q_i} =& \sqrt{\frac{\phi_i(q_i)}{R_i^2}} \\
d\omega_i =& \frac{1}{R_i}\sqrt{\phi_i(q_i)}dq_i \\
\omega =& \sum_{i=1}^{n}\int \frac{1}{R_i}\sqrt{\phi_i(q_i)}dq_i \qquad (4.38) \\
Where \qquad \phi_i(q_i) =& 2R_i(hf_i - \nu_i + \alpha_i). \qquad (4.39)
\end{aligned}
$$

This solution constants $(n+1)$ constants $\alpha_1, \alpha_2, ..., \alpha_n, h$ but $\sum \alpha_i = 0$, suggests that one α_i can be eliminated leaving m independent constants.

Hence equation (4.35) can be solved for the complete solution of the modified Hamilton Jacobi equation.

To find the solution for the motion of the system

$$\sum_{i=1}^{n} \alpha_i = 0$$

$$\alpha_n = -\alpha_1, -\alpha_2, ..., -\alpha_n$$

$$\omega = \omega_i(\alpha_i) = \omega_1(\alpha_1) + ... + \omega_n(-\alpha_1, -\alpha_2, ..., -\alpha_n).$$

Now
$$\frac{\partial \omega}{\partial \alpha_i} = \frac{\partial \omega}{\partial \alpha_i} + \frac{\partial \omega_n}{\partial \alpha_n}\frac{\partial \alpha_n}{\partial \alpha_i}$$
$$= \frac{\partial \omega_i}{\partial \alpha_i} - \frac{\partial \omega_n}{\partial \alpha_n} \qquad i = 1, 2, ..., n-1. \tag{4.40}$$

From $-\beta_i = \frac{\partial}{\partial \alpha_i}$

$$\frac{\partial \omega}{\partial \alpha_i} = \sum_{i=1}^{n} \int \frac{\partial}{\partial \alpha_i}\left(\frac{1}{R_i}\sqrt{\phi_i(q_i)}\right) dq_i$$
$$= \sum_{i=1}^{n} \int \frac{2R_i}{R_i\sqrt{\phi_i(q_i)}} dq_i$$
$$= \sum_{i=1}^{n} \frac{dq_i}{\sqrt{\phi_i(q_i)}}.$$

From $t - \beta_n = \frac{\partial}{\partial \alpha_n}$

$$\therefore \frac{\partial \omega}{\partial \alpha_i} = \sum_{i=1}^{n} \int \frac{dq_i}{\sqrt{\phi_i(q_i)}} - \int \frac{dq_n}{\sqrt{\phi_n(q_n)}} = -\beta_i. \tag{4.41}$$

W.K.T

$$\frac{\partial \omega}{\partial \alpha_n} = t - \beta_n$$
$$\frac{\partial \omega}{\partial h} = \sum_{i=1}^{n} \int \frac{f_i dq_i}{\sqrt{\phi_i(q_i)}}.$$

Hence

$$\sum_{i=1}^{n} \int \frac{f dq_i}{\sqrt{\phi_i(q_i)}} = t - \beta_n. \tag{4.42}$$

Equation (4.38) and (4.39) represent the solution to the problem and it represent the path . The path can also be found from the equation

$$\begin{aligned} p_i =& \frac{\partial \omega}{\partial q_i} \\ =& \frac{\sqrt{\phi_i(q_i)}}{R_i}. \end{aligned}$$

4.3.2 Stackel's Theorem

The orthogonal system specified by

$$T = \frac{1}{2}\sum_{i=1}^{n} m_i \dot{q_i}^2 = \frac{1}{2}\sum_{i=1}^{n} c_i p_i^2.$$

Where $c_i(q_1, q_2, ..., q_n) > 0$ is a stacked are met, namely that a non-singular $n \times n$ matrix $[p_{ij}(q_i)]$ and a column matrix $\psi(q_i)$ exist such that

$$c^T \Phi = (1, 0, ...0) \tag{4.43}$$

$$\text{and} \qquad c^T \psi = V, \tag{4.44}$$

where $v(q_1, q_2, ..., q_n)$ is the potential energy and c is a column matrix.

Proof: Let us that the orthogonal system is separable. Then we have $\omega = \sum_{i=1}^{n} \omega_i(q_i)$. W.K.T

$$H = \alpha_1$$

$$T + V = \alpha_1$$

$$\frac{1}{2}\sum_{i=1}^{n} c_i p_i^2 + V = \alpha_1 \tag{4.45}$$

$$\frac{1}{2}\sum_{i=1}^{n} c_i (\frac{\partial \omega}{\partial q_i})^2 + V = \alpha_1. \tag{4.46}$$

Now $\frac{\partial \omega}{\partial q_i}$ is a function $(q_i, \alpha_1, ..., \alpha_n)$. Let us consider the general form,

$$(\frac{\partial \omega}{\partial q_i})^2 = -2\psi_i(q_i) + 2\sum_{i=1}^{n} \Phi_j(q_i)\alpha_j. \tag{4.47}$$

Substitute (4.44) in (4.43), we get

$$\frac{1}{2}\sum_{i=1}^{n} c_i(-2\psi_i(q_i) + 2\sum_{i=1}^{n}\Phi_{ij}(q_i)\alpha_j) + V = \alpha_1$$
$$-\sum_{i=1}^{n} c_i\psi_i(q_i) + \sum_{i=1}^{n}\sum_{j=1}^{n} c_i\Phi_{ij}(q_i)\alpha_j + V = \alpha_1.$$

In matrix form,

$$\begin{aligned} -c^T\psi + c^T\Phi\alpha + V =& \alpha_1 \\ -c^T\psi + c^T\Phi\alpha + V =& (1,0,...,0)\alpha_1 \\ -c^T\psi + V =& 0 \qquad and \qquad c^T\Phi = (1,0,...,0) \\ c^T\psi =& V \qquad and \qquad c^T\Phi = (1,0,...,0). \end{aligned}$$

conversely,

Assume that stackrel conditions are satisfied, define aa column matrix $'a'$ by

$$a_i = (\frac{\partial\omega}{\partial q_i})^2, \qquad (i = 1,2,...n). \tag{4.48}$$

From modified Hamilton's Jacobi's equation we know that

$$\begin{aligned} H =& \alpha_1 \\ T + V =& \alpha_1 \\ \frac{1}{2}\sum_{i=1}^{n} c_i p_i^2 + V =& \alpha_1 \\ \frac{1}{2}\sum_{i=1}^{n} c_i(\frac{\partial\omega}{\partial q_i})^2 + V =& \alpha_1 \\ \frac{1}{2}\sum_{i=1}^{n} c_i(a_i)^2 + V =& \alpha_1 \\ \frac{1}{2}c^T + V = (1,0,...,0)\alpha& \\ c^T(\frac{1}{2}a + \psi) =& (1,0,...,0), \alpha \\ \text{since} \qquad c^T\Phi =& (1,0,...,0) \\ c^T =& (1,0,...,0)(\Phi)^{-1} \\ (ie.,) \qquad (1,0,...,0)(\Phi)^{-1}(\frac{1}{2}a + \psi) =& (1,0,...,0)\alpha, \end{aligned}$$

which has a solution $a = -2\psi + q\Phi\alpha$. This results is idendified with (4.44) and indicates that the system is separable.

Let us sum up

1. We have discuss the Lioville's system.
2. We have derive the Stackel's theorem.

Check your progress

5. Explain the Liouville's system.
6. State the Stackel's theorem.

Summary

- Introduce the canonical integral.
- Derive Pfaffian differential form.
- Discuss the Hamilton Jacobi's equation.
- Derive the Jacobi's equation.
- Derive the modified Hamilton Jacobi's equation.
- Discuss the Lioville's system.
- Derive the Stackel's theorem.

Glossary

- **Canonical integral:** The canonical integral of the form $s(q_0, q_1, t_n, t_1) = \int_{t_0}^{t_1} Ldt$.
- **Ignorable coordinates:** It is a generalized co-ordinates of a mechanical system that does not appear in the systems of characteristic functions.
- **Seperability:** The index of seperability associated with the solution of P.D.E by a solution to that is by expressing the solution interms of integrals each involving only one variable.

• **Orthogonal system:** Orthogonal system is conservative holonomic system whose K.E function contains only squard forms.

Self-Assessment Questions

Short-Answer Questions:

1) Discuss the Pfaffian differential form.

2) Derive Modified Hamilton –Jacobi's equation.

3) State and Prove Jacobi's theorem.

4) Explain the separability of a system.

5) For Keppler's problem using spherical polar co-ordinates, verify stackles condition for separability.

6) Using Hamiltonian–Jacobi's method, solve the mass spring problem.

Long-Answer Questions:

1) Prove that any complete solution of the Hamilton-Jacobi's equation leads to a solution of the Hamiltonian problem.

2) Define Liouville's system and prove that the Liouville's condition are sufficient for separability of the given system.

3) Discuss Hamiltonian principle function.

4) Prove that necessary and sufficient condition of Stackel's theorem.

5) Explain Pfaffian differential form and first Pfaffian's system,

6) Analyse Keppler's problem using Hamilton Jacobis method.

7) Prove that any complete solution of the Hamilton-Jacobi's equation leads to a solution of the Hamiltonian problem.

8) Establish Stackel's theorem.

9) Illustrate the Hamilton-Jacobi's method by an example.

10) Write a brief note on separability.

Objective Questions

1) Which transformation preserve the Hamiltonian form of the form of the equations of motion in the new variables?

a) Canonical transformation b) Noncanonical transformation

c) Legendre transformation d) Laplace transformation

2) In case of canonical transformation ....

a) The form of the Hamilton's equation is need not preserved

b) Hamilton's principle is satisfied in old as well as in new coordinates

c) The form of the Hamilton's equation cannot be preserved

d) The form of the Hamilton's equation may or may not be preserved

3) The complete solution of Hamilton's canonical equations, commonly known as the solution of the ....

a) Newton's problem b) Euler's problem

c) Hamilton's problem d) Lagrange's problem

4) The complete solution for a holonomic system having n degrees of freedom is obtained by finding $2n$ independent function known as ....

a) Principle function b) Generating function

c) Charecteristic function d) Integrals of the motion

5) The function $S(q_0, q_1, t_0, t_1) = \int_{t_0}^{t_1} Ldt$ is assumed to be twice differentiable in all its arguments and is known as ....

a) Generating function b) Hamilton's principle function

c) Charecteristic function d) D'Alembert's Principle

6) In which space, the solution of the Lagrange problem $q_i1 = q_i1(q_0, p_0, t_0, t_1)$ $(i = 1, 2, 3, \ldots .n)$gives the motion as a function of time?

a) Configuration space b) Phase space

c) Eigen space d) Euclidean space

7) Name of the space the solution of the Hamilton problem $p_{i1} = p_{i1}(q_0, p_0, t_0, t_1)$ (i=1,2,3,....n) gives the motion as a function of time?

a) Configuration space b) Phase space

c) Eigen space d) Euclidean space

8) The pfaffian differential form Ω in m variables $x_1, x_2, \ldots, x_m$ is given by $\Omega = X_1(x)dx_1 + ...X_m(x)dx_m$ is an exact differential if $C_{ij} = \frac{\partial X_i}{\partial x_j} - \frac{\partial X_j}{\partial x_i}$ then all the $c's$ are .....

a) Zero b) Non zero c) Greater than zero d) Less than zero

9) $\frac{\partial s}{\partial t} + H(q, \frac{\partial s}{\partial t}, t) = 0$ is known as ......

a) Hamilton – Jacobi equation b) Euler – Lagrange equation

c) Modified Jacobi equation d) D'Alembert's equation

10) Hamilton Jacobi equation is

a) 1st order ODE b) 2nd order ODE

c) 1st order PDE d) 2nd order PDE

11) Which of the following is known as the modified Hamilton – Jacobi equation

a) $\frac{\partial s}{\partial t} + H = 0$ b) $H(q, \frac{\partial w}{\partial q}) \neq a_n$

c) $H(q, \frac{\partial w}{\partial q}) = a_n$ d) $\frac{\partial s}{\partial t} - H = 0$

12) The Hamiltonian function of the mass spring system is ......

a)$H = \frac{p^2}{2m} - \frac{1}{2}kX^2$ b)$H = \frac{p^2}{2m} + \frac{1}{2}kX^2$

c) $H = \frac{p}{2m} - \frac{1}{2}kX^2$ d)$H = \frac{p^2}{m} - \frac{1}{2}kX^2$

13) Find the generalized momentam in the kepler problem?

a) $p_r = r, \quad p_\theta = r\theta$ b) $p_r = 0, \quad p_\theta = r^2\theta$

c) $p_r = r, \quad p_\theta = r^2\theta$ d) $p_r = r, \quad p_\theta = 0$

14) In which system, whose kinetic energy function contains only squared terms in the q's

a) Rheonomic system b) Non holonomic system

c) Holonomic system d) Orthogonal system

15) In what type of system, the kinetic energy function contains no inertial coupling terms?

a) Rheonomic system b) Orthogonal system

c) Non holonomic system d) Holonomic system

16) The Liouville conditions are necessary for a separability of an orthogonality system

for the special case in which . . . ..

a) n = 2 b) n = 0 c) n = 1 d) n = 3

17) Which conditions are sufficient for a reparability of an orthogonal system?

a) Jacobi b) Hamilton c) Euler d) Liouville

18) In the Liouville system, represents this expression $\frac{R_1(p_1)^2+...+R_n(p_1)^2}{2(f_1+...+f_n)}$ form of Liouville system

a) Kinetic energy b) Potential energy

c) Generalizing energy d) Total energy

19) In the Liouville system represents the expression $\frac{v_1(q_1)+...+v_n(q_n)}{f_1(q_1)+...+f_n(q_n)}$

a) Kinetic energy b) Potential energy

c) Generalizing energy d) Total energy

20) The conservative holonomic system whose kinetic energy functin contains only squard terms in the q's and p's and no product terms is called

a) Lioville's system b) Separable system

c) Orthogonal system d) Modified system

Answers for Check Your Progress

1. The canonical integral of the form $s(q_0, q_1, t_n, t_1) = \int_{t_0}^{t_1} Ldt$. The function s is a twice differential in all its arguments and is known as Hamilton's principle function.

2. A pfaffian form is defined by $\Omega = \sum_{i=1}^{m} X_i(x)dx_i$.

3. If $s(q, \alpha, t)$ is any complete solution of the Hamilton Jacobi equation

$\frac{\partial s}{\partial t} + H(q, \frac{\partial s}{\partial t}, t) = 0, \quad -\beta_i = \frac{\partial s}{\partial \alpha_i}, \quad p_i = \frac{\partial s}{\partial q_i}$, where $\beta' s$ are arbitrary constant are used to solve for $q_i(\alpha, \beta, t)$ and $p_i(\alpha, \beta, t)$. Then these expressions provide the general solution of the canonical equation associated with Hamiltonian $H(q, p, t)$.

4. $H(q, \frac{\partial s}{\partial q_i}) = \alpha_n$ is the modified - Hamilton Jacobi equation.

5. It is an orthogonal system which has kinetic and potential energy of the form $T = \frac{1}{2}\sum_{i=1}^{n} f_i(q_i)(\sum_{i=1}^{n} \frac{\dot{q_i}^2}{R_i(q_i)})$ and $V = \sum_{i=1}^{n} \frac{\nu_i(q_i)}{f_i(q_i)}$ where f_i, q_i and ν_i are each function of q_i. We assume that $\sum_{i=1}^{n} f_i(q_i) > 0$ and$R_i(q_i) > 0$.

6. The orthogonal system specified by $T = \frac{1}{2}\sum_{i=1}^{n} m_i\dot{q_i}^2 = \frac{1}{2}\sum_{i=1}^{n} c_i p_i^2$, where $c_i(q_1, q_2, ..., q_n) > 0$ is a stacked are met, namely that a non-singular $n \times n$ matrix $[p_{ij}(q_i)]$ and a column matrix $\psi(q_i)$ exist such that $c^T\pi = (1, 0, ...0)$ and $c^T\psi = V$, where $v(q_1, q_2, ..., q_n)$ is the potential energy and c is a column matrix.

Suggested Readings

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Unit 5

CANONICAL TRANSFORMATIONS

Objectives

After the successful completion of this unit; the students are expected

- To gain the knowledge of differential form and generating functions.
- To discuss the canonimal transformations with illustrate examples.
- To understand the concepts of the Hamilton-Jacobi's method.
- To analyse the special transformations like identity, orthogonal, translation transformation.
- To understand the homogeneous canonical, point and momentum transformation.
- To develope the concepts of Lagrange and Poisson brackets.
- To derive the Bilinear covarient.

5. Introduction

Dear students, in this last chapter we have discussed primarily with the use of Hamilton Jacobi's method in obtaing the principal function $S(q, \alpha, t)$ and for the conservative system, the characteristic function $W(q, \alpha)$. In both two cases we found the solution of Hamilton problem (ie) the solution of Hamilton canonical equation of motion. The solution represents a canonical transformation between 2 points in phase space namely a moving point (q, p) and a fixed point $(\alpha.\beta)$. The principle function is the generating function for this transformation.

5.1 Differential Forms and Generating Functions

Dear students, in this section we will discuss about the canonical transformations, principle forms of generating functions and also futher comments on the Hamilton-Jacobi's method.

5.1.1 Canonical transformations

Consider a holonomic system described by the generalized coordinates $q_1, q_2, ..., q_n$. W.K.T

$$\delta \int_{t_0}^{t_1} L(q, \dot{q}, t)dt = 0. \tag{5.1}$$

Let us consider a new set of coordinates $Q_1, Q_2, ..., Q_n$ related by a point transformation

$$q_i = q_i(Q, t), \qquad (i = 1, 2, ..., n). \tag{5.2}$$

The lagrangian is given by

$$L^*(Q, \dot{Q}, t) = L(q, \dot{q}, t) = T - V. \tag{5.3}$$

L and $L^*$are same in the value. Applying Hamilton's principle to L^*, we get

$$\delta \int_{t_0}^{t_1} L^*(Q, \dot{Q}, t)dt = 0. \tag{5.4}$$

Now let us consider a new lagrangian function

$$L^*(Q, \dot{Q}, t) = L(q, \dot{q}, t) - \frac{d}{dt}(\phi(q, Q, t)), \tag{5.5}$$

where $\phi(q, Q, t)$ is twice differentiable. Now

$$\delta \int_{t_0}^{t_1} L^*(Q, \dot{Q}, t) = \delta \int_{t_0}^{t_1} L(q, \dot{q}, t) - \delta[\phi(q, Q, t)]_{t_0}^{t_1}. \tag{5.6}$$

Since $\delta q's$ and $\delta Q's$ are zero the term vanishes. Using eqn (5.1) and (5.5) becomes

$$\delta \int_{t_0}^{t_1} L^*(Q, \dot{Q}, t)dt = 0. \tag{5.7}$$

Hence $L^*(Q, \dot{Q}, t)$ describes the given system as effectively as $L(q, \dot{q}, t)$

Now let us consider two Hamiltonian description of the given holonomic system. The Hamiltonian functions are given by

$$H(q, p, t) = \sum_{i=1}^{n} p_i, q_i - L(q, \dot{q}, t) \tag{5.8}$$

$$K(Q, p, t) = \sum_{i=1}^{n} p_i, Q_i - L^*(Q, \dot{Q}, t). \tag{5.9}$$

Where the generalised momenta are given by,

$$p_i = \frac{\partial L}{\partial \dot{q}_i}.$$
$$P_i = \frac{\partial L^*}{\partial \dot{Q}_i}. \tag{5.10}$$

Since, the Hamilton's principle apply to $L(q, \dot{q}, t)$ and $L^*(Q, \dot{Q}, t)$, we have the canonical equations as follows

$$\dot{p}_i = -\frac{\partial H}{\partial \dot{q}_i}.$$
$$\dot{q}_i = \frac{\partial H}{\partial \dot{p}_i}, (i = 1, 2, 3, ..., n). \tag{5.11}$$
$$\dot{P}_i = -\frac{\partial K}{\partial Q_i}.$$
$$\dot{Q}_i = \frac{\partial K}{\partial P_i}, (i = 1, 2, 3, ..., n). \tag{5.12}$$

A transformation form (q, p) to (Q, P) which preserves the canonical form of the equation of motion is known as canonical transformation

Next let us consider a system which has a Hamiltonian function $H(q, p, t)$. The transformation equations are of the form

$$Q_i = Q_i(q, p, t), \qquad P_i = P_i(q, p, t), \qquad i = 1, 2, ...n, \tag{5.13}$$

where each function is twice differentiable. Substituting the values of L and L* from eqns (5.8) and (5.9) in eqn (5.5), we get

$$\sum_{i=1}^{n} P_iQ_i = \sum_{i=1}^{n} p_iq_i - H(q,p,t) - \frac{d}{dt}\phi(q,Q,t)$$

$$\frac{d}{dt}\phi(q,Q,t) = \sum_{i=1}^{n} p_iq_i - H(q,p,t) - \sum_{i=1}^{n} p_iQ_i + K(Q,P,t)$$

$$\frac{d}{dt}\phi(q,Q,t) = \sum_{i=1}^{n} p_idq_i - H(q,p,t)dt - \sum_{i=1}^{n} p_idQ_i + K(Q,P,t)dt$$

$$d\phi = \sum_{i=i}^{n} p_idq_i - Hdt - \sum_{i=1}^{n} p_idQ_i + Kdt. \tag{5.14}$$

The exact differential $d\phi$ is equal to the difference two pfaffian differential forms canonical transformation form the variables (q,p) and the associated Hamiltonian function $H(q,p,t)$ to the new variables (Q,P) and the Hamiltonian $K(Q,P,t)$ is called the generating functions for the transformation.

Now let us consider a function $\psi(q,p,t)$ which is equal in the value to the generating function ϕ

$$\psi(q,p,t) = \phi(q,Q,t). \tag{5.15}$$

Then eqn (5.14) becomes

$$p_idqi - Hdt - \sum_{i=1}^{n} p_idQi + Kdt = d\psi. \tag{5.16}$$

Here the function $\psi(q,p,t)$ is not a generating function because it contains none of the new variables.

Next let us consider a generating function $\phi(q,p,t)$ which is arbitrary. Now

$$d\phi = \sum_{i=1}^{n} \frac{\partial\phi}{\partial q_i}dq_i + \sum_{i=1}^{n} \frac{\partial\phi}{\partial Q_i}dQ_i + \frac{\partial\phi}{\partial t}dt. \tag{5.17}$$

Comparing eqn (5.14) and eqn (5.17)

$$p_i = \frac{\partial\phi}{\partial q_i}, \qquad (i = 1,2,3,.....,n). \tag{5.18}$$

$$P_i = -\frac{\partial\phi}{\partial Q_i}, \qquad (i = 1,2,3,.....,n). \tag{5.19}$$

$$K = H + \frac{\partial\phi}{\partial t}. \tag{5.20}$$

Eqn (5.20) and (5.19) can be used to solve $q_i(Q,P,t)$ and $p_i(Q,P,t)$ or conversely for $Q_i(q,p,t)$ and $P_i(q,p,t)$. The new Hamiltonian function $K(Q,P,t)$ is found by using the transformation equations and eqn (5.20). Hence, if an arbitrary generating $\phi(q,Q,t)$ is given, eqn (5.18) and (5.19) gives the transformation equations. Further the time t is unchanged in eqn (5.14) (in the transformation) and hence it may be regraded as a independent parameter. Now dt can be set to zero.The eqn (5.14) and eqn (5.16) reduces to

$$\sum_{i=1}^{n} p_i\delta q_i - \sum_{i=1}^{n} P_i\delta Q_i = \delta\phi. \tag{5.21}$$

$$\sum_{i=1}^{n} p_i\delta q_i - \sum_{i=1}^{n} P_i\delta Q_i = \delta\psi. \tag{5.22}$$

Now, consider $\psi(q,p,t)$ and $Q(q,p,t)$

$$\delta\psi = \sum_{i=1}^{n} \frac{\partial\psi}{\partial q_i}\delta q_i + \sum_{i=1}^{n} \frac{\partial\psi}{\partial p_i}\delta p_i. \tag{5.23}$$

$$\delta Q_j = \sum_{j=1}^{n} \frac{\partial Q_j}{\partial q_i}\delta q_i + \sum_{j=1}^{n} \frac{\partial Q_j}{\partial p_i}\delta p_i. \tag{5.24}$$

$$\sum_{j=1}^{n} P_j\delta Q_j = \sum_{j=1}^{n} P_j(\sum_{j=1}^{n} \frac{\partial Q_j}{\partial q_i}\delta q_i + \sum_{j=1}^{n} \frac{\partial Q_j}{\partial p_i}\delta p_i). \tag{5.25}$$

Now from (5.21) and (5.25)

$$\begin{aligned}\delta\psi = \delta\phi &= \sum_{i=1}^{n} p_i\delta q_i - \sum_{j=1}^{n}\sum_{i=1}^{n} P_j\frac{\partial Q_j}{\partial q_i}\delta q_i - \sum_{j=1}^{n}\sum_{i=1}^{n} P_j\frac{\partial Q_j}{\partial p_i}\delta p_i \\ &= \sum_{i=1}^{n}[p_i - \sum_{i=1}^{n} P_j\frac{\partial Q_j}{\partial q_i}]\delta q_i - \sum_{j=1}^{n}\sum_{i=1}^{n} P_j\frac{\partial Q_j}{\partial p_i}\delta p_i. \end{aligned} \tag{5.26}$$

Compare eqn (5.23) and eqn (5.26). Hence

$$\frac{\partial\psi}{\partial q_i} = p_i - \sum_{j=1}^{n} P_j\frac{\partial Q_j}{\partial q_i} \tag{5.27}$$

$$and \qquad \frac{\partial\psi}{\partial p_i} = -\sum_{j=1}^{n} P_j\frac{\partial Q_j}{\partial p_i}. \tag{5.28}$$

Consider the total differentials of $\psi(p,q,t)$ and $Q(q,p,t)$

$$d\psi = \sum_{i=1}^{n} \frac{\partial \psi}{\partial q_i} dq_i + \sum_{i=1}^{n} \frac{\partial \psi}{\partial p_i} dp_i + \frac{\partial \psi}{\partial t} dt. \tag{5.29}$$

$$dQ_j = \sum_{i=1}^{n} \frac{\partial Q_j}{\partial q_i} dq_i + \sum_{i=1}^{n} \frac{\partial Q_j}{\partial p_i} dp_i + \frac{\partial Q_j}{\partial t} dt. \tag{5.30}$$

$$\sum_{j=1}^{n} P_j dQ_j = \sum_{j=1}^{n} P_j (\sum_{i=1}^{n} \frac{\partial Q_j}{\partial q_i} dq_i + \sum_{i=1}^{n} \frac{\partial Q_j}{\partial p_i} dp_i + \frac{\partial Q_j}{\partial t} dt).$$

From eqn (5.16)

$$d\psi = d\phi = \sum_{i=1}^{n} p_i dq_i - Hdt - \sum_{j=1}^{n} P_j dQ_j + Kdt$$

$$= \sum_{i=1}^{n} p_i dq_i - Hdt - \sum_{i=1}^{n} \sum_{j=1}^{n} P_j dQ_j dq_i - \sum_{j=1}^{n} \sum_{i=1}^{n} P_j dQ_j dP_i - \sum_{j=1}^{n} P_j \frac{\partial Q_j}{\partial t} dt + Kdt$$

$$d\psi = \sum_{i=1}^{n} [p_i - \sum_{j=1}^{n} P_j \frac{\partial Q_j}{\partial q_i}] dq_i - \sum_{i=}^{n} \sum_{j=1}^{n} P_j \frac{\partial Q_j}{\partial p_i} dp_i - Hdt + Kdt - \sum_{j=1}^{n} P_j \frac{\partial Q_j}{\partial t}. \tag{5.31}$$

From eqn (5.29) and eqn (5.30), equating the coefficients of dt, we get

$$H + K - \sum_{j=1}^{n} P_j \frac{\partial Q_j}{\partial \partial t} = \frac{\partial \psi}{\partial t}$$

$$K = H + \sum_{j=1}^{n} P_j \frac{\partial Q_j}{\partial \partial t} + \frac{\partial \psi}{\partial t}.$$

Now eqn (5.14) indicates the canonical transformation from the old variables (q,p) to (Q,P). Now the symmetry of the eqn (5.14) and eqn (5.18), eqn (5.19) and eqn (5.20) shows that the inverse of a given canonical transformation is itself canonical and is generated by the negative of $\phi(q,p,t)$. Also sum of two exact differentials is exact. Hence the two canonical transformations performed in sequence are equivalent to a single canonical transformation. Further the identity transformation is canonical. Hence the canonical transformations forms the group.

Example 1: Consider the transformation $Q = \frac{1}{2}(q^2+p^2)$ and $P = -\tan^{-1}\left(\frac{q}{p}\right)$. To show that the transformation is canonical. The old transformation function is $H = \frac{1}{2}(q^2+p^2)$.

Solution :

$$\delta Q = \frac{1}{2}\delta\left(q^2+p^2\right) = q\delta q + p\delta p.$$

Now $p\delta q - P\delta Q = p\delta q + \tan^{-1}\left(\frac{q}{p}\right)(q\delta q + p\delta p)$

$$= \left[p\delta q + \tan^{-1}\left(\frac{q}{p}\right)\right]\delta q + \tan^{-1}\left(\frac{q}{p}\right)\delta p. \tag{5.32}$$

Consider

$$\frac{\partial}{\partial p}\left(p + q\tan^{-1}\left(\frac{q}{p}\right)\right) = 1 + q\frac{1}{1+\frac{q^2}{p^2}}\left(\frac{-q}{p^2}\right) = \frac{p^2}{p^2+q^2}. \tag{5.33}$$

$$\frac{\partial}{\partial q}\left[p\tan^{-1}\left(\frac{q}{p}\right)\right] = p\frac{1}{1+\frac{q^2}{p^2}}\left(\frac{1}{p}\right) = \frac{p^2}{p^2+q^2}. \tag{5.34}$$

Since eqn (5.32) = eqn (5.33). Therefore eqn (5.31) is exact. Hence the transformation is canonical. Now

$$d\psi = p + q\tan^{-1}\left(\frac{q}{p}\right) + + p\tan^{-1}\left(\frac{q}{p}\right)dp.$$

Consider,

$$\int q\tan^{-1}\left(\frac{q}{p}\right)dq = \int pdq + \int q\tan^{-1}\left(\frac{q}{p}\right)dq = pq + \int qtan^{-1}\left(\frac{q}{p}\right)dq. \tag{5.35}$$

Consider, $\int q\tan^{-1}\left(\frac{q}{p}\right)dq$

$$\begin{aligned}
u &= \tan^{-1}\left(\frac{q}{p}\right) \qquad \int dv = \int qdq \\
du &= \frac{p}{p^2+q^2}dq \qquad v = \frac{q^2}{2} \\
\therefore \int qtan^{-1}\left(\frac{q}{p}\right)dq &= \frac{q^2}{2}\tan^{-1}\left(\frac{q}{p}\right) - \int \frac{q^2}{2}\frac{p}{p^2+q^2}dq = \frac{q^2}{2}\tan^{-1}\left(\frac{q}{p}\right) - \frac{p}{2}\int\frac{p^2+q^2-p^2}{p^2+q^2}dq \\
&= \frac{q^2}{2}\tan^{-1}\left(\frac{q}{p}\right) - \frac{pq}{2} + \frac{p^3}{2}\int\frac{dq}{p^2+q^2} \\
\int q\tan^{-1}\left(\frac{q}{p}\right)dq &= \frac{q^2}{2}\tan^{-1}\left(\frac{q}{p}\right) - \frac{pq}{2} + \frac{p^2}{2}\tan^{-1}\left(\frac{q}{p}\right).
\end{aligned} \tag{5.36}$$

$\therefore$ Substitute (5.36) in (5.35)

$$\begin{aligned}
\int p + qtan^{-1}\left(\frac{q}{p}\right)dq &= pq + \frac{q^2}{2}tan^{-1}\left(\frac{q}{p}\right) - \frac{pq}{2} + \frac{p^2}{2}tan^{-1}\left(\frac{q}{p}\right) = \frac{pq}{2} + \left(\frac{p^2+q^2}{2}\right)tan^{-1}\left(\frac{q}{p}\right) \\
\psi(q,p) &= \frac{pq}{2} + \left(\frac{p^2+q^2}{2}\right)tan^{-1}\left(\frac{q}{p}\right) \\
\phi(q,Q,t) &= Qsin^{-1}\left(\frac{q}{2Q}\right) + \sqrt{2Q-q^2}\left(\frac{q}{2}\right),
\end{aligned}$$

where ϕ is a generating function and $\phi = \psi$.

$$\text{W.K.T } K = H + \frac{\partial \phi}{\partial t}$$
$$\text{But} \quad \frac{\partial \phi}{\partial t} = 0,$$
$$K = H = \frac{1}{2}\left(q^2 + p^2\right) = Q$$
$$\text{Hence} \quad K = Q.$$

Canonical equation

$$\dot{p} = \frac{\partial K}{\partial Q} = -1$$
$$K = Q \quad \text{constant}$$
$$\dot{Q} = \frac{\partial K}{\partial P} = 0.$$

Example 2: Canonical for Rhenomic transformation

Show that the transformation is canonical for rhenomic transformation given by,

$$Q = \sqrt{2q}e^t \cos p \text{ and } P = \sqrt{2q}e^{-t} \sin p.$$

Solution:

$$p\delta q - P\delta Q = p\delta q - (\sqrt{2q}e^{-t}\sin p)\delta(\sqrt{2q}e^t \cos p)$$
$$\delta(\sqrt{2q}e^t \cos p) = \frac{2e^t \cos p}{2\sqrt{2q}}\delta q - \sqrt{2q}e^{-t}\sin p\delta p$$
$$\therefore p\delta q - P\delta Q = p\delta q - (\sqrt{2q}e^{-t} sinp)\left[\frac{2e^t \cos p}{2\sqrt{2q}}\delta q - \sqrt{2q}e^{-t} sinp\delta p\right]$$
$$= (p - \sin p \cos p)\delta q + 2q \sin^2 p\delta p. - - - - - - - - - - - - - - (A)$$
$$\frac{\partial}{\partial p}(p - \sin p \cos p) = 2\sin^2 p. \tag{5.37}$$
$$\frac{\partial}{\partial q}(2q \sin^2 p) = 2\sin^2 p. \tag{5.38}$$

eqn (5.37) = eqn (5.38) $\Rightarrow A$ is exact.

$$\int (p - \sin p \cos p)\delta q = pq - q \sin p \cos p$$
$$\therefore \psi = \phi.$$

The generating function ϕ is given by

$$\begin{aligned}\phi(q,Q,t) =& q\cos^{-1}\left(\frac{Qe^{-t}}{\sqrt{2q}}\right) - q\left(\frac{Q}{e^t\sqrt{2q}}\right)\left(\sqrt{1-\frac{Q^2}{2qe^{2t}}}\right)\\ =& q\cos^{-1}\left(\frac{Qe^{-t}}{\sqrt{2q}}\right) - \frac{Qe^{-t}\sqrt{2q-Q^2e^{-2t}}}{2}.\end{aligned}$$

5.1.2 Principle forms of generating functions

To determine the various types of generating functions.

Let us designate $\Phi(q,Q,t)$ as the first type $F_1(q,Q,t) = \phi(q,Q,t)$. The other types of generating functions are $F_2(q,P,t)$, $F_3(p,Q,t)$ and $F_4(p,P,t)$.

To find the relationship between $F_1(q,Q,t)$ and $F_2(q,P,t)$.

Now

$$dF_1 = \sum_{i=1}^{n} p_i dq_i - Hdt - \sum_{i=1}^{n} P_i dQ_i + Kdt. \tag{5.39}$$

Replace Q' s by P's after considering

$$d(\sum_{i=1}^{n}) = \sum_{i=1}^{n} Q_i dP_i + \sum_{i=1}^{n} P_i dQ_i. \tag{5.40}$$

$$\sum_{i=1}^{n} P_i dQ_i = d(\sum_{i=1}^{n} Q_i P_i) - \sum_{i=1}^{n} Q_i dP_i. \tag{5.41}$$

Substitute eqn (5.40) in eqn (5.41)

$$\begin{aligned} dF_1 &= \sum_{i=1}^{n} p_i dq_i - Hdt - d(\sum_{i=1}^{n} Q_i P_i) + \sum_{i=1}^{n} Q_i dP_i + Kdt\\ dF_1 + d(\sum_{i=1}^{n} Q_i P_i) &= \sum_{i=1}^{n} p_i dq_i - Hdt + \sum_{i=1}^{n} Q_i dP_i + Kdt\\ d(F_1 + \sum_{i=1}^{n} Q_i P_i) &= \sum_{i=1}^{n} p_i dq_i - Hdt + \sum_{i=1}^{n} Q_i dP_i + Kdt\end{aligned}$$

$$dF_2 = \sum_{i=1}^{n} p_i dq_i - Hdt + \sum_{i=1}^{n} Q_i dP_i + Kdt. \tag{5.42}$$

$$\text{Where } F_2 = F_1 + \sum_{i=1}^{n} Q_i P_i \tag{5.43}$$

$$F_2(q,P,t) = F_1(q,Q,t) + \sum_{i=1}^{n} Q_i P_i.$$

Taking the total differential of $F_2(q,P,t)$ we get

$$dF_2 = \sum_{i=1}^{n} \frac{\partial F_2}{\partial q_i} dq_i + \sum_{i=1}^{n} \frac{\partial F_2}{\partial P_i} dP_i + \frac{\partial F_2}{\partial t} dt. \qquad (5.44)$$

Form (5.42) and (5.44)

$$p_i = \frac{\partial F_2}{\partial q_i}. \qquad (5.45)$$

$$Q_i = \frac{\partial F_2}{\partial P_i}. \qquad (5.46)$$

$$-H + K = \frac{\partial F_2}{\partial t}$$

$$K = H + \frac{\partial F_2}{\partial t}. \qquad (5.47)$$

Thus we have obtained a generating function $F_2(q,P,t)$ with differential form given by eqn (5.42) and the canonical transformation equations form eqn(5.46) and the Hamiltonian function is given by eqn (5.47). Consider $F_1 - \sum_{i=1}^{n} p_i q_i$

$$\begin{aligned} d(F_1 - \sum_{i=1}^{n} p_i q_i) &= dF_1 - \sum_{i=1}^{n} dp_i q_i - \sum_{i=1}^{n} p_i dq_i \\ &= \sum_{i=1}^{n} p_i dq_i - Hdt - \sum_{i=1}^{n} P_i dQ_i + Kdt - \sum_{i=1}^{n} dp_i q_i - \sum_{i=1}^{n} p_i dq_i \\ &= -\sum_{i=1}^{n} P_i dQ_i - \sum_{i=1}^{n} dp_i q_i - Hdt + Kdt \\ dF_3(p,Q,t) &= -\sum_{i=1}^{n} dp_i q_i - Hdt - \sum_{i=1}^{n} P_i dQ_i + Kdt \end{aligned} \qquad (5.48)$$

Consider the total differential of $F_3(p,Q,t)$

$$dF_3 = \sum_{i=1}^{n} \frac{\partial F_3}{\partial p_i} dp_i + \sum_{i=1}^{n} \frac{\partial F_3}{\partial Q_i} dQ_i + \frac{\partial F_3}{\partial t} dt. \qquad (5.49)$$

Comparing (5.48) and (5.49)

$$\begin{aligned} \frac{\partial F_3}{\partial p_i} &= -q_i \\ \frac{\partial F_3}{\partial Q_i} &= -p_i, \quad (i = 1,2,3,......,n). \qquad (5.50) \\ \frac{\partial F_3}{\partial t} &= -H + K \\ K &= H + \frac{\partial F_3}{\partial t}. \qquad (5.51) \end{aligned}$$

Thus we have obtained a generating function $F_3(P,Q,t) = F_1 - \sum_{i=1}^{n} p_i q_i$ with differential form given by (5.48) and the canonical transformation equations given by (5.50) and the Hamiltonian function given by (5.51). Consider $F_2 - \sum_{i=1}^{n} q_i p_i$

$$\begin{aligned} d(F_2 - \sum_{i=1}^{n} q_i p_i) &= dF_2 - \sum_{i=1}^{n} dq_i p_i - \sum_{i=1}^{n} q_i dp_i \\ &= \sum_{i=1}^{n} p_i dq_i - H dt - \sum_{i=1}^{n} Q_i dP_i + K dt - \sum_{i=1}^{n} dq_i p_i - \sum_{i=1}^{n} q_i dp_i \\ &= \sum_{i=1}^{n} Q_i dP_i - \sum_{i=1}^{n} dp_i q_i - H dt + K dt \\ dF_4(p,P,t) &= - \sum_{i=1}^{n} dp_i q_i - H dt - \sum_{i=1}^{n} Q_i dP_i + K dt. \end{aligned} \tag{5.52}$$

Consider the total differential of $F_4(p,P,t)$

$$dF_4 = \sum_{i=1}^{n} \frac{\partial F_4}{\partial p_i} dp_i + \sum_{i=1}^{n} \frac{\partial F_4}{\partial P_i} dP_i + \frac{\partial F_4}{\partial t} dt. \tag{5.53}$$

Comparing (5.52) and (5.53)

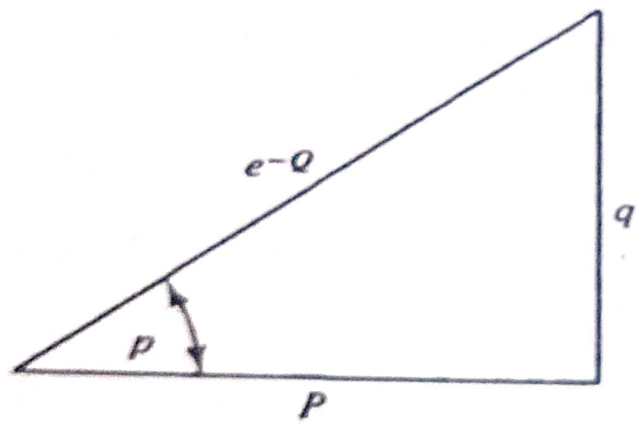
$$\begin{aligned} \frac{\partial F_4}{\partial p_i} &= -q_i \\ \frac{\partial F_4}{\partial P_i} &= Q_i, \qquad (i = 1,2,3,......,n). \qquad (5.54) \\ \frac{\partial F_4}{\partial t} &= -H + K \\ K &= H + \frac{\partial F_4}{\partial t}. \qquad (5.55) \end{aligned}$$

Thus we have obtained a generating function $F_4 = F_2 - \sum_{i=1}^{n} q_i p_i$ with differential form given by (5.52) and the canonical transformation equations given by (5.54) and the Hamiltonian function given by (5.55).

Example 3: Consider the transformation

$$\begin{aligned} Q &= \log(\frac{\sin p}{q}). \qquad ----(I) \\ P &= q \cot p. \qquad -----(II) \end{aligned}$$

Obtain the four types of generating functions.



Solution :

To show that the transformation is canonical

$$p\delta q - P\delta Q = p\delta q - (q\cot p)\delta(\log(\frac{\sin p}{q}))$$
$$\delta(\log(\frac{\sin p}{q})) = \frac{1}{\frac{\sin p}{q}}\left[\frac{q\cos p \quad \delta p - \sin p\delta q}{q^2}\right] = \cot p\delta p - \frac{1}{q}\delta q$$
$$p\delta q - P\delta Q = p\delta q - (q\cot p)\left(\cot p\delta p - \frac{1}{q}\delta q\right) = (p+\cot p)\delta q - q\cot^2\delta p. \tag{5.56}$$

To show the exactness

$$\frac{\partial}{\partial p}(p+\cot p) = 1 - cosec^2 p. \tag{5.57}$$
$$\frac{\partial}{\partial q}(-q\cot^2 p) = -\cot^2 p = 1 - cosec^2 p. \tag{5.58}$$

Since eqn (5.57) and eqn (5.58) are equal. The given equation is exact. To find ψ

$$\psi = \int(p+\cot p)\delta q = pq + (\cot p)q$$
$$From(I) \quad e^Q = \frac{\sin p}{q} \Rightarrow p = \sin^{-1}(qe^Q).$$

Now

$$\sin p = qe^Q \Rightarrow \cos p = \sqrt{1-\sin^2 p}. Then$$
$$p = \cos^{-1}\sqrt{1-qe^{2Q}}.$$

Now

$$
\begin{aligned}
\cot p =& \frac{\sqrt{1-q^2e^{2Q}}}{qe^Q} = \frac{\sqrt{e^{-2Q}}-q^2}{q} \\
\phi(q,Q) =& q\cos^{-1}(\sqrt{1-q^2e^{2Q}}) + \frac{\sqrt{e^{-2Q}-q^2}.q}{q} \\
(ie.,)F_1(q,Q) =& q\cos^{-1}(\sqrt{1-q^2e^{2Q}}) + \sqrt{e^{-2Q}-q^2}.
\end{aligned}
$$

Now

$$
F_2(q,Q) = F_1 + QP = pq + cotp.q + QP
$$

$$
pq = q[\tan^{-1}\left(\frac{q}{p}\right)]
$$

$$
QP = P[-\log\sqrt{p^2+q^2}].
$$

$$
\begin{aligned}
\therefore \quad F_2(q,Q) =& q\tan^{-1}\left(\frac{q}{P}\right) + P - P\log\sqrt{P^2+q^2} \\
=& q\tan^{-1}\left(\frac{q}{P}\right) + P(1-\log\sqrt{P^2+q^2}) \\
\frac{\partial F_2}{\partial q} =& tan^{-1}\left(\frac{q}{P}\right) + \frac{q}{1+\frac{q^2}{p^2}}\frac{1}{P} - \frac{2Pq}{2\sqrt{P^2+q^2}\sqrt{P^2+q^2}} \\
=& \tan^{-1}\left(\frac{q}{P}\right) + \frac{qP}{(P^2+q^2)} - \frac{Pq}{P^2+q^2} = \tan^{-1}\left(\frac{q}{p}\right) \\
=& \tan^{-1}\left(\frac{q}{P}\right) = P. \\
\frac{\partial F_2}{\partial P} =& \frac{q}{1+\frac{P^2}{q^2}}\left(\frac{-q}{P^2}\right) + 1 - \log\sqrt{P^2+q^2} + p\left(\frac{-1}{\sqrt{P^2+q^2}}.\frac{2P}{\sqrt{P^2+q^2}}\right) \\
=& \frac{q^2}{P^2+q^2} + 1 - \log\sqrt{P^2+q^2} - \frac{P^2}{\sqrt{P^2+q^2}} \\
=& \log\sqrt{P^2+q^2} = Q.
\end{aligned}
$$

Now

$$
\begin{aligned}
F_3(p,Q) =& F_1 - pq \\
=& pq + cotp.q - pq = cotp.q \\
=& e^{-Q}cosp \\
\frac{\partial F_3}{\partial p} =& e^{-Q}sinp = -q \\
\frac{\partial F_3}{\partial Q} =& -e^{-Q}cosp = -P.
\end{aligned}
$$

Now

$$
\begin{aligned}
F_4(p,P) =& F_2 - qp\\
=& pq + cotp.q + QP - pq = cotp.q + QP = P + QP\\
=& P + log\left(\frac{cosp}{P}\right)P\\
\frac{\partial F_4}{\partial p} =& \frac{P}{cosp}\left(\frac{-\sin p}{P}\right)P = -P\tan p = -q\\
\frac{\partial F_4}{\partial P} =& 1 + \log\left(\frac{\cos p}{P}\right) + \frac{P^2\cos p}{\cos p}\frac{-1}{P^2} = \log\left(\frac{\cos p}{P}\right) = Q.
\end{aligned}
$$

5.1.3 Further comments on Hamilton-Jacobi's method

We know that the Hamilton Jacobi partial differential equation is given by

$$\tfrac{\partial s}{\partial t} + H(q, \tfrac{\partial s}{\partial q}, t) = 0.$$

Where

$$
\begin{aligned}
p_i &= \frac{\partial F_1}{\partial q_i}, \qquad (i = 1, 2, 3, ..., n).\\
and \qquad -\beta_i &= \frac{\partial F_1}{\partial \alpha_i}.
\end{aligned}
$$

Considering $F_2(q, \alpha, t)$ as the generating function. The Hamiltonian-Jacobi equation is given by

$$\tfrac{\partial F_2}{\partial t} + H(q, \tfrac{\partial F_2}{\partial q}, t) = 0,$$

where

$$
\begin{aligned}
p_i &= \frac{\partial F_2}{\partial q_i} \qquad (i = 1, 2, 3, ..., n)\\
and \qquad \beta_i &= \frac{\partial F_2}{\partial \alpha_i}.
\end{aligned}
$$

Similarly the Hamiltonian-Jacobi equation in terns of F_3 and F_4 are given by

$$
\begin{aligned}
\frac{\partial F_3}{\partial t} + H(-\frac{\partial F_3}{\partial q}, p, t) = 0.\\
\frac{\partial F_4}{\partial t} + H(-\frac{\partial F_4}{\partial q}, p, t) = 0.
\end{aligned}
$$

Here the generating function are of the form $F_3(p, \alpha, t)$ and $F_3(p, \alpha, t)$.

Next let us consider the characteristic function $W(q, \alpha)$ as generating function then the Hamiltonian Jacobi equation is $H(q, \frac{\partial w}{\partial q}) = \alpha_n$.

Let $P_i = P_i(\alpha)$, where P's are the function of $\alpha's$ or conversely $\alpha_i = \alpha_i(p)$.

Then the generating function $W(q, \alpha)$ takes the form $W(q, P)$ resembling F_2.

The transformation equation are,

$$P_i = \frac{\partial w}{\partial q_i}.$$
$$Q_i = \frac{\partial w}{\partial p_i}.$$

The Hamiltonian takes the form $K(p) = \alpha_n(p)$

$$\dot{Q}_i = \frac{\partial K}{\partial P_i} = \nu_i, \qquad (i = 1, 2, 3, ..., n).$$
$$\dot{P}_i = -\frac{\partial K}{\partial p_i} = 0, \qquad (i = 1, 2, 3, ..., n).$$

Now

$$Q_i = \nu_i$$

$$Q_i = \nu_i(t) + \beta_i,$$

where ν_i are functions of P's. The characteristic function has resulted in a new set of coordinates which in general vary with time.

Let us sum up

1. We have introduce the cononical transformations with few examples.
2. We have discuss the principle forms of generating functions.
3. Also we discuss the comments on Hamilton-Jacobi's method.

Check your progress

1. Write the Hamilton-Jacobi's equation in terms of F_3 and F_4.
2. What is mean by canonical transformation.
3. State principle forms of generating functions.

5.2 Some Special Transformations

Dear students, in this section we will discuss some simple canonical transformations. Also we will discuss the identity, orthogonal, translation, point and momentum transformations.

5.2.1 Simple canonical transformations

1. Let us consider the identity transformation.

Consider $F_2 = \sum\limits_{i=1}^{n} q_i P_i$.

Now

$$P_i = \frac{\partial F_2}{\partial q_i} = p_i.$$
$$Q_i = \frac{\partial F_2}{\partial P_i} = q_i.$$

Thus

$$P_i = p_i$$
$$Q_i = q_i, \qquad (i = 1, 2, 3, ..., n).$$

Confirming the identity transformation. Consider

$$F_3 = -\sum_{i=1}^{n} p_i Q_i,$$
$$\text{where} \qquad P_i = -\frac{\partial F_3}{\partial Q_i} = -(-p_i) = p_i$$
$$\text{and} \qquad q_i = -\frac{\partial F_3}{\partial p_i} = -(-Q_i) = Q_i, \qquad (i = 1, 2, 3, ..., n).$$

The functions of the form $F_1(q, Q, t)$ or $F_4(p, P, t)$ cannot be used to generate identity transformation for the variables in the generating function are directly related

2. Let us consider a transformation that results in translation. Let

$$F_2 = \sum_{i=1}^{n}(q_iP_i + c_i - d_iq_i).$$

$$\text{Now} \qquad p_i = \frac{\partial F_2}{\partial q_i} = P_i - d_i$$

$$P_i = p_i + d_i$$

$$\text{and} \qquad Q_i = \frac{\partial F_2}{\partial P_i} = q_i + c_i,$$

$$Q_i = q_i + c_i.$$

Here $P_i = p_i + d_i$ and $Q_i = q_i + c_i$ gives the required translation

3.The transformation that interchanges the roles of co-ordinates and momenta. Consider

$$F_1 = \sum_{i=1}^{n} q_iQ_i.$$

$$\text{Now} \qquad p_i = \frac{\partial F_1}{\partial q_i} = Q_i,$$

$$\text{and} \qquad P_i = -\frac{\partial F_1}{\partial Q_i} = -q_i, \qquad (i = 1, 2, 3, ..., n).$$

The presence of minus sign shows that the canonical equations are not symmetrical w.r.t the interchange of co-ordinates and momenta, where $\alpha's$ are constants meeting the orthogonality condition,

$$aa^T = a^Ta = I$$

$$\sum_{i=1}^{n} a_{ij}a_{ik} = \delta_{jk}.$$

$$\text{W.K.T} \qquad p_j = \frac{\partial F_2}{\partial q_j} = \sum_{i=1}^{n} a_{ij}P_i$$

$$Q_j = \frac{\partial F_2}{\partial P_i} = \sum_{i=1}^{n} a_{ij}q_j.$$

Hence the transformation are given by

$$P_i = \frac{p_j}{\sum_{i=1}^{n} a_{ij}} = \sum_{j=1}^{n} a_{ij}p_j + Q_i = \sum_{j=1}^{n} a_{ij}q_j.$$

Cases

1. If $|a| = 1$, then the equations, represent the rotations.

2. If $a_{ij} = \delta_{ij}$ then it represents an identity transformation with zero rotation.

5.2.2 Homogeneous canonical transformation (or) Mathieu transformation (or) contact transformation

Consider the differential form

$$\delta\psi = \sum_{i=1}^{n} p_i \delta q_i - \sum_{i=1}^{n} P_i \delta Q_i, \tag{5.59}$$

here $\delta\psi$ is an exact differential and the transformation from (q, p) to (Q, P) is called the canonical transformation. Consider the case, where ϕ and ψ are identically zero. Then

$$\sum_{i=1}^{n} p_i \delta q_i - \sum_{i=1}^{n} P_i \delta Q_i = 0, \tag{5.60}$$

and the corresponding transformation is called a homogeneous canonical transformation.

Important features of homogeneous canonical transformation:

$$L^*(Q, \dot{Q}, t) = L(q, \dot{q}, t). \tag{5.61}$$

$$\text{W.K.T} \qquad K = H + \frac{\partial \psi}{\partial t} + \sum_{i=1}^{n} P_i \frac{\partial Q_i}{\partial t}$$

$$\because \qquad \psi = 0 \qquad K = H + \sum_{i=1}^{n} P_i \frac{\partial Q_i}{\partial t}, \tag{5.62}$$

$$\text{and} \qquad P_i - \sum_{j=1}^{n} P_j \frac{\partial Q_j}{\partial q_i} = 0$$

$$\sum_{j=1}^{n} P_j \frac{\partial Q_j}{\partial q_i} = 0. \tag{5.63}$$

Where P's are not all identically equal to zero

From eqn (5.63) $|\frac{\partial Q_j}{\partial p_i}| = 0$

p's cannot be solved as functions of (q, Q, t). Consider,

$$\Omega_j(q, Q, t) = 0 \tag{5.64}$$

Differentiate

$$\sum_{i=1}^{n} \frac{\partial \Omega_j}{\partial q_i} \delta q_i + \sum_{i=1}^{n} \frac{\partial \Omega_j}{\partial Q_i} \delta Q_i = 0$$

By Lagrange's multipliers,

$$F_1^*(\lambda, q, Q, t) = \sum_{j=1}^{n} \lambda_j \Omega_j(q, Q, t), \tag{5.65}$$

arbitrary variations w.r.t Q's and q's

$$\sum_{i=1}^{n} \frac{\partial F_1^*}{\partial q_i} \delta q_i + \sum_{i=1}^{n} \frac{\partial F_1^*}{\partial Q_i} \delta Q_i = \sum_{j=1}^{m} \lambda_j [\sum_{i=1}^{n} \frac{\partial \Omega_j}{\partial q_i} \delta q_i + \sum_{i=1}^{n} \frac{\partial \Omega_j}{\partial Q_i} \delta Q_i]$$

$$\text{But } p_i = \frac{\partial F_1^*}{\partial q_i} \text{ and } P_i = \frac{\partial F_1^*}{\partial Q_i}.$$

Therefore we have

$$\sum_{i=1}^{n} p_i \delta q_i + \sum_{i=1}^{n} (-P_i) \delta Q_i = \sum_{j=1}^{m} [\sum_{i=1}^{n} \lambda_j \frac{\partial \Omega}{\partial q_i} \delta q_i + \sum_{i=1}^{n} \frac{\partial \Omega}{\partial Q_i} \delta Q_i].$$

$$(ie)., p_i = \sum_{j=1}^{m} \lambda_j \frac{\partial \Omega}{\partial q_i}. \tag{5.66}$$

$$P_i = \sum_{j=1}^{m} \lambda_j \frac{\partial \Omega}{\partial Q_i}. \tag{5.67}$$

Eqns (5.64),(5.66) and (5.67) can be used to solve $\lambda' s$,P's and Q's as functions of (q, p, t).

Arbitrary giving variations of equation(5.65) w.r.t 't'

$$\frac{\partial F_1^*}{\partial t} \delta t = \sum_{j=1}^{m} \lambda_j \frac{\partial \Omega_j}{\partial t} \delta t.$$

$$\text{W.K.T} \qquad K - H = \frac{\partial F_1^*}{\partial t}$$

$$K = H + \sum_{j=1}^{m} \lambda_j \frac{\partial \Omega_j}{\partial t}.$$

5.2.3 Point transformation

Consider

$$\Omega_j(q, Q, t) = 0. \tag{5.68}$$

$$\sum_{i=1}^{n}\frac{\partial\Omega_j}{\partial q_i}\delta q_i+\sum_{i=1}^{n}\frac{\partial\Omega_j}{\partial Q_i}\delta Q_i=0.$$

If

$$|\frac{\partial\Omega_j}{\partial q_i}|\neq 0 \qquad and \qquad |\frac{\partial\Omega_j}{\partial Q_i}|\neq 0.$$

Then Q's can be represent a point transformation. They represent a mapping of points in configuration space. Now from

$$p_i=\sum_{j=1}^{n}P_j\frac{\partial Q_i}{\partial q_i}.$$

We get

$$p_i=\sum_{j=1}^{n}P_j\frac{\partial f_i}{\partial q_i}, \qquad (i=1,2,3,...,n),$$

where p's are linear function of P's and vise versa. Now define $\Omega's$ of the form

$$\Omega's=Q_j-f_j(q,t), \qquad (j=1,2,3,...,n).$$

W.K.T

$$\begin{aligned}P_i&=-\sum_{j=1}^{n}\lambda_j\frac{\partial\Omega_j}{\partial q_i}\\&=\lambda_i.\end{aligned}$$

W.K.T

$$\begin{aligned}K&=H+\sum_{j=1}^{n}\lambda_j[-\frac{\partial f_j}{\partial t}]\\&=H+\sum_{j=1}^{n}P_j(\frac{\partial f_j}{\partial t}),\end{aligned}$$

K and H are equal only for scleronomic system.

Point transformation neednot imply homogeneous canonical transformation

Consider

$$F_1^*(\lambda,q,Q,t)=F_1(q,Q,t)+\sum_{j=1}^{n}\lambda_j\Omega_j(q,Q,t).$$

$$p_i=\frac{\partial F_1^*}{\partial q_i}=\frac{\partial F_1}{\partial q_i}+\sum_{j=1}^{n}\lambda_j\frac{\partial\Omega_j}{\partial q_i}.$$

$$p_i=-\frac{\partial F_1^*}{\partial Q_i}=-\frac{\partial F_1}{\partial Q_i}-\sum_{j=1}^{n}\lambda_j\frac{\partial\Omega_j}{\partial Q_i}.$$

The Hamiltonian functions are given by

$$K = H + \frac{\partial F_1^*}{\partial t} = H + \frac{\partial F_1}{\partial t} + \sum_{j=1}^{n} \lambda_j \frac{\partial \Omega_j}{\partial t}.$$

Consider a non-homogeneous point transformation $\psi(q, p, t)$

$$\delta\psi = \sum_{i=1}^{n} p_i \delta q_i - \sum_{i=1}^{n} P_i \delta Q_i. \tag{5.69}$$

$$\delta\psi = \sum_{i=1}^{n} \frac{\partial \psi}{\partial q_i} \delta q_i + \sum_{i=1}^{n} \frac{\partial \psi}{\partial p_i} \delta p_i \tag{5.70}$$

$$Q_j = f_j(q, t)$$

$$\delta Q_j = \sum_{i=1}^{n} \frac{\partial f_j}{\partial q_i} \delta q_i. \tag{5.71}$$

$$\sum_{j=1}^{n} P_j \delta Q_j = \sum_{i=1}^{n} \sum_{j=1}^{n} P_j \frac{\partial f_j}{\partial q_i} \delta q_i. \tag{5.72}$$

Substitute(5.72) in (5.69)

$$\delta\psi = \sum_{i=1}^{n} p_i \delta q_i - \sum_{i=1}^{n} \sum_{j=1}^{n} P_j \frac{\partial f_j}{\partial q_i} \delta q_i \tag{5.73}$$

From (5.70) and (5.73) we get

$$\frac{\partial \psi}{\partial q_i} = p_i - \sum_{j=1}^{n} P_j \frac{\partial f_j}{\partial q_i}$$

$$p_i = \frac{\partial \psi}{\partial q_i} + \sum_{j=1}^{n} P_j \frac{\partial f_j}{\partial q_i}$$

$$\text{and} \qquad \frac{\partial \psi}{\partial p_i} = 0$$

$\Rightarrow \psi$ is not a function of p's.

5.2.4 Momentum transformation

Consider the momentum transformation equation is given by $p_i = h_i(p, t)$ This represents a point transformation in momentum space and it is called a momentum transformation. Define the function

$$\omega_j(P, p, t) = 0, \quad (i = 1, 2, 3, ..., n). \tag{5.74}$$

$$\text{W.K.T} \quad p_i = \frac{\partial F_1}{\partial q_i} + \sum_{j=1}^{n} \lambda_j \frac{\partial \Omega_j}{\partial q_i}.$$

$$\text{Then} \quad q_i = -\frac{\partial F_4}{\partial p_i} - \sum_{j=1}^{n} \lambda_j \frac{\partial \omega_j}{\partial p_i}. \tag{5.75}$$

$$\text{W.K.T} \quad P_i = -\frac{\partial F_1}{\partial Q_i} - \sum_{j=1}^{n} \lambda_j \frac{\partial \Omega_j}{\partial Q_i}.$$

$$\text{Then} \quad Q_i = -\frac{\partial F_4}{\partial P_i} - \sum_{j=1}^{n} \lambda_j \frac{\partial \omega_j}{\partial Pi}. \tag{5.76}$$

$$\text{W.K.T} \quad K = H + \frac{\partial F_1}{\partial t} + \sum_{j=1}^{n} \lambda_j \frac{\partial \Omega_j}{\partial t}.$$

$$\text{Then} \quad K = H + \frac{\partial F_1}{\partial t} + \sum_{j=1}^{n} \lambda_j \frac{\partial \omega_j}{\partial t} \qquad (m \le n). \tag{5.77}$$

Consider $\omega_j = p_j - h_j(p,t),\ (j = 1, 2, 3, ..., n)$. From (5.75)

$$q_i = -\frac{\partial F_4}{\partial p_i} + \sum_{j=1}^{n} \lambda_j \frac{\partial h_j}{\partial p_i}.$$

$$Q_i = \frac{\partial F_4}{\partial P_i} + \sum_{j=1}^{n} \lambda_j \frac{\partial \omega_j}{\partial p_i}$$

$$\frac{\partial \omega_j}{\partial p_i} Q_i = \frac{\partial F_4}{\partial P_i} + \lambda_i. \tag{5.78}$$

Hence we have $Q_i = \lambda_i$. Then

$$q_i = -\tfrac{\partial F_4}{\partial p_i} + \textstyle\sum_{j=1}^{n} Q_j \tfrac{\partial h}{\partial p_i}.$$

W.K.T from (5.72)

$$K = H + \frac{\partial F_4}{\partial t} - \sum_{j=1}^{n} \lambda_j \frac{\partial h_j}{\partial t}$$

$$K = H + \frac{\partial F_4}{\partial t} - \sum_{j=1}^{n} Q_j \frac{\partial h_j}{\partial t} \qquad [\because \lambda_j = Q_j]$$

(5.73) becomes,

$$\lambda_i = \frac{\partial F_4}{\partial P_i} + \lambda_i$$

$$\frac{\partial F_4}{\partial P_i} = 0.$$

Let us sum up

1. We have introduce the some special transformations.

2. We have classified the canonical transformation namely identity, orthogonal, translation, homogenous, point and momentum transformation.

Check your progress

4. What is the another name of homogenous canonical transformation.

5. Write the equation of identity transformation.

6. Define momentum transformation.

5.3 Lagrange and Poisson Brackets

Dear students, in this section we will discuss the Lagrange and Poisson brackets. Also we will discuss the properties of Poisson brackets and poisson theorem. Finially, we will discuss the bilinear covarient.

5.3.1 Lagrange Bracket

Suppose we are given the transformation equation of the form $Q_i = Q_i(q,p,t)$ and $P_i = P_i(q,p,t)$. If u and v are functions of $Q_1, Q_2, ..., Q_n$ and $P_1, P_2, ..., P_n$ then the lagrangian brackets $[u,v]$ is defined by

$$\begin{aligned}[u,v] &= \sum_{i=1}^{n}(\frac{\partial Q_i}{\partial u}\frac{\partial P_i}{\partial v} - \frac{\partial Q_i}{\partial v}\frac{\partial P_i}{\partial u}) \\ &= \sum_{i=1}^{n}(\frac{\partial q_i}{\partial u}\frac{\partial p_i}{\partial v} - \frac{\partial q_i}{\partial v}\frac{\partial p_i}{\partial u}).\end{aligned} \tag{5.79}$$

Consider

$$\begin{aligned}
\delta\psi &= \sum_{j=1}^{n} p_j \delta q_j - \sum_{j=1}^{n} P_j \delta Q_j \\
&= \sum_{j=1}^{n} p_j \delta q_j - \sum_{i=1}^{n} P_i (\sum_{i=1}^{n} (\frac{\partial Q_i}{\partial q_j}\delta q_j + \frac{\partial Q_i}{\partial p_j}\delta p_j)) \\
&= \sum_{j=1}^{n} (p_j - \sum_{i=1}^{n} P_i \frac{\partial Q_i}{\partial q_i}\delta q_j - \sum_{i=1}^{n}\sum_{j=1}^{n} P_i \frac{\partial Q_i}{\partial p_j}\delta p_j) \\
&= \sum_{j=1}^{n} (p_j - \sum_{i=1}^{n} P_i \frac{\partial Q_i}{\partial q_i}\delta q_j - \sum_{i=1}^{n}\sum_{j=1}^{n} P_i \frac{\partial Q_i}{\partial p_j}\delta p_j).
\end{aligned} \tag{5.80}$$

To check the exactness

$$\begin{aligned}
A. \qquad \frac{\partial}{\partial p_k}(\sum_{i=1}^{n} P_i \frac{\partial Q_i}{\partial p_j}) &= \frac{\partial}{\partial p_j}(\sum_{i=1}^{n} P_i \frac{\partial Q_i}{\partial p_k}) \\
\sum_{i=1}^{n} P_i (\frac{\partial^2 Q_i}{\partial p_j \partial p_k}) + \sum_{i=1}^{n} \frac{\partial P_i}{\partial p_k}\frac{\partial Q_i}{\partial p_j} &= \sum_{i=1}^{n} P_i (\frac{\partial^2 Q_i}{\partial p_j \partial p_k}) + \sum_{i=1}^{n} \frac{\partial P_i}{\partial p_j}\frac{\partial Q_i}{\partial p_k} \\
\sum_{i=1}^{n} (\frac{\partial P_i}{\partial p_k}.\frac{\partial Q_i}{\partial p_j} - \frac{\partial P_i}{\partial p_j}.\frac{\partial Q_i}{\partial p_k}) &= 0.
\end{aligned} \tag{5.81}$$

$$\begin{aligned}
B. \qquad \frac{\partial}{\partial p_k}(p_j - \sum_{i=1}^{n} P_i \frac{\partial Q_i}{\partial p_j}) &= \frac{\partial}{\partial q_j}(p_k - \sum_{i=1}^{n} P_i \frac{\partial Q_i}{\partial q_k}) \\
-\sum_{i=1}^{n} \frac{\partial P_i}{\partial q_k}\frac{\partial Q_i}{\partial q_j} - \sum_{i=1}^{n} P_i \frac{\partial^2 Q_i}{\partial q_j \partial q_k} &= \sum_{i=1}^{n} \frac{\partial P_j}{\partial q_j}\frac{\partial Q_i}{\partial q_k} - \sum_{i=1}^{n} P_i \frac{\partial^2 Q_i}{\partial q_j \partial q_k} \\
\sum_{i=1}^{n} (\frac{\partial P_i}{\partial q_k}.\frac{\partial Q_i}{\partial q_j} - \frac{\partial P_i}{\partial q_j}.\frac{\partial Q_i}{\partial q_k}) &= 0.
\end{aligned} \tag{5.82}$$

$$\begin{aligned}
C. \qquad \frac{\partial}{\partial p_k}(p_i - \sum_{i=1}^{n} P_i \frac{\partial Q_i}{\partial p_j}) &= \frac{\partial}{\partial q_j}(-\sum_{i=1}^{n} P_i \frac{\partial Q_i}{\partial p_k}) \\
\frac{\partial P_i}{\partial p_k} - \sum_{i=1}^{n} \frac{\partial P_i}{\partial p_k}.\frac{\partial Q_i}{\partial q_j} - \sum_{i=1}^{n} P_i \frac{\partial^2 Q_i}{\partial q_j \partial q_k} &= \sum_{i=1}^{n} (\frac{\partial P_i}{\partial p_k}.\frac{\partial Q_i}{\partial q_j} + P_i \frac{\partial^2 Q_i}{\partial q_j \partial P_k}) \\
\sum_{i=1}^{n} (\frac{\partial P_i}{\partial p_k}.\frac{\partial Q_i}{\partial q_j} - \frac{\partial P_i}{\partial q_j}.\frac{\partial Q_i}{\partial p_k}) &= \delta_{jk}.
\end{aligned} \tag{5.83}$$

Using the differential of Lagrange Bracket equation (5.82), (5.83) and (5.84) can be modified as

$$
\begin{aligned}
(5.82) \Longrightarrow \sum_{i=1}^{n}(\frac{\partial P_i}{\partial q_k}.\frac{\partial Q_i}{\partial q_j} - \frac{\partial P_i}{\partial q_j}.\frac{\partial Q_i}{\partial q_k}) &= 0 \\
\Longrightarrow [p_j, p_k] &= 0 \\
(5.83) \Longrightarrow \sum_{i=1}^{n}(\frac{\partial Q_i}{\partial q_j}.\frac{\partial P_i}{\partial q_k} - \frac{\partial P_i}{\partial q_j}.\frac{\partial Q_i}{\partial q_k}) &= 0 \\
\Longrightarrow [q_j, q_k] &= 0 \\
(5.84) \Longrightarrow \sum_{i=1}^{n}(\frac{\partial Q_i}{\partial q_j}.\frac{\partial P_i}{\partial p_k} - \frac{\partial P_i}{\partial q_j}.\frac{\partial Q_i}{\partial p_k}) &= 0 \\
\Longrightarrow [q_j, p_k] &= 0.
\end{aligned}
$$

Alternating the Lagrangian brackets can be defined using Jacobi determinants (ie.,) $[u, v] = \sum_{i=1}^{n} \frac{\partial(Q_i, P_i)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial Q_i}{\partial u} & \frac{\partial Q_i}{\partial u} \\ \frac{\partial Q_i}{\partial u} & \frac{\partial Q_i}{\partial u} \end{vmatrix}$

Properties of Lagrange's bracket

1. $[u, v] = 0$.
2. $[v, u] = 0$.
3. $[u, v] = -[v, u]$.

5.3.2 Poisson Brackets

suppose we have two functions namely, $u(q, p, t)$ and $v(q, p, t)$. The Poisson of (u, v) is defined as

$(u, v) = \sum_{i=1}^{n}(\frac{\partial u}{\partial q_i}.\frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i}.\frac{\partial v}{\partial q_i})$.

Properties of Poisson bracket

1. $(u, v) = 0$, 2. $(v, u) = 0$, 3. $(u, v) = -(v, u)$.

Necessary and sufficient for the transformation to be canonical

To prove $(P_j, p_k) = 0$, $(Q_j, Q_k) = 0$, $(Q_j, P_k) = \delta_{jk}$. Consider the transformation

$Q_i = Q_i(q, p, t)$, $P_i = Q_i(q, p, t)$, $p_j = p_j(Q, P, t)$, $q_j = q_j(Q, P, t)$.

$$\delta q_j = \sum_{i=1}^{n}(\frac{\partial q_j}{\partial Q_i}\delta Q_i + \frac{\partial q_i}{\partial P_i}\delta P_i). \tag{5.84}$$

$$\delta p_j = \sum_{i=1}^{n}(\frac{\partial p_j}{\partial Q_i}\delta Q_i + \frac{\partial p_i}{\partial P_i}\delta P_i). \tag{5.85}$$

$$Where \qquad \delta Q_i = \sum_{k=1}^{n}(\frac{\partial Q_i}{\partial q_k}\delta q_k + \frac{\partial Q_i}{\partial p_k}\delta p_k). \tag{5.86}$$

$$\delta P_i = \sum_{k=1}^{n}(\frac{\partial P_i}{\partial q_k}\delta q_k + \frac{\partial P_i}{\partial p_k}\delta p_k). \tag{5.87}$$

Substitute (5.87), (5.88) in (5.85) and (5.86)

$$\delta q_j = \sum_{i=1}^{n}\frac{\partial q_j}{\partial Q_i}\left(\sum_{k=1}^{n}(\frac{\partial Q_i}{\partial q_k}\delta q_k + \frac{\partial Q_i}{\partial p_k}\delta p_k)\right) + \sum_{i=1}^{n}\frac{\partial q_i}{\partial P_i}\left(\sum_{k=1}^{n}(\frac{\partial P_i}{\partial q_k}\delta q_k + \frac{\partial P_i}{\partial p_k}\delta p_k)\right).$$

$$\sum_{i=1}^{n}\left(\frac{\partial q_i}{\partial Q_i}.\frac{\partial Q_i}{\partial q_k} + \frac{\partial q_j}{\partial P_i}.\frac{\partial P_i}{\partial q_k}\right) = \delta_{jk}. \tag{5.88}$$

$$\sum_{i=1}^{n}\left(\frac{\partial q_j}{\partial Q_i}.\frac{\partial Q_i}{\partial p_k} + \frac{\partial q_i}{\partial P_i}.\frac{\partial P_i}{\partial p_k}\right) = 0. \tag{5.89}$$

similarly
$$\sum_{i=1}^{n}\left(\frac{\partial p_j}{\partial Q_i}.\frac{\partial Q_i}{\partial p_k} + \frac{\partial p_j}{\partial P_i}.\frac{\partial P_i}{\partial p_k}\right) = \delta_{jk}. \tag{5.90}$$

$$\sum_{i=1}^{n}\left(\frac{\partial p_j}{\partial Q_i}.\frac{\partial Q_i}{\partial q_k} + \frac{\partial p_j}{\partial P_i}.\frac{\partial P_i}{\partial q_k}\right) = 0. \tag{5.91}$$

Assume that the Lagrangian brackets satisfies

$$(i) \qquad [q_j, p_k] = \delta_{jk}$$

$$[q_j, p_k] = \sum_{i=1}^{n}\left(\frac{\partial Q_i}{\partial q_j}.\frac{\partial P_i}{\partial p_k} - \frac{\partial Q_i}{\partial p_k}.\frac{\partial P_i}{\partial q_j}\right) = \delta_{jk}. \tag{5.92}$$

Compare (5.91) and (5.93)

$$\frac{\partial Q_i}{\partial q_j} = \frac{\partial p_j}{\partial P_i}, \qquad \frac{\partial P_i}{\partial q_j} = -\frac{\partial p_j}{\partial Q_i}$$

$$(ii) \qquad [q_k, p_j] = \delta_{jk}$$

$$[q_j, p_k] = \sum_{i=1}^{n}\left(\frac{\partial Q_i}{\partial q_k}.\frac{\partial P_i}{\partial p_i} - \frac{\partial Q_i}{\partial p_j}.\frac{\partial P_i}{\partial q_k}\right). \tag{5.93}$$

Compare (5.94) and (5.88)

$$\frac{\partial P_i}{\partial p_j} = \frac{\partial q_j}{\partial Q_i}, \qquad -\frac{\partial Q_i}{\partial P_j} = \frac{\partial q_j}{\partial P_i}.$$

Consider,

$$\begin{aligned}[q_j, q_k] =& \sum_{i=1}^{n} \left(\frac{\partial Q_i}{\partial q_j} . \frac{\partial P_i}{\partial q_k} - \frac{\partial Q_i}{\partial q_k} . \frac{\partial P_i}{\partial q_j} \right) \\ =& \sum_{i=1}^{n} \left(-\frac{\partial P_j}{\partial p_i} . \frac{\partial P_k}{\partial Q_i} + \frac{\partial P_k}{\partial p_i} . \frac{\partial P_j}{\partial Q_i} \right).\end{aligned}$$

Since $[q_j, q_k] = 0 \qquad \Longrightarrow \qquad (p_j, p_k) = 0$

Consider,

$$\begin{aligned}[p_j, p_k] =& \sum_{i=1}^{n} \left(-\frac{\partial Q_i}{\partial p_k} . \frac{\partial P_k}{\partial Q_i} + \frac{\partial P_k}{\partial p_i} . \frac{\partial P_j}{\partial Q_i} \right) \\ =& \sum_{i=1}^{n} \left(-\frac{\partial q_k}{\partial P_i} . \frac{\partial q_j}{\partial Q_i} - \frac{\partial q_j}{\partial P_i} . \frac{\partial q_k}{\partial Q_i} \right) \\ =& (q_j, q_k).\end{aligned}$$

Since $[p_j, p_k] = 0 \qquad \Longrightarrow \qquad (q_j, q_k) = 0$

Consider,

$$\begin{aligned}[q_j, p_k] =& \sum_{i=1}^{n} \left(-\frac{\partial Q_i}{\partial p_j} . \frac{\partial P_i}{\partial p_k} - \frac{\partial Q_i}{\partial p_k} . \frac{\partial P_i}{\partial p_j} \right) \\ =& \sum_{i=1}^{n} \left(-\frac{\partial P_j}{\partial p_i} . \frac{\partial q_k}{\partial p_k} - \frac{\partial q_k}{\partial P_i} . \frac{\partial P_j}{\partial Q_i} \right) \\ =& (q_k, p_j)\end{aligned}$$

Since $[q_j, p_k] = \delta_{jk} \qquad \Longrightarrow \qquad (q_k, p_j) = \delta_{jk}$.

Hence the poisson brackets confirms that the transformation is canonical.

Special properties of Poisson brackets

1. Writing hamilton's canonical equation using Poisson brackets. Consider (q_i, H)

$$\begin{aligned}
(q_i, H) &= \sum_{i=1}^{n} \left(-\frac{\partial q_i}{\partial q_k} . \frac{\partial H}{\partial p_k} - \frac{\partial H}{\partial q_k} . \frac{\partial q_i}{\partial p_k} \right) \\
&= \frac{\partial H}{\partial p_i} \\
&= \dot{q}_i. \\
(p_i, H) &= \sum_{i=1}^{n} \left(-\frac{\partial p_i}{\partial q_k} . \frac{\partial H}{\partial p_k} - \frac{\partial H}{\partial q_k} . \frac{\partial p_i}{\partial p_k} \right) \\
&= -\frac{\partial H}{\partial q_i} \\
&= \dot{p}_i \\
\dot{q}_i &= (q_i, H) \\
\dot{p}_i &= (p_i, H).
\end{aligned}$$

2. Consider a dynamical system specified by the function $f(q, p, t)$

Now

$$\begin{aligned}
\frac{\partial f}{\partial t} &= \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i} . \dot{q}_i - \frac{\partial f}{\partial p_i} \dot{p}_i \right) + \frac{\partial f}{\partial t} \\
&= (f, H) + \frac{\partial f}{\partial t}.
\end{aligned}$$

If f is not an explicit function of time t

$\frac{\partial f}{\partial t} = 0$ then $\frac{\partial f}{\partial t} = (f, t)$

If f is a constant of motion, then $(f, H) = 0$

Poisson Theorem:

If $u(q, p)$ and $v(q, p)$ are integrals of a Hamiltonian system, then the Poisson bracket (u, v) is an integral. (ie..,) (u, v) is a constant of the motion

Proof: Now $u(p, q)$

$\frac{du}{dt} = 0 \qquad (\therefore$ u is constant) (ie.,) $(u,H)+\frac{\partial u}{\partial t}=0$. Also,

$$\frac{\partial}{\partial t}(u,v)=((u,v),H)+\frac{\partial}{\partial t}(u,v).$$

5.3.3 Bilinear covariant

Consider the pfaffian differential form

$$\Omega=\sum_{i=1}^{n}X_i(x)dx_i. \tag{5.94}$$

Bilinear covariant system is given by

$$\delta\Omega-d\omega=\sum_{i=1}^{n}\sum_{j=1}^{n}c_{ij}\delta x_j dx_j. \tag{5.95}$$

Consider the canonical transformation from (q,p) to (Q,P)

$$d\psi=\sum_{i=1}^{n}p_idq_i-\sum_{i=1}^{n}P_idQ_i. \tag{5.96}$$

$$\delta(d\psi)=\sum_{i=1}^{n}(p_i\delta q_i+\delta p_idq_i-\delta p_idq_i-P_id\delta Q_i). \tag{5.97}$$

Now $$\delta\psi=\sum_{i=1}^{n}p_i\delta q_i-\sum_{i=1}^{n}P_i\delta Q_i. \tag{5.98}$$

$$d(\delta\psi)=\sum_{i=1}^{n}(dp_i\delta q_i+p_id\delta q_i-dP_i\delta Q_i-P_id\delta Q_i). \tag{5.99}$$

(5.98)-(5.100)

$$\sum_{i=1}^{n}(\delta p_idq_i-dp_i\delta q_i)=\sum_{i=1}^{n}(\delta P_idQ_i-P_id\delta Q_i).$$

$\sum_{i=1}^{n}(\delta p_idq_i-dp_i\delta q_i)$ is a bilinear covariant.
Hence the bilinear covariant is invariant w.r.t the canonical transformation.

Relationship between Lagrange and poisson brackets

Prove that Lagrange and poisson brackets are reciprocal quantities(or)

Prove that $LP=I$ (or) prove that $L=P^{-1}I$

$$\sum_{k=1}^{2n}[u_i;u_k](u_j,u_k)=\sum_{k=1}^{2n}[(\sum_{r=1}^{n}\frac{\partial q_r}{\partial u_i}\frac{\partial p_r}{\partial u_k}-\frac{\partial p_r}{\partial u_i}\frac{\partial q_r}{\partial u_k})\sum_{j=1}^{n}(\frac{\partial u_j}{\partial q_s}\frac{\partial u_k}{\partial p_s}-\frac{\partial u_j}{\partial p_s}\frac{\partial u_k}{\partial p_s})]$$

$$=\sum_{k=1}^{2n}\sum_{r=1}^{n}\sum_{j=1}^{n}[\frac{\partial q_r}{\partial u_i}.\frac{\partial p_r}{\partial u_k}.\frac{\partial u_j}{\partial q_s}.\frac{\partial u_k}{\partial p_s}-\frac{\partial q_r}{\partial u_i}.\frac{\partial p_r}{\partial u_k}.\frac{\partial u_j}{\partial p_s}.\frac{\partial u_k}{\partial p_s}-\frac{\partial p_r}{\partial u_i}.\frac{\partial q_r}{\partial u_k}.\frac{\partial u_j}{\partial q_s}.\frac{\partial u_k}{\partial p_j}+\frac{\partial p_r}{\partial u_i}.\frac{\partial q_r}{\partial u_k}.\frac{\partial u_j}{\partial p_s}.\frac{\partial u_k}{\partial p_s}]. \tag{5.100}$$

We have

$$\sum_{k=1}^{2n}(\frac{\partial q_r}{\partial u_k},\frac{\partial u_k}{\partial q_s}) = \delta_{rs} = 1$$
$$\sum_{k=1}^{2n}(\frac{\partial p_r}{\partial u_k},\frac{\partial u_k}{\partial p_s}) = \delta_{rs} = 1$$
$$\sum_{k=1}^{2n}(\frac{\partial q_r}{\partial u_k},\frac{\partial u_k}{\partial p_s}) = 0$$
$$\sum_{k=1}^{2n}(\frac{\partial p_r}{\partial u_k},\frac{\partial u_k}{\partial q_s}) = 0. \tag{5.101}$$

substitute (5.102) in (5.101)

$$\begin{aligned}
\sum_{k=1}^{2n}[u_i;u_k](u_j,u_k) &= \sum_{r=1}^{n}\sum_{j=1}^{n}[\frac{\partial q_r}{\partial u_i}.\frac{\partial u_j}{\partial q_s}+\frac{\partial p_r}{\partial u_i}\frac{\partial u_j}{\partial p_s}] \\
&= \sum_{r=1}^{n}(\sum_{r=1}^{n}\sum_{j=1}^{n}[\frac{\partial q_r}{\partial u_i}.\frac{\partial u_j}{\partial q_r}+\frac{\partial p_r}{\partial u_i}\frac{\partial u_j}{\partial p_r}]) \\
&= \frac{\partial u_j}{\partial u_i} \\
&= \delta_{ij} = 1 \\
\because \sum_{k=1}^{2n}[u_i;u_k](u_j,u_k) &= 1 \\
Consider \quad L_{ik} &= [u_i,u_k] \\
P_{kj} &= (u_j,u_k) \\
\therefore LP &= 1 \\
L &= p^{-1}.
\end{aligned}$$

Jacobi identity

$(u,(v,w))+(v,(w,u))+(w,(u,v))=0$.

Consider

$$\begin{aligned}
(u,(v,w))-(v,(u,w)) &= (u,\sum_k(\frac{\partial v}{\partial q_k}.\frac{\partial w}{\partial p_k}-\frac{\partial w}{\partial q_k}\frac{\partial v}{\partial p_k}))-(v,\sum_k(\frac{\partial u}{\partial q_k}.\frac{\partial w}{\partial p_k}-\frac{\partial w}{\partial q_k}\frac{\partial w}{\partial p_k})) \\
&= (u,\sum_k(\frac{\partial v}{\partial q_k}.\frac{\partial w}{\partial p_k}))-(u,\sum_k(\frac{\partial w}{\partial q_k}.\frac{\partial v}{\partial p_k}))(v,\sum_k(\frac{\partial u}{\partial q_k}.\frac{\partial w}{\partial p_k}))+(v,\sum_k(\frac{\partial w}{\partial q_k}.\frac{\partial w}{\partial p_k})).
\end{aligned} \tag{5.102}$$

By the property, $(u,(v,w)) = (u,v)w + (u,w)v$ (5.101) becomes

$$= \sum_k \frac{\partial w}{\partial p_k}[(u, \sum_k \frac{\partial v}{\partial q_k}) - (v, \sum_k \frac{\partial u}{\partial q_k})] + \sum_k \frac{\partial w}{\partial q_k}[(v, \sum_k \frac{\partial u}{\partial p_k}) - (u, \sum_k \frac{\partial v}{\partial p_k})] +$$
$$\sum_k \frac{\partial w}{\partial q_k}[(u, \sum_k \frac{\partial w}{\partial p_k}) - \frac{\partial v}{\partial p_k}(u, \sum_k \frac{\partial w}{\partial q_k})] + \sum_k [\frac{\partial u}{\partial p_k}(v, \sum_k \frac{\partial w}{\partial q_k}) - \frac{\partial u}{\partial q_k}(v, \sum_k \frac{\partial w}{\partial p_k})]$$
$$= \sum_k [\frac{\partial w}{\partial p_k}[(u, \frac{\partial v}{\partial p_k}) + (\frac{\partial u}{\partial q_k}, v] - \frac{\partial w}{\partial q_k}[(\frac{\partial u}{\partial p_k}, v) + (u, \frac{\partial v}{\partial p_k})]] +$$
$$\sum_k [\frac{\partial v}{\partial q_k}(u, \frac{\partial w}{\partial p_k}) - \frac{\partial v}{\partial p_k}(u, \frac{\partial w}{\partial q_k}) + \frac{\partial u}{\partial p_k}(v, \frac{\partial w}{\partial q_k}) - \frac{\partial u}{\partial q_k}(v, \frac{\partial w}{\partial p_k})].$$

W.K.T

$$\begin{aligned} \frac{\partial}{\partial x}(u,v) &= (\frac{\partial u}{\partial x}, v) + (u, \frac{\partial v}{\partial x}) \\ &= \sum_k [-\frac{\partial w}{\partial q_k}.\frac{\partial}{\partial p_k}(u,v) + \frac{\partial w}{\partial p_k}.\frac{\partial}{\partial q_k}(u,v) + 0] \\ &= -\sum_k [\frac{\partial w}{\partial q_k}.\frac{\partial}{\partial p_k}(u,v) - \frac{\partial}{\partial q_k}(u,v)\frac{\partial w}{\partial p_k}] \\ &= -(w,(u,v)) \end{aligned}$$

$$\therefore \qquad (u,(v,w)) - (v,(u,w)) = -(w,(u,v))$$

$$(u,(v,w))) + (v,(w,u)) + (w,(u,v)) = 0.$$

Let us sum up

1. We have discuss the Lagrange and Poisson bracket.
2. We have derive the Poisson theorem.
3. We have discuss the Bilinear covariant.

Check your progress

7. Define Lagrangian brackets.
8. Explain skew symmetry.

Summary

- Introduce the cononical transformations with few examples.
- Discuss the principle forms of generating functions.

- Discuss the comments on Hamilton-Jacobi's method.
- Introduce the some special transformations.
- Classified the canonical transformation namely identity, orthogonal, translation, homogenous, point and momentum transformation.
- Discuss the Lagrange and Poisson bracket.
- Derive the Poisson theorem.
- Discuss the bilinear covariant.

Glossary

- **Homogenous canonical transformation:** Consider the function ϕ and ψ are identically zero. Then $\sum_{i=1}^{n}(P_i \delta q_i - P_i \delta Q_i) = 0$ and the corresponding transformation is called a homogenous canonical transformation. This transformation is also known as Mathieu transformation or constant transformation.
- **Lagrangian brackets:** Expression of two variable (u, v) by using the notation $[u, v] = \sum_{i=1}^{n} \left(\frac{\partial Q_i}{\partial u}\frac{\partial P_i}{\partial v} - \frac{\partial P_i}{\partial u}\frac{\partial Q_i}{\partial v}\right)$, where u and v are any two variables $q_1, q_2, .., q_n,\ p_1, p_2, .., p_n$.
- **Poisson brackets:** The function of the dynamical variable and time namely $u = u(q, p, t)$ and $v = v(q, p, t)$. The Poission bracket expression for function is $(u, v) = \sum_{i=1}^{n} \left(\frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i}\right)$.
- **Poisson theorm:** If $u(q, p)$ and $v(q, p)$ are integrals of a Hamiltonian system, then the Poisson bracket (u, v) is also an integral, that is (u, v) is constant of the motion.

Self-Assessment Questions

Short-Answer Questions

1. Consider the transformation $Q = \sqrt{e^{-2q} - p^2}$, $P = \cos^{-1}(pe^{q})$. Use poisson bracket

to show that it is canonical.

2. Show that $Q = \sqrt{2q}e^t \cos p$, $P = \sqrt{2q}e^{-t} \sin p$ is canonical.

3. Let the transformation $Q = \frac{1}{2}(q^2 + p^2)$, $P = -\tan^{-1}\left(\frac{q}{p}\right)$. Show that this transformation is canonical.

4. Prove that canonical transformation is invariant under canonical transformation.

5. Derive the equation $\delta\Omega - d\theta = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} dx_i \delta x_j$ under bilinear co-variant.

6. Explain the differential forms of Pfaffian differential equation.

Long-Answer Questions:

1. Explain Poisson brackets.

2. Derive the principle of generating functions.

3. State and prove Poisson's theorem.

4. Consider the transformation $Q = \log \frac{\sin p}{q}$, $P = q \cdot p$. Obtain the four major types of generating function associated with the transformation.

5. Explain the types of transformation.

6. Write bilinear co-variant with the differential form of pfaffian function Ω.

7. Show that the value of a Lagrangian bracket is invariant under canonical transformation.

8. Prove that poisson brackets is Jacobi's identity.

Objective Questions

1) Hamiltonian's canonical equations in terms of Poisson brackets are

a) $q_i = (\dot{q}_i, H), P_i = (\dot{p}_i, H)$ b) $\dot{q}_i = (q_i, H), P_i = (P_i, H)$

c) $\dot{q}_i = (q_i, H), \dot{p}_i = (P_i, H)$ d) $(q_j, q_k) = 0, (P_j, P_k) = 0, (q_j, p_k) = \delta_{jk}$

2) In a canonical transformation the first generating function is a function of

a) (p, q, t) b) (q, Q, t) c) (p, Q, t) d) (q, P, t)

3) If the poisson bracket of a function with the Hamiltonian vanishes

a) The function depends upon time b) The function is a constant of motion

c) The function is not the constant of motion d) The function is canonical function

4) The Poisson bracket expression for the function $u(q,p,t)$ and $v(q,p,t)$ is $(u,v) =$

a)$\sum_{i=1}^{n}(\frac{\partial u}{\partial q_i}\frac{\partial u}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial u}{\partial q_i})$ b)$\sum_{i=1}^{n}(\frac{\partial u}{\partial q_i}\frac{\partial q_i}{\partial t} - \frac{\partial u}{\partial t}\frac{\partial u}{\partial p_i})$

c)$\sum_{i=1}^{n}(\frac{\partial^2 u}{\partial q_i \partial p_i} - \frac{\partial^2 u}{\partial p_i \partial q_i})$ d)$\sum_{i=1}^{n}(\frac{\partial^2 u}{\partial q_i \partial p_i} - \frac{\partial^2 u}{\partial p_i \partial q_i})$

5) Poisson bracket is

a) Invariant under canonical transformation b) Variant under canonical transformation.

c) Both (a) and (b) d) Canonical transformation

6) In a Canonical transformation the third type generating function is a function of

a) (q,p,t) b) (p,P,t) c) (q,Q,t) d) (p,Q,t)

7) A transformation from (q,p) to (Q,P) which preserves the canonical form of the equation of motion is known as

a) Canonical transformation b) Point transformation

c) Momentum transformation d) Identity transformation

8) Given transformation equations $Q = q^m \cos np$ and $P = q^m \sin np$.

a) for $m = \frac{1}{2}$ and $p = 2$ the transformation equation become canonical

b) for $m = \frac{1}{2}$ and $p = 2$. It is not canonical.

c) for $m = 2$, $p = \frac{1}{2}$. It is canonical.

d) for $m = 2$, $P = 2$.It is canonical.

9) In a Canonical transformation the fourth type generating function is a function of

a) (p,P,t) b) (q,Q,t) d) (P,Q,t) d) (q,P,t)

10) Contact transformation is also known as

a) Momentum transformation b) Identity transformation

c) Orthogonal transformation d) Homogeneous Canonical transformation.

11) Homogeneous Canonical transformation is also known as

a) Mathieu transformation b) Point transformation

c) Momentum transformation d) Orthogonal transformation.

12) The Momentum transformation is a ————————- in a momentum space.

a) Legendre transformation b) Point transformation

c) Canonical transformation d) Co-ordinate transformation.

13) The term $(u,(v,w))+(v,(w,u))+(w,(u,v))=0$is called

a) Jacobi's identify b) Poisson bracket c)Lagrange bracket d) Jacobi's identify

14) Lagrange's bracket is a) Canonical invariant b) Canonical variant c) Non-Invariant d) Invariant under Canonical transformation

15) The relation between matrix form of Lagrange and Poisson brackets are

a) LP=1 b) L=P-1 c) Both a) and b) d) L=P

16) How many different forms of generating functions are there

a) 2 b) 3 c) 4 d) 5

17) In a canonical transformation the second generating function is a function of

a) (q,Q,t) b) (q,P,t) c) (p,P,t) d) (p,Q,t)

18) The principal forms of generating functions of F_2 is

a)$F_1(q,Q,t)+\sum_{i=1}^{n}Q_iP_i$ b)$F_1(q,Q,t)-\sum_{i=1}^{n}q_iP_i$

c)$F_1(q,Q,t)-\sum_{i=1}^{n}q_iP_i$ d)$F_1(q,Q,t)-\sum_{i=1}^{n}Q_iP_i$.

Answers for Check Your Progress

1. $\frac{\partial F_3}{\partial t}+H\left(-\frac{\partial F_3}{\partial p},p,t\right)=0$ and $\frac{\partial F_4}{\partial t}+H\left(-\frac{\partial F_4}{\partial p},p,t\right)=0$.

2. The Hamilton canonical function $\dot{Q}_i=\frac{\partial K}{\partial P_i}$, $\dot{P}_i=\frac{\partial H}{\partial Q_i}$, $i=1,2,..,n$ A transformation from (q,p) to (Q,P) which preserves the canonical form of the equation of motion is known as canonical transformation.

3. The various types of generating functions namely $F_1(q,Q,t)$, $F_2(q,P,t)$, $F_3(p,Q,t)$ and $F_4(p,P,t)$. the relationship of the generating function F_1,F_2,F_3 and F_4 are called the principle of generating function.

4. Homogenous canonical transformation is also known as Mathieu transformation or Contact transformation.

5. $F_3=\sum_{i=1}^{n}P_iQ_i$.

6. The momentum transformation of the form $\omega_j=Q_j-f_j(q,t), \quad j=1,2,..n$.

7. Lagrangian brackets expression of two variables (u,v) by using the notation $[u,v]=\sum_{i=1}^{n}\left(\frac{\partial Q_i}{\partial u}\frac{\partial P_i}{\partial v}-\frac{\partial P_i}{\partial u}\frac{\partial Q_i}{\partial v}\right)$, where u and v are any two variables $q_1,q_2,.,q_n,p_1,p_2,..,p_n$.

8. A consequence of a two skew symmetry of the Lagrangian bracket is $[u, v] = -[v, u]$ and $[u, u] = [v, v] = 0$.

Suggested Readings

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